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STABLE ALGEBRAS OF ENTIRE FUNCTIONS

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Abstract. Suppose that h and g belong to the algebra $\mathcal{B}$ generated by the rational functions and an entire function f of finite order on $\mathbb{C}^n$ and that h/g has algebraic polar variety. We show that either $h/g \in \mathcal{B}$ or $f = q_1 e^p + q_2$, where p is a polynomial and q_1, q_2 are rational functions. In the latter case, h/g belongs to the algebra generated by the rational functions, e^p and e^{-p}.

The stability property is related to the problem of algebraic dependence of entire functions over the ring of polynomials. The case of algebraic dependence over $\mathbb{C}$ of two entire or meromorphic functions on $\mathbb{C}^n$ is completely resolved in this paper.

1. Introduction

In this paper we approach the problem of algebraic independence of entire functions on $\mathbb{C}^n$ over $\mathbb{C}$ or over the ring of polynomials $\mathcal{P}^n$ on $\mathbb{C}^n$. Two entire functions f and g are algebraically dependent if there is a non-zero polynomial P in $\mathbb{C}[x,y]$ or in $\mathcal{P}^n[x,y]$ such that $P(f,g) \equiv 0$.

The first case, i.e., $P \in \mathbb{C}[x,y]$, can be answered completely, and this is done in Section 5. We prove in Theorem 5.1 that if f and g are algebraically dependent entire functions over $\mathbb{C}$, then there is an entire function h such that either $f, g \in \mathbb{C}[h]$ or $f, g \in \mathbb{C}[e^h, e^{-h}]$. In Theorem 5.2 such a description is given for meromorphic functions f and g.

To approach the seemingly difficult problem of algebraic dependence of functions over $\mathcal{P}^n$ we study in this paper the dependence relation $P(f,g) = P_1(f)g + P_0(f) \equiv 0$, where $P_1, P_0 \in \mathcal{P}^n[x]$. In other words, if f is entire we are asking when the ratio $P_0(f)/P_1(f)$ is also entire.

This problem has a rather long history. Following [BD], we say that a subalgebra $\mathcal{B}$ of an algebra $\mathcal{A}$ is *stable in* $\mathcal{A}$ if whenever $g, h \in \mathcal{B}$ and $h/g \in \mathcal{A}$, then $h/g \in \mathcal{B}$.

In 1929 Ritt ([R1], [R2]) proved that the algebra of quasipolynomials $\mathcal{Q}$ is stable in the algebra of entire functions on $\mathbb{C}$. Quasipolynomials are linear combinations of exponentials $e^{\lambda_j z}$. In [S] Shields improved Ritt's theorem by showing that $\mathcal{Q}$ is also stable in the algebra of meromorphic functions whose number of poles in $|z| < r$ is $o(r)$ as $r \to \infty$. Gordon and Levin considered in [GL] algebras of more general quasipolynomials on angles in $\mathbb{C}$ and proved the stability property in these cases.

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In [BD] Berenstein and Dostal considered the field $\mathcal{R}^n$ of rational functions on $\mathbb{C}^n$ and the algebra of entire $\mathcal{R}^n$-quasipolynomials. These are functions of the form $\sum_{j=1}^k r_j(z)e^{\langle\theta_j,z\rangle}$, where $\theta_j \in \mathbb{C}^n$ and $r_j \in \mathcal{R}^n$. They proved that this algebra is stable in the algebra of all entire functions. The choice of $\mathcal{R}^n$ over $\mathcal{P}^n$ was done to accommodate the example of the function $z^{-1}\sin z$ on $\mathbb{C}$.

One may think that the stability of algebras of quasipolynomials is due to the special features of exponentials. However, in this paper we show that in fact the stability is a generic property of algebras of entire functions. Let $\mathcal{M}_0^n$ be the algebra of meromorphic functions on $\mathbb{C}^n$ whose polar varieties are algebraic. We prove the following theorem:

Theorem 1.1. *Let $\mathcal{R}^n[f] \subset \mathcal{M}_0^n$ be the subalgebra generated by $\mathcal{R}^n$ and an entire function f of finite order on $\mathbb{C}^n$. Then either $\mathcal{R}^n[f]$ is stable in $\mathcal{M}_0^n$ or $\mathcal{R}^n[f] = \mathcal{R}^n[e^p]$ for some $p \in \mathcal{P}^n$ and the algebra $\mathcal{R}^n[e^p, e^{-p}]$ is stable in $\mathcal{M}_0^n$.*

This theorem is proved in Section 4. For this we show in Theorem 4.1 and Corollary 4.2 that an algebra $\mathcal{R}^n[f]$ contains non-trivial (i.e., non-rational) invertible elements in $\mathcal{M}_0^n$ if and only if $f = q_1e^p + q_2$, where $q_1, q_2 \in \mathcal{R}^n$ and $p \in \mathcal{P}^n$.

The proof of our basic tool, namely Theorem 4.1, relies on solving in Section 3 algebraic equations in an angle in $\mathbb{C}$. Let P be a monic polynomial with slowly growing holomorphic coefficients in an angle. We show in Theorem 3.1 that under some suitable growth conditions on e^p the equation $P(f) = e^p$ has a holomorphic solution if and only if $P(w) = (w + q)^m$.

The results in Section 3 use Phragmen–Lindelöf type theorems presented in Section 2. They relate the growth of holomorphic functions in an angle with the behavior of their indicators.

2. Preliminary results

We will need the following standard estimate for the roots of a polynomial.

Proposition 2.1. *If $P(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0 = 0$ and $\|P\| = \max\{1, |a_0|, \ldots, |a_{m-1}|\}$, then $|a_0|(m\|P\|)^{-1} \leq |z| \leq m\|P\|$.*

For $\alpha < \beta$ and $r_0 \geq 0$ let

$$S(\alpha, \beta, r_0) = \{z = re^{i\theta} : r > r_0,\ \alpha < \theta < \beta\}.$$

We say that a holomorphic function f on $S(\alpha, \beta, r_0)$ is *of order at most* $\rho \geq 0$ if $|f(z)| \leq C(\rho')e^{|z|^{\rho'}}$ for every $\rho' > \rho$. For every θ, $\alpha < \theta < \beta$, the indicator of f of order ρ is defined by

$$h(\theta) = h_{f,\rho}(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r^\rho}.$$

The indicator may have infinite values. If $h_{f,\rho}(\theta) < \infty$ for $\alpha < \theta < \beta$, then we say that the function f has a *finite indicator of order* ρ.

The following lemma is a version of [L, Lemma 6, p. 52] in our setting and is proved in the same way.

Lemma 2.2. *Let f be a holomorphic function on $S(\alpha, \beta, r_0)$. Suppose that $\alpha < \theta_1 < \theta < \theta_2 < \beta$ and $\theta_2 - \theta_1 < \pi/\rho$. If f is of order at most $\rho > 0$ and has finite indicator $h = h_{f,\rho}$, then*

$$h(\theta) \leq \frac{h(\theta_1)\sin\rho(\theta_2 - \theta) + h(\theta_2)\sin\rho(\theta - \theta_1)}{\sin\rho(\theta_2 - \theta_1)}.$$

It follows from this lemma that a finite indicator is continuous. The following lemma can be easily obtained using elementary conformal mappings.

Lemma 2.3. *Let $D_r = S(-\alpha, \alpha, r_0) \cap \{|z| < r\}$ and $\gamma_r = \{z = re^{i\theta} : -\alpha < \theta < \alpha\}$. If ω_r is the harmonic measure of γ_r in D_r and K is a compact set in $S(-\alpha, \alpha, r_0)$, then there are positive constants C_1 and C_2 depending on K such that for all r sufficiently large we have*

$$C_1 r^{-\sigma} \leq \omega_r(z) \leq C_2 r^{-\sigma},\ z \in K,$$

where $\sigma = \pi/(2\alpha)$.

Using this lemma and the Two Constants Theorem, the following results related to the Phragmen-Lindelöf principle can be easily derived.

Corollary 2.4. *Let f be a holomorphic function on $\overline{S}(-\alpha, \alpha, r_0)$ of order at most $\rho < \pi/(2\alpha)$.*

(i) If the holomorphic function e^{-g} is also of order at most ρ and $|f(z)| \leq \left|e^{g(z)}\right|$ on $\partial S(-\alpha, \alpha, r_0)$, then $|f(z)| \leq \left|e^{g(z)}\right|$ in $S(-\alpha, \alpha, r_0)$.

(ii) If $h_{f,\rho}(\pm\alpha) < -\delta < 0$, then there is a constant $C > 0$ such that $|f(z)| \leq Ce^{-\delta|z|^\rho}$ on $\overline{S}(-\alpha, \alpha, r_0)$.

(iii) If $h_{f,\rho}(\pm\alpha) \leq 0$, then for any $a > 0$ there is a constant $C > 0$ such that $|f(z)| \leq Ce^{a|z|^\rho}$ on $\overline{S}(-\alpha, \alpha, r_0)$.

While Corollary 2.4 gives upper bounds on the absolute values of holomorphic functions on small angles, the following lemma shows that such functions cannot be too small on big angles.

Lemma 2.5. *Let f be holomorphic on $S(-\alpha, \alpha, r_0)$ and let $\rho > \pi/(2\alpha)$.*

(i) If $|f(z)| \leq Ce^{-a|z|^\rho}$ on $S(-\alpha, \alpha, r_0)$, where $a, C > 0$, then $f \equiv 0$.

(ii) If f is of order at most $\rho > 0$, $h_{f,\rho}(\theta) \leq 0$ for all $\theta \in (-\alpha, \alpha)$ and $h_{f,\rho}(\theta_0) < 0$ for some $\theta_0 \in (-\alpha, \alpha)$, then $f \equiv 0$.

Proof. The first part follows immediately from Lemma 2.3 and the Two Constants Theorem. To prove the second part we note that by Lemma 2.2 $h_{f,\rho}(\theta) < 0$ for all $\theta \in (-\alpha, \alpha)$. Choose $0 < \alpha' < \alpha$ such that $2\alpha'\rho > \pi$. Since $h_{f,\rho}$ is a continuous function, there is $\delta > 0$ such that $h_{f,\rho}(\theta) < -\delta$ on $[-\alpha', \alpha']$. Take θ_1 and θ_2 such that $-\alpha' \leq \theta_1 < \theta_2 \leq \alpha'$ and $\theta_2 - \theta_1 < \pi/\rho$. By the second part of Corollary 2.4 there is a constant $C > 0$ such that $|f(z)| \leq Ce^{-\delta|z|^\rho}$ on $\overline{S}(\theta_1, \theta_2, r_0)$. The region $S(-\alpha', \alpha', r_0)$ can be covered by finitely many regions $S(\theta_1, \theta_2, r_0)$ with $\theta_2 - \theta_1 < \pi/\rho$. Therefore $|f(z)| \leq Me^{-\delta|z|^\rho}$ on $S(-\alpha', \alpha', r_0)$. By the first part of this lemma $f \equiv 0$. □

3. Algebraic equations in an angle

Let $\mathcal{A} = \mathcal{A}_{\alpha,\beta,r_0,\rho}$ be the set of functions f holomorphic on $S(\alpha, \beta, r_0)$ of order at most ρ and so that $h_{f,\rho}(\theta) \leq 0$ for $\alpha < \theta < \beta$. Note that $\mathcal{A}$ is an algebra. Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. We denote by $\mathcal{B}[w]$ the algebra of polynomials with coefficients in $\mathcal{B}$.

Let p be a holomorphic function on $S(\alpha, \beta, r_0)$. We say that the indicator of order $\rho > 0$ of the function e^p is *almost sinusoidal* if e^p and e^{-p} are of order at most ρ, $h_{e^{-p},\rho}(\theta) < 0$ on some interval $(\alpha_1, \beta_1) \subset (\alpha, \beta)$, while $h_{e^p,\rho}(\theta) \leq 0$ for all remaining θ. This definition is justified by a theorem of Cartwright [C], which states that if $h_{e^p,\rho}$ is *sinusoidal* and $\beta-\alpha>\pi/\rho$, then e^p is of completely regular growth and has all the above properties.

Theorem 3.1. *Let $\rho>0$ and $\beta-\alpha>\pi/\rho$. Suppose that there is a monic polynomial $P\in\mathcal{B}[w]$ of degree $m\geq 1$ and a function f holomorphic on $S(\alpha,\beta,r_0)$ such that $P(f)=e^p$. If e^p has an almost sinusoidal indicator of order ρ, then $P(w)=(w+q)^m$ for some $q\in\mathcal{B}$ and $f=\varepsilon e^{p/m}-q$, where $\varepsilon^m=1$.*

Proof. The case $m=1$ is trivial. Let $m\geq 2$ and

$$P(f)=P_mf^m+P_{m-1}f^{m-1}+\cdots+P_0=e^p,$$

where $P_j\in\mathcal{B}$, $0\leq j\leq m-1$, and $P_m\equiv 1$. If $f_1=f+P_{m-1}/m$, then f_1 verifies the equation

$$f_1^m+R_{m-2}f_1^{m-2}+\cdots+R_0=e^p, \tag{1}$$

where

$$R_k=\sum_{j=k}^m\binom{j}{k}(-1)^{j-k}m^{k-j}P_jP_{m-1}^{j-k}\in\mathcal{B}.$$

The function $f_2=e^{-p/m}f_1$ satisfies the equation

$$f_2^m+R_{m-2}e^{-2p/m}f_2^{m-2}+\cdots+R_0e^{-p}=1. \tag{2}$$

Take any $\alpha_1'<\beta_1'$ in (α_1,β_1). Then $h_{e^{-p},\rho}(\theta)<-\delta<0$ for some $\delta>0$ on $[\alpha_1',\beta_1']$. By Lemma 2.5 $\beta_1-\alpha_1\leq\pi/\rho$, otherwise $e^{-p}\equiv 0$. Thus $\beta_1'-\alpha_1'<\pi/\rho$. Corollary 2.4 shows that there is a positive constant C_1 such that $\left|e^{-p(z)}\right|\leq C_1e^{-\delta|z|^\rho}$ on $\overline{S}(\alpha_1',\beta_1',r_0)$. Since $h_{R_j,\rho}(\theta)\leq 0$, it follows from the same corollary that there is a constant $C_2>0$ such that for $z\in\overline{S}(\alpha_1',\beta_1',r_0)$

$$|R_{m-2}(z)|\leq C_2e^{a|z|^\rho},\ \left|R_{m-j}(z)e^{-(j-2)p(z)/m}\right|\leq C_2e^{-a|z|^\rho},\ j\geq 3,$$

where $a=\delta/(2m)$.

These estimates, together with Proposition 2.1 and (2), show that the function f_2 is bounded on $\overline{S}(\alpha_1',\beta_1',r_0)$. Moreover, we see that

$$f_2^m=1+G_1e^{-2p/m},$$

where G_1 is holomorphic and $|G_1(z)|<C_3e^{a|z|^\rho}$ on $\overline{S}(\alpha_1',\beta_1',r_0)$. We conclude that there exist $r_1\geq r_0$ and ε, $\varepsilon^m=1$, such that

$$f_2=\varepsilon+Ge^{-2p/m}, \tag{3}$$

where G is holomorphic and

$$|G(z)|<C_4e^{a|z|^\rho} \tag{4}$$

on $\overline{S}(\alpha_1',\beta_1',r_1)$. Clearly ε does not depend on our choice of α_1' and β_1', while the constants C_4 and δ may depend on the indicated choice.

Multiplying (3) by $e^{p/m}$ we obtain that

$$f_1=\varepsilon e^{p/m}+g,\ g=e^{-p/m}G;$$

hence g is analytic on $S(\alpha,\beta,r_0)$. Using (4) and the fact that $h_{e^{-p},\rho}(\theta)<-\delta$ on $[\alpha_1',\beta_1']$, we obtain $h_{g,\rho}(\theta)<0$ for $\theta\in(\alpha_1,\beta_1)$.

Letting $R_m\equiv 1$, $R_{m-1}\equiv 0$, and plugging $f_1=\varepsilon e^{p/m}+g$ into (1) we get the equation

$$g^m+Q_{m-1}g^{m-1}+\cdots+Q_1g+Q_0=0,$$

where

$$Q_0 = \sum_{j=0}^{m-2} \left(\varepsilon e^{p/m}\right)^j R_j, \quad Q_k = \sum_{j=k}^{m} \binom{j}{k} \left(\varepsilon e^{p/m}\right)^{j-k} R_j, \; 1 \le k \le m-1.$$

By Proposition 2.1 the order of g does not exceed ρ. If $\theta \notin (\alpha_1, \beta_1)$, then $h_{e^p,\rho}(\theta) \le 0$. Hence $h_{Q_j,\rho}(\theta) \le 0$. Again Proposition 2.1 yields that $h_{g,\rho}(\theta) \le 0$. Thus the function g satisfies the conditions of Lemma 2.5 on $S(\alpha, \beta, r_0)$, and, therefore, $g \equiv 0$ and $f_1 = \varepsilon e^{p/m}$.

Plugging $f_1 = \varepsilon e^{p/m}$ into (1) and dividing by $e^{\frac{m-2}{m}p}$, we obtain

$$\varepsilon^{m-2} R_{m-2} + \varepsilon^{m-3} R_{m-3} e^{-p/m} + \cdots + R_0 e^{\frac{2-m}{m}p} = 0.$$

This immediately implies that $h_{R_{m-2},\rho}(\theta) < 0$ when $\theta \in (\alpha_1, \beta_1)$. By Lemma 2.5 $R_{m-2} \equiv 0$. Continuing this process we conclude that $R_{m-2} \equiv \cdots \equiv R_0 \equiv 0$ on $S(\alpha, \beta, r_0)$.

Thus $f = \varepsilon e^{p/m} - P_{m-1}/m$ and $P(w) = (w + P_{m-1}/m)^m$. □

Corollary 3.2. *Let $\rho > 0$ and $\beta - \alpha > \pi/\rho$. Suppose that there is a monic polynomial $P \in \mathcal{B}[w]$ of degree $m \ge 1$ and a function f holomorphic on $S(\alpha, \beta, r_0)$ such that $P(f) = e^p$. If the function e^p is of completely regular growth of order ρ, then the conclusions of Theorem* 3.1 *hold.*

Proof. Since the density of zeroes of e^p is zero, by [L, Corollary, p. 155] $h_{e^{\pm p},\rho}(\theta) = \pm a \sin \rho(\theta - \theta_0)$. Hence we can choose an interval $(\alpha', \beta') \subset (\alpha, \beta)$, $\beta' - \alpha' > \pi/\rho$, where the hypotheses of Theorem 3.1 are verified. □

Let $\mathcal{A}^0 = \mathcal{A}^0_{r_0,\rho}$ be the algebra of functions f holomorphic on $\{|z| > r_0\}$, of order at most ρ and with indicator $h_{f,\rho}(\theta) \le 0$ for all θ.

Corollary 3.3. *Suppose that $P \in \mathcal{A}^0[w]$ is a monic polynomial of degree $m \ge 1$ and f, p are holomorphic functions on $\{|z| > r_0\}$, such that $P(f) = e^p$. If f is of order at most ρ, then either p extends analytically at ∞ or the conclusions of Theorem* 3.1 *hold.*

Proof. It follows that the function e^p is of order at most ρ. Thus the function p has a pole at infinity of order $n \le \rho$. If $n = 0$, then p extends analytically at ∞. If $n \ge 1$, then e^p is of completely regular growth of order n. So the previous corollary applies. □

4. Division theorems

We denote by $\mathcal{P}^n$ and $\mathcal{R}^n$ the spaces of polynomials, respectively rational functions, on $\mathbb{C}^n$. Let $\mathcal{M}_0^n$ be the set of meromorphic functions on $\mathbb{C}^n$ of the form h/p, where h is entire and $p \in \mathcal{P}^n$. Equivalently, $\mathcal{M}_0^n$ consists of the meromorphic functions on $\mathbb{C}^n$ whose polar variety is algebraic. Note that $\mathcal{M}_0^n$ is an algebra and $\mathcal{R}^n \subset \mathcal{M}_0^n$. Moreover, the invertible elements of $\mathcal{M}_0^n$ are the functions Re^h, where $R \in \mathcal{R}^n$ and h is entire. If h is not a constant, we will call Re^h a *non-trivial invertible function* in $\mathcal{M}_0^n$.

We let $\mathcal{R}^n[w]$ be the algebra of polynomials in w with coefficients in $\mathcal{R}^n$. If $P \in \mathcal{R}^n[w]$, the degree of P will mean the degree of P in w. For a function $f \in \mathcal{M}_0^n$ we denote by $\mathcal{R}^n[f] \subset \mathcal{M}_0^n$ the algebra generated by $\mathcal{R}^n$ and f. We will omit the index n in our notation if $n = 1$.

Theorem 4.1. *Let f be an entire transcendental function on $\mathbb{C}^n$ of finite order. If $P(f) = Re^p$ is a non-trivial invertible function in $\mathcal{M}_0^n$, where $P \in \mathcal{R}^n[w]$ and $R \in \mathcal{R}^n$, then $p \in \mathcal{P}^n$, $\deg p \geq 1$, and there exist an integer $m \geq 1$ and $p_1, p_2 \in \mathcal{R}^n$ such that*

$$P(w) = Rp_2^m \left(w + p_1\right)^m, \ f = p_2^{-1} e^{p/m} - p_1.$$

Proof. Let $m = \deg P$ and $P(f) = P_m f^m + \cdots + P_0 = Re^p$, where $P_j \in \mathcal{R}^n$. Clearly $m \geq 1$, since $P(f)$ is a non-trivial invertible function. As the order of f is finite, $p \in \mathcal{P}^n$. If p is a constant, it follows that f is a polynomial. Hence $\deg p \geq 1$ and the order of f is positive.

Let us assume at first that $n = 1$. There is $r_0 > 0$ such that the functions P_j and R have no zeros or poles when $|z| > r_0$. If $Q = R/P_m$, then Qe^p is a function of completely regular growth of order $\deg p$. By Corollary 3.2, $P(w) = P_m(w + p_1)^m$. Thus $mP_m p_1 = P_{m-1}$, so $p_1 \in \mathcal{R}$. It follows that $P_m = Rp_2^m$, $p_2 \in \mathcal{R}$, and $f = p_2^{-1} e^{p/m} - p_1$.

If $n > 1$ we let $z = (z', z_n)$, $z' \in \mathbb{C}^{n-1}$. There exists a point $z_0 \in \mathbb{C}^n$ so that the functions P_m, R are holomorphic and non-vanishing in a neighborhood of z_0, and the gradient of p is also non-vanishing there. Without loss of generality, we may assume that there is $\varepsilon > 0$ such that for any fixed z', $|z'| < \varepsilon$, the polynomial $p(z', z_n)$ is not constant and the functions $P_m(z', z_n)$ and $R(z', z_n)$ are well defined rational functions not identically equal to 0.

From the one-dimensional case we see that

$$P(w) = P_m(z', z_n)(w + p_1(z', z_n))^m,$$

for each z', $|z'| < \varepsilon$. Thus $mP_m p_1 = P_{m-1}$, so $p_1 \in \mathcal{R}^n$. Hence $P(w) = P_m(w+p_1)^m$ on an open set in $\mathbb{C}^n$ and, consequently, this equality holds everywhere. We conclude that $P_m = Rp_2^m$ and $f = p_2^{-1} e^{p/m} - p_1$, with $p_2 \in \mathcal{R}^n$. □

This result allows us to describe all algebras $\mathcal{R}^n[f]$ with non-trivial invertible elements in $\mathcal{M}_0^n$.

Corollary 4.2. *Let f be an entire transcendental function on $\mathbb{C}^n$ of finite order. The following statements are equivalent:*

(i) The algebra $\mathcal{R}^n[f]$ contains non-trivial invertible elements in $\mathcal{M}_0^n$.

(ii) The function $f = q_1 e^p + q_2$, where $q_1, q_2 \in \mathcal{R}^n$ and $p \in \mathcal{P}^n$.

(iii) $\mathcal{R}^n[f] = \mathcal{R}^n[e^p]$, where $p \in \mathcal{P}^n$.

Moreover, if one of these statements holds and $g \in \mathcal{R}^n[f]$ is invertible in $\mathcal{M}_0^n$, then $g = re^{mp}$, where $m \geq 0$ is an integer and $r \in \mathcal{R}^n$.

Proof. We have that (i) implies (ii) by Theorem 4.1. Clearly, (ii) implies (iii), and (iii) implies (i).

If one of these statements holds and $g \in \mathcal{R}^n[f]$ is invertible in $\mathcal{M}_0^n$, then $g = re^q$, where $r \in \mathcal{R}^n$ and $q \in \mathcal{P}^n$. By Theorem 4.1, $q = mp$ for some integer $m \geq 0$. □

We will need the following lemma for the proof of our division theorems.

Lemma 4.3. *Let f be an entire function on $\mathbb{C}^n$ and P, Q be relatively prime polynomials of $\mathcal{R}^n[w]$. If the function $P(f)/Q(f) \in \mathcal{M}_0^n$, then $Q(f)$ is an invertible element in $\mathcal{M}_0^n$.*

Proof. It suffices to show that the zero variety $Z(Q(f))$ of the meromorphic function $Q(f)$ is algebraic. Assuming the contrary, there exists an irreducible component

X of $Z(Q(f))$ which is not algebraic. Since $P(f)/Q(f) \in \mathcal{M}_0^n$, it follows that $X \subset Z(P(f))$. As P, Q are relatively prime, there exist polynomials $P_1, Q_1 \in \mathcal{R}^n[w]$ so that $PP_1 + QQ_1 = 1$, hence $P(f)P_1(f) + Q(f)Q_1(f) = 1$. Note that the polar varieties of the meromorphic functions $P_1(f), Q_1(f)$ are algebraic. As X is not algebraic, we can find a point $z_0 \in X$ so that the functions $P(f), P_1(f), Q(f), Q_1(f)$ are holomorphic in a neighborhood of z_0. Since $P(f), Q(f)$ vanish at z_0, we get a contradiction. $\square$

Let $\mathcal{A}$ be an algebra, $\mathcal{B} \subset \mathcal{A}$ a subalgebra and $g \in \mathcal{B}$. It is an interesting problem to compare the ideal $\mathcal{B}g$ generated by g in $\mathcal{B}$ to the ideal $\mathcal{A}g \cap \mathcal{B}$. Clearly, $\mathcal{B}g \subseteq \mathcal{A}g \cap \mathcal{B}$, but the inclusion is in general strict. If $\mathcal{A} = \mathcal{M}_0^n$ and $\mathcal{B} = \mathcal{R}^n[f]$, we are going to describe $\mathcal{M}_0^n g \cap \mathcal{R}^n[f]$ completely.

Given a function $g \in \mathcal{R}^n[e^p]$, where $p \in \mathcal{P}^n$ is not constant, then $g = P(e^p)$ for a unique $P \in \mathcal{R}^n[w]$. Let $m \geq 0$ denote the order of the zero of $P(w)$ at $w = 0$. Then g can be written uniquely in the form $g = Q(e^p)e^{mp}$, where $m \geq 0$, $Q \in \mathcal{R}^n[w]$, $Q(0) \neq 0$.

Theorem 4.4. *(i) Let $g \in \mathcal{R}^n[e^p]$, where $p \in \mathcal{P}^n$ is not constant. If $g = Q(e^p)e^{mp}$, where $m \geq 0$, $Q \in \mathcal{R}^n[w]$, $Q(0) \neq 0$, then*

$$\mathcal{M}_0^n g \cap \mathcal{R}^n[e^p] = \mathcal{R}^n[e^p]Q(e^p).$$

(ii) Let $g \in \mathcal{R}^n[f]$, where f is an entire function on $\mathbb{C}^n$ such that $\mathcal{R}^n[f]$ does not contain non-trivial invertible elements in $\mathcal{M}_0^n$. Then

$$\mathcal{M}_0^n g \cap \mathcal{R}^n[f] = \mathcal{R}^n[f]g.$$

Proof. (i) Clearly, $Q(e^p) \in \mathcal{M}_0^n g$, so $\mathcal{R}^n[e^p]Q(e^p) \subset \mathcal{M}_0^n g \cap \mathcal{R}^n[e^p]$.

Suppose $h \in \mathcal{M}_0^n$ and $hg = P_0(e^p)$, where $P_0 \in \mathcal{R}^n[w]$. Let $P \in \mathcal{R}^n[w]$ be the greatest common divisor of P_0, Q, and write $P_0 = PQ_0$, $Q = PQ_1$, where $Q_j \in \mathcal{R}^n[w]$, $j = 0, 1$. Then $he^{mp} = Q_0(e^p)/Q_1(e^p) \in \mathcal{M}_0^n$, so by Lemma 4.3, $Q_1(e^p)$ is invertible in $\mathcal{M}_0^n$. Corollary 4.2 implies that $Q_1(e^p) = re^{dp}$, where $r \in \mathcal{R}^n$ and $d \geq 0$. Since $Q(0) \neq 0$ it follows that $d = 0$. Hence $P_0 = QQ_2$, for some $Q_2 \in \mathcal{R}^n[w]$, and $hg = Q_2(e^p)Q(e^p)$. So we have shown that $\mathcal{M}_0^n g \cap \mathcal{R}^n[e^p] \subset \mathcal{R}^n[e^p]Q(e^p)$.

(ii) We have $\mathcal{R}^n[f]g \subset \mathcal{M}_0^n g \cap \mathcal{R}^n[f]$. The opposite inclusion is clearly true if f is a polynomial, so we assume that f is transcendental.

Suppose that $g = P_1(f)$ and $hg = P_0(f)$ for some $h \in \mathcal{M}_0^n$, where $P_0, P_1 \in \mathcal{R}^n[w]$. We write as in (i) $P_j = PQ_j$, where $P, Q_j \in \mathcal{R}^n[w]$, $j = 0, 1$, and Q_0, Q_1 are relatively prime. Then we conclude by Lemma 4.3 that $Q_1(f)$ is invertible in $\mathcal{M}_0^n$. Since $\mathcal{R}^n[f]$ does not contain non-trivial invertible elements in $\mathcal{M}_0^n$ and since f is transcendental, it follows that $\deg Q_1 = 0$. So $P_0 = P_1Q_2$, where $Q_2 \in \mathcal{R}^n[w]$. Hence $hg = Q_2(f)g \in \mathcal{R}^n[f]g$. $\square$

The immediate consequence of Theorem 4.4 is the division theorem for algebras $\mathcal{R}^n[f]$, which finishes the proof of Theorem 1.1.

Corollary 4.5. *Let f be an entire function on $\mathbb{C}^n$, $h_1, h_0 \in \mathcal{R}^n[f]$, and assume that $g = h_0/h_1 \in \mathcal{M}_0^n$.*

(i) If $\mathcal{R}^n[f]$ does not contain non-trivial invertible elements in $\mathcal{M}_0^n$, then $g \in \mathcal{R}^n[f]$.

(ii) If $\mathcal{R}^n[f] = \mathcal{R}^n[e^p]$, where $p \in \mathcal{P}^n$ is not constant, then $g \in \mathcal{R}^n[e^p, e^{-p}]$.

For functions $g_1, g_2 \in \mathcal{M}_0^n$ we say that $g_1 \sim g_2$ if $g_1/g_2, g_2/g_1 \in \mathcal{M}_0^n$. The following corollary is an immediate consequence of the previous results.

Corollary 4.6. *(i) Let $p \in \mathcal{P}^n$ be non-constant and $g_1, g_2 \in \mathcal{R}^n[e^p]$. Then $g_1 \sim g_2$ if and only if $g_2 = Re^{mp}g_1$, for some $R \in \mathcal{R}^n$ and $m \in \mathbb{Z}$.*

(ii) Let f be an entire function on $\mathbb{C}^n$ and $g_1, g_2 \in \mathcal{R}^n[f]$. If $\mathcal{R}^n[f]$ does not contain non-trivial invertible elements in $\mathcal{M}_0^n$, then $g_1 \sim g_2$ if and only if $g_2 = Rg_1$, for some $R \in \mathcal{R}^n$.

5. Algebraic dependence over $\mathbb{C}$

Two meromorphic functions f, g on $\mathbb{C}^n$ are called algebraically dependent if there exists a non-trivial polynomial P on $\mathbb{C}^2$ so that $P(f, g) = 0$. We now give a complete characterization of such pairs of functions.

Theorem 5.1. *The entire functions f, g on $\mathbb{C}^n$ are algebraically dependent if and only if there exists an entire function h on $\mathbb{C}^n$ such that one of the following holds:*

(i) $f, g \in \mathbb{C}[h]$.

(ii) $f, g \in \mathbb{C}[e^h, e^{-h}]$.

Proof. Two functions f, g as in (i) or (ii) are clearly algebraically dependent. Conversely, let $P(z_1, z_2)$ be an irreducible polynomial of degree d so that $P(f, g) = 0$. If $d = 1$, then case (i) in the statement clearly holds, so we may assume that $d \geq 2$. Moreover, we may assume that none of the maps f, g is constant. Let $[z_0 : z_1 : z_2]$ denote the homogeneous coordinates on the complex projective space $\mathbb{P}^2$, and consider the standard embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$, $(z_1, z_2) \to [1 : z_1 : z_2]$.

Let $X \subset \mathbb{P}^2$ be the algebraic curve defined by P,

$$X = \{[z_0 : z_1 : z_2] : \widetilde{P}(z_0, z_1, z_2) = 0\}, \ \widetilde{P}(z_0, z_1, z_2) = z_0^d P(z_1/z_0, z_2/z_0).$$

We denote by $X_\infty = X \cap \{z_0 = 0\}$ the set of points where X intersects the line at infinity. We then have a holomorphic mapping

$$F : \mathbb{C}^n \to X \setminus X_\infty, \ F(\zeta) = [1 : f(\zeta) : g(\zeta)].$$

Consider the normalization of X, $\sigma : \widetilde{X} \to X$, where $\widetilde{X}$ is a compact Riemann surface (see e.g. [G]), and let $S = \sigma^{-1}(X_\infty)$. Finally, let $\pi : Y \to \widetilde{X}$ be the universal covering of $\widetilde{X}$, where $Y = \mathbb{P}^1$, or $Y = \mathbb{C}$, or $Y = \Delta$ (the unit disk), and let $Z = \pi^{-1}(S)$. There exists a non-constant holomorphic lifting of F, $G : \mathbb{C}^n \to Y \setminus Z$, $F = \sigma \circ \pi \circ G$. Therefore $Y \neq \Delta$. Suppose $Y = \mathbb{C}$. Since $S \neq \emptyset$ it follows that Z is an infinite discrete subset of $\mathbb{C}$, so G is constant by Picard's theorem.

We conclude that $Y = \widetilde{X} = \mathbb{P}^1$, and we have a non-constant holomorphic map $G : \mathbb{C}^n \to \mathbb{P}^1 \setminus S$, $F = \sigma \circ G$. Picard's theorem implies that $|S| \leq 2$. Now $\sigma : \mathbb{P}^1 \to \mathbb{P}^2$ is a holomorphic map, with $\sigma(\mathbb{P}^1) = X$. It follows that σ is a rational map; i.e., there exist homogeneous polynomials P_0, P_1, P_2 of degree m so that

$$\sigma([x_0 : x_1]) = [P_0(x_0, x_1) : P_1(x_0, x_1) : P_2(x_0, x_1)], \ [x_0 : x_1] \in \mathbb{P}^1.$$

We have two cases.

Case 1. $|S| = 1$. Composing with a Möbius map, we may assume that $S = \{[0 : 1]\}$. It follows that $G(\zeta) = [1 : h(\zeta)]$, for some entire function h on $\mathbb{C}^n$. Since $S = \sigma^{-1}(\{z_0 = 0\})$, we see that $P_0(x_0, x_1) = 0$ if and only if $x_0 = 0$; hence $P_0(x_0, x_1) = x_0^m$. Therefore

$$F(\zeta) = \sigma([1 : h(\zeta)]) = [1 : P_1(1, h(\zeta)) : P_2(1, h(\zeta))],$$

and we conclude that f, g verify case (i) from the statement.

Case 2. $|S| = 2$. We may now assume that $S = \{[0 : 1], [1 : 0]\}$, so $G(\zeta) = [1 : e^{h(\zeta)}]$, where h is an entire function on $\mathbb{C}^n$. As in Case 1 we see that $P_0(x_0, x_1) = 0$ if and only if $x_0 = 0$ or $x_1 = 0$; hence $P_0(x_0, x_1) = x_0^k x_1^{m-k}$, for some $1 \le k < m$. This yields

$$F(\zeta) = [1 : e^{-Nh(\zeta)}P_1(1, e^{h(\zeta)}) : e^{-Nh(\zeta)}P_2(1, e^{h(\zeta)})],$$

where $N = m - k > 0$, so the functions f, g verify (ii). □

Theorem 5.2. *The meromorphic functions f, g on $\mathbb{C}^n$ are algebraically dependent if and only if one of the following holds:*

(i) There exists a meromorphic function h on $\mathbb{C}^n$ and rational functions R_1, R_2 on $\mathbb{C}$, so that $f = R_1 \circ h$, $g = R_2 \circ h$.

(ii) There exists an entire function h on $\mathbb{C}^n$, and elliptic functions φ_1, φ_2 with the same periods, so that $f = \varphi_1 \circ h$, $g = \varphi_2 \circ h$.

Proof. Two functions f, g as in (i) are clearly algebraically dependent. If f, g are as in (ii), then $\sigma = [1 : \varphi_1 : \varphi_2]$ induces a holomorphic map from a complex torus into $\mathbb{P}^2$. The image of this map is an analytic variety, hence algebraic by Chow's theorem. This implies that f, g are algebraically dependent.

Conversely, assume that f, g are algebraically dependent and non-constant, and let $P(z_1, z_2)$ be an irreducible polynomial of degree d so that $P(f, g) = 0$. We use the same notation as in the proof of Theorem 5.1: $X = \{P = 0\} \subset \mathbb{P}^2$, $\sigma : \widetilde{X} \to X$ is the normalization of X, and $\pi : Y \to \widetilde{X}$ is the universal covering of $\widetilde{X}$.

Let A be the union of the polar varieties of f and g. We define the meromorphic map $F = [f_1 g_1 : f_0 g_1 : g_0 f_1] : \mathbb{C} \dashrightarrow X$, where $f = f_0/f_1$, $g = g_0/g_1$, and f_j, g_j are entire functions. We have $F = [1 : f : g]$ on $\mathbb{C}^n \setminus A$. If I is the indeterminacy set of F, then I is an analytic subvariety of $\mathbb{C}^n$ of codimension at least 2 and $I \subset A$. Hence $D = \mathbb{C}^n \setminus I$ is simply connected, so F has a non-constant holomorphic lifting $G : D \to Y$, $F = \sigma \circ \pi \circ G$. If $Y = \Delta$, then, since $\operatorname{codim} I \ge 2$, G extends holomorphically to $\mathbb{C}^n$, so it is constant. Therefore we have two cases.

Case 1. $Y = \widetilde{X} = \mathbb{P}^1$. Then $F = \sigma \circ G$, $G = [h_0 : h_1] : D \to \mathbb{P}^1$, where h_j are holomorphic on D, so they extend holomorphically to $\mathbb{C}^n$. Moreover $\sigma = [P_0 : P_1 : P_2]$, for some homogeneous polynomials P_0, P_1, P_2 of degree m. It follows that f, g verify case (i) from the statement, with $R_1(x) = P_1(1, x)/P_0(1, x)$, $R_2(x) = P_2(1, x)/P_0(1, x)$, $h = h_1/h_0$.

Case 2. $Y = \mathbb{C}$, so $\widetilde{X}$ is a complex torus. Then G extends to an entire function on $\mathbb{C}^n$, since $\operatorname{codim} I \ge 2$. The holomorphic map $\sigma : \widetilde{X} \to X \subset \mathbb{P}^2$ is given by $\sigma = [\psi_0 : \psi_1 : \psi_2]$, where ψ_j are meromorphic functions on $\widetilde{X}$. It follows that $\rho_j = \psi_j \circ \pi$ are elliptic functions with the same periods, and f, g verify (ii) with $\varphi_1 = \rho_1/\rho_0$, $\varphi_2 = \rho_2/\rho_0$. □

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