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INCLUSION THEOREMS FOR ABSOLUTELY SUMMING HOLOMORPHIC MAPPINGS

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Abstract. For linear operators, if $1 \leq p \leq q < \infty$, then every absolutely p-summing operator is also absolutely q-summing. On the other hand, it is well known that for $n \geq 2$, there are no general "inclusion theorems" for absolutely summing n-linear mappings or n-homogeneous polynomials. In this paper we deal with situations in which the spaces of absolutely p-summing and absolutely q-summing linear operators coincide, and prove that for $1 \leq p \leq q \leq 2$ and $n \geq 2$, we have inclusion theorems for absolutely summing n-linear mappings/n-homogeneous polynomials/holomorphic mappings. It is worth mentioning that our results hold precisely in the opposite direction from what is expected in the linear case, i.e., we show that, in some situations, as p increases, the classes of absolutely p-summing mappings becomes smaller.

1. Introduction

The theory of absolutely p-summing operators goes back to 1950, with Grothendieck's seminal paper [7]. In the sixties, with the works of Pietsch [14] and Lindenstrauss-Pełczyński [8], absolutely p-summing linear operators have become an important field of investigation in functional analysis. For the general theory of absolutely p-summing linear operators, we refer to the book by Diestel-Jarchow-Tonge [6]. From now on, the ideal of absolutely p-summing linear operators will be denoted by $\mathcal{L}_{as,p}$, and if $T : X \to Y$ is absolutely p-summing, we write $T \in \mathcal{L}_{as,p}(X;Y)$.

It is well known that, if $1 \leq p \leq q$, then every absolutely p-summing linear operator is absolutely q-summing. Results of this type are called "inclusion theorems".

Since Pietsch's paper [15], several generalizations of absolutely summing operators to polynomials and multilinear mappings were introduced, such as absolutely summing polynomials and n-linear mappings ([1, 9]) and fully (or multiple) summing n-linear mappings ([3, 11]). We will recall the necessary definitions in Sections 2 and 3.

For fully summing n-linear mappings, there is an inclusion theorem for some particular situation (see [13]), asserting that if an n-linear mapping is fully p-summing and $1 \leq p \leq q < 2$, then it is fully q-summing. But, for absolutely summing mappings there is no inclusion theorem. Indeed, for $n \geq 2$, we have

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$\mathcal{L}(^n l_2, \mathbb{C}) = \mathcal{L}_{as,1}(^n l_2, \mathbb{C})$ but $\mathcal{L}(^n l_2, \mathbb{C}) \neq \mathcal{L}_{as,2}(^n l_2, \mathbb{C})$, and thus $\mathcal{L}_{as,1}(^n l_2, \mathbb{C})$ is not contained in $\mathcal{L}_{as,2}(^n l_2, \mathbb{C})$. In the linear case, the situation is sometimes simpler and we have more than just inclusions: if X has cotype 2 and $1 \leq r \leq q \leq 2$, then

$$\mathcal{L}_{as,q}(X;Y) = \mathcal{L}_{as,r}(X;Y)$$

for every Y. Surprisingly, we will show that (if X has cotype 2 and $n \geq 2$) there are inclusion theorems, but they occur in the opposite direction from what is usual in the linear case; i.e., we will show that if p increases, the ideal decreases. For example, we prove that if X has cotype 2 and $n \geq 2$, then

$$\mathcal{L}_{as,q}(^n X;Y) \subseteqq \mathcal{L}_{as,r}(^n X;Y),$$

for every Y and $1 \leq r \leq q \leq 2$. We also obtain similar results for homogeneous polynomials and holomorphic mappings.

In the following, $X, X_1, ..., X_n, Y$ will denote Banach spaces over the complex scalar field and $\mathbb{N}$ will denote the set of all positive integers. By X' we denote the topological dual of X, and $B_{X'}$ represents its closed unit ball.

For $p \geq 1$, the vector space of all sequences $(x_j)_{j=1}^{\infty}$ in X such that $\|(x_j)_{j=1}^{\infty}\|_p = (\sum_{j=1}^{\infty} \|x_j\|^p)^{\frac{1}{p}} < \infty$ is denoted by $l_p(X)$. We denote by $l_p^w(X)$ the linear space of the sequences $(x_j)_{j=1}^{\infty}$ in X such that $(\varphi(x_j))_{j=1}^{\infty} \in l_p$ for every continuous linear functional $\varphi : X \to \mathbb{C}$. This is a Banach space under the norm

$$\|(x_j)_{j=1}^{\infty}\|_{w,p} = \sup_{\varphi \in B_{X'}} \|(\varphi(x_j))_{j=1}^{\infty}\|_p.$$

As usual, $l_p^u(X)$ denotes the subspace of $l_p^w(X)$ composed by the sequences $(x_j)_{j=1}^{\infty} \in l_p^w(X)$ such that

$$\lim_{k \to \infty} \|(x_j)_{j=k}^{\infty}\|_{w,p} = 0.$$

Given a natural number $n \in \mathbb{N}$, the Banach space of all continuous n-linear mappings from X to Y endowed with the sup norm will be denoted by $\mathcal{L}(^n X;Y)$ ($\mathcal{L}(X;Y)$ if $n = 1$). The set of all continuous n-homogeneous polynomials with the sup norm will be denoted by $\mathcal{P}(^n X;Y)$.

The paper is organized as follows: In Section 2 we obtain connections between the theory of absolutely summing holomorphic mappings and absolutely summing polynomials; in Section 3, by using multilinear interpolation techniques, we obtain inclusion theorems for absolutely summing multilinear mappings; in Section 4 we adapt our results to homogeneous polynomials; and, in the last section, the results are extended to holomorphic mappings.

2. Absolutely summing holomorphic mappings

The theory of absolutely summing holomorphic mappings was introduced by the second author in [10].

From now on, A will always be an open set of X and p, q, r, s will be real numbers greater than or equal to 1. If $a \in A$, $B_\delta(a)$ will denote the closed ball of center a and radius δ.

Definition 1. A holomorphic mapping $f : A \subset X \to Y$ at a point $a \in A$ is absolutely $(r;s)$-summing at a if there is a $\delta > 0$ such that $B_\delta(a) \subset A$ and

$$(f(a + x_j) - f(a))_{j=1}^{\infty} \in l_r(Y)$$

whenever $(x_j)_{j=1}^{\infty} \in l_s^u(X)$ with $\|x_j\| \leq \delta$ for every $j \in \mathbb{N}$.

If f is an n-homogeneous polynomial and $a = 0$, it is equivalent to saying that $(f(x_j))_{j=1}^{\infty} \in l_r(Y)$ whenever $(x_j)_{j=1}^{\infty} \in l_s^u(X)$. The following characterization will be useful:

Theorem 1 ([9]). *If $P \in \mathcal{P}(^nX;Y)$, the following statements are equivalent:*
(1) P is absolutely $(r;s)$-summing.
(2) There exists $L > 0$ such that

$$(\sum_{j=1}^{\infty} \|P(x_j)\|^r)^{\frac{1}{r}} \leq L\|(x_j)_{j=1}^{\infty}\|_{w,s}^n \text{ for all } (x_j)_{j=1}^{\infty} \in l_s^u(X).$$

The infimum of the possible constants $L > 0$ (denoted by $\|.\|_{as(r;s)}$) is a norm for the space of absolutely $(r;s)$-summing polynomials (represented as $\mathcal{P}_{as(r;s)}(^nX;Y)$).

For the theory of absolutely summing polynomials we refer to [9].

If $a \in X$, let us denote by $\mathcal{H}_{as(r;s)}^{(a)}(X;Y)$ the set of all holomorphic mappings at $a \in X$ which are absolutely $(r;s)$-summing at a. The next theorem (that will be very useful in the final section) states a nice connection between the study of absolutely summing holomorphic mappings and absolutely summing homogeneous polynomials.

Theorem 2. *A holomorphic mapping $f : A \subset X \to Y$ at a point $a \in A$ belongs to $\mathcal{H}_{as(r;s)}^{(a)}(X;Y)$ if, and only if,*

$$\frac{1}{n!}\widehat{d}^n f(a) \in \mathcal{P}_{as(r;s)}(^nX;Y) \tag{2.1}$$

and there are finite constants $K \geq 0$ and $k > 0$ such that

$$\left\|\frac{1}{n!}\widehat{d}^n f(a)\right\|_{as(r;s)} \leq Kk^n \tag{2.2}$$

for every $n \in \mathbb{N}$.

Proof. Let us suppose that (2.1) and (2.2) hold. Since f is holomorphic at a, there is $\delta > 0$ such that

$$f(a+x) - f(a) = \sum_{n=1}^{\infty} \frac{1}{n!}\widehat{d}^n f(a)(x)$$

for every $x \in E$ with $\|x\| \leq \delta$.

If $(x_j)_{j=1}^{\infty} \in l_s^u(X)$, with

$$\left\|(x_j)_{j=1}^{\infty}\right\|_{w,s} < \min\{\delta, \frac{1}{2k}\},$$

we have

$$\begin{aligned}\left\|(f(a+x_j)-f(a))_{j=1}^{\infty}\right\|_r &\leq \sum_{n=1}^{\infty}\left\|\left(\frac{1}{n!}\widehat{d}^n f(a)(x_j)\right)_{j=1}^{\infty}\right\|_r \leq K\sum_{n=1}^{\infty}\left(k\left\|(x_j)_{j=1}^{\infty}\right\|_{w,s}\right)^n \\ &\leq K\sum_{n=1}^{\infty} k^n \frac{1}{(2k)^n} < \infty,\end{aligned}$$

and hence f is absolutely summing at a.

Conversely, suppose that f is absolutely summing at a. Hence, from [10, Theorem 3.5], there are $C_1 \geq 0$ and $\delta_1 > 0$ such that

$$f(a+x) - f(a) = \sum_{n=1}^{\infty} \frac{1}{n!} \widehat{d}^n f(a)(x)$$

for $x \in E$ with $\|x\| \leq \delta_1$, and

$$\left\| (f(a+z_j) - f(a))_{j=1}^{\infty} \right\|_r^r \leq C_1 \left\| (z_j)_{j=1}^{\infty} \right\|_{w,s}^s$$

for $(z_j)_{j=1}^{\infty} \in l_s^u(X)$ with $\left\| (z_j)_{j=1}^{\infty} \right\|_{w,s} \leq \delta_1$.

So, if $(z_j)_{j=1}^{\infty} \in l_s^u(X)$ and $0 \neq \left\| (z_j)_{j=1}^{\infty} \right\|_{w,s} \leq \delta_1$, we have, for every n,

$$\begin{aligned}
\sum_{j=1}^{\infty} \left\| \frac{1}{n!} \widehat{d}^n f(a)(z_j) \right\|^r &= \sum_{j=1}^{\infty} \left\| \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{f(a+\lambda z_j) - f(a)}{\lambda^{n+1}} d\lambda \right\|^r \\
&\leq \sum_{j=1}^{\infty} \sup_{|\lambda|=1} \|f(a+\lambda z_j) - f(a)\|^r \\
&\leq C_1 \sup_{|\lambda|=1} \left\| (\lambda z_j)_{j=1}^{\infty} \right\|_{w,s}^s = C_1 \left\| (z_j)_{j=1}^{\infty} \right\|_{w,s}^s .
\end{aligned}$$

Hence $\frac{1}{n!} \widehat{d}^n f(a) \in \mathcal{P}_{as(r;s)}(^n X; Y)$.

To simplify notation, we will write $P_n = \frac{1}{n!} \widehat{d}^n f(a)$.

If $0 \neq (x_j)_{j=1}^{\infty} \in l_s^u(X)$, we have

$$\left\| \left(\frac{\delta_1 x_j}{\|(x_t)_{t=1}^{\infty}\|_{w,s}} \right)_{j=1}^{\infty} \right\|_{w,s} = \delta_1$$

and thus

$$\sum_{j=1}^{\infty} \left\| P_n \left(\frac{\delta_1 x_j}{\|(x_t)_{t=1}^{\infty}\|_{w,s}} \right) \right\|^r \leq C_1 \left\| \left(\frac{\delta_1 x_j}{\|(x_t)_{t=1}^{\infty}\|_{w,s}} \right)_{j=1}^{\infty} \right\|_{w,s}^s .$$

Hence

$$\left(\sum_{j=1}^{\infty} \left\| P_n \left(\frac{\delta_1 x_j}{\|(x_t)_{t=1}^{\infty}\|_{w,s}} \right) \right\|^r \right)^{\frac{1}{r}} \leq C_1^{\frac{1}{r}} \left\| \left(\frac{\delta_1 x_j}{\|(x_t)_{t=1}^{\infty}\|_{w,s}} \right)_{j=1}^{\infty} \right\|_{w,s}^{\frac{s}{r}} ,$$

and we obtain

$$\left(\sum_{j=1}^{\infty} \|P_n(x_j)\|^r \right)^{\frac{1}{r}} \leq C_1^{\frac{1}{r}} \frac{\|(x_t)_{t=1}^{\infty}\|_{w,s}^n}{\delta_1^n} \delta_1^{s/r} = \left(C_1^{\frac{1}{r}} \delta_1^{s/r} \right) \left(\frac{1}{\delta_1} \right)^n \|(x_t)_{t=1}^{\infty}\|_{w,s}^n$$

and finally

$$\left\| \frac{1}{n!} \widehat{d}^n f(a) \right\|_{as(r,s)} \leq \left(C_1^{\frac{1}{r}} \delta_1^{s/r} \right) \left(\frac{1}{\delta_1} \right)^n$$

for every $n \in \mathbb{N}$. $\square$

3. Inclusion theorem for multilinear mappings

Recall that an n-linear map $T \in \mathcal{L}(^nX;Y)$ is absolutely $(r;s)$-summing ($T \in \mathcal{L}_{as(r;s)}(^nX;Y)$) if

$$T(x_j^{(1)},...,x_j^{(n)})_{j=1}^{\infty} \in l_r(Y)$$

whenever $(x_j^{(k)})_{j=1}^{\infty} \in l_s^w(X)$, $k = 1,...,n$. In the case $r = s$ we simply say that T is r-summing, and instead of $\mathcal{L}_{as(r;r)}$ we write $\mathcal{L}_{as,r}$. The characterization of absolutely r-summing mappings and the definition of the norm is similar to those of Theorem 1 (for details we refer to [1]). In particular, the following characterization of absolutely r-summing mappings holds true:

Proposition 1. *$T \in \mathcal{L}(^nX;Y)$ is absolutely r-summing if and only if the mapping $\widehat{T} : l_r^u(X) \times \cdots \times l_r^u(X) \to l_r(Y)$ given by*

$$\widehat{T}\left((x_j^{(1)})_{j=1}^{\infty},...,(x_j^{(n)})_{j=1}^{\infty}\right) = \left(T(x_j^{(1)},...,x_j^{(n)})\right)_{j=1}^{\infty}$$

is well defined and bounded. In this case we have

$$\|T\|_{as,r} = \left\|\widehat{T}\right\|.$$

Now, we are ready to prove an inclusion theorem. As usual, the number $C_2(X)$ denotes the cotype constant of the Banach space X.

Theorem 3 (Inclusion Theorem)**.** *If X has cotype 2 and $n \geq 2$, then*

$$\mathcal{L}_{as,q}(^nX;Y) \subseteqq \mathcal{L}_{as,r}(^nX;Y)$$

holds true for $1 \leq r \leq q \leq 2$. Besides,

$$\|.\|_{as,r} \leq 16^n C_2(X)^{(\frac{7}{2}-\theta)n} \|.\|^{1-\theta} \|.\|_{as,q}^{\theta},$$

where θ is given by

$$\frac{1}{r} = (1-\theta) + \frac{\theta}{q}.$$

Proof. From [4, Theorem 2.5] we know that

$$\mathcal{L}_{as,1}(^nX;Y) = \mathcal{L}(^nX;Y) \text{ and } \|.\|_{as,1} \leq C_2(X)^n \|.\|$$

hold true for $n \geq 2$ and all Banach spaces Y, provided that X has cotype 2. So it suffices to prove the following sandwich-type result and then to apply it to $p = 1$:

- If X has cotype 2 and $T \in \mathcal{L}_{as,p}(^nX;Y) \cap \mathcal{L}_{as,q}(^nX;Y)$ for some $1 \leq p \leq q \leq 2$, then

  $$T \in \mathcal{L}_{as,r}(^nX;Y)$$

  for every r with $p \leq r \leq q$. Besides,

  $$\|.\|_{as,r} \leq 16^n C_2(X)^{5n/2} \|.\|_{as,p}^{1-\theta} \|.\|_{as,q}^{\theta},$$

  with θ given by

  $$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}. \tag{3.1}$$

We are going to prove the above result using the interpolation technique. By assumption on T and by Proposition 1 the map T generates bounded operators

$$\begin{cases} \widehat{T_p} : l_p^u(X) \times \cdots \times l_p^u(X) \to l_p(Y), \\ \widehat{T_q} : l_q^u(X) \times \cdots \times l_q^u(X) \to l_q(Y). \end{cases}$$

Applying the complex interpolation method to these n-linear operators we get a linear operator

$$\widehat{T_{(\theta)}} : \left[l_p^u(X), l_q^u(X)\right]_\theta \times \cdots \times \left[l_p^u(X), l_q^u(X)\right]_\theta \to [l_p(Y), l_q(Y)]_\theta$$

with

$$\left\|\widehat{T_{(\theta)}}\right\| \leq \left\|\widehat{T_p}\right\|^{1-\theta} \left\|\widehat{T_q}\right\|^{\theta}.$$

This operator satisfies

$$\widehat{T_{(\theta)}}\left((x_j^{(1)})_{j=1}^\infty, ..., (x_j^{(n)})_{j=1}^\infty\right) = \left(T(x_j^{(1)}, ..., x_j^{(n)})\right)_{j=1}^\infty$$

for all sequences $(x_j^{(k)})_{j=1}^\infty$ in $\left[l_p^u(X), l_q^u(X)\right]_\theta$, $1 \leq k \leq n$.

By [2] we have $[l_p(Y), l_q(Y)]_\theta = l_r(Y)$ isometrically (with r as in (3.1)), and, using the natural isometric identification $l_p^u(X) = l_p \widetilde{\otimes}_\varepsilon X$, we obtain from [5, Lemma 2 and Proposition 8], as a particular case, a natural isomorphism

$$J : l_r^u(X) \to \left[l_p^u(X), l_q^u(X)\right]_\theta$$

with

$$\|J\| \leq 16 C_2(X)^{\frac{5}{2}}.$$

With this isomorphism we can identify the operator $\widehat{T_{(\theta)}}$ with the map

$$\widehat{T_r} : l_r^u(X) \times \cdots \times l_r^u(X) \to l_r(Y),$$

and this gives us $T \in \mathcal{L}_{as,r}(^n X; Y)$ and

$$\begin{aligned} \|T\|_{as,r} = \left\|\widehat{T_r}\right\| &\leq \|J\|^n \left\|\widehat{T_{(\theta)}}\right\| \leq 16^n C_2(X)^{\frac{5n}{2}} \left\|\widehat{T_p}\right\|^{1-\theta} \left\|\widehat{T_q}\right\|^{\theta} \\ &= 16^n C_2(X)^{\frac{5n}{2}} \|T\|_{as,p}^{1-\theta} \|T\|_{as,q}^{\theta}. \end{aligned}$$

In our special case, $p = 1$ and $\|T\|_{as,1} \leq C_2(X)^n \|T\|$. Hence

$$\|T\|_{as,r} \leq 16^n C_2(X)^{\frac{5n}{2}} \left(C_2(X)^n \|T\|\right)^{1-\theta} \|T\|_{as,q}^{\theta},$$

and the result follows. □

Remark 1. In the linear case, for spaces X of cotype 2 we have $\mathcal{L}_{as,r}(X;Y) = \mathcal{L}_{as,2}(X;Y)$ for all numbers r with $1 \leq r \leq 2$ (cf. [6, Corollary 11.16]).

Remark 2. The same reasoning can be adapted to n-linear mappings defined on $X_1 \times \cdots \times X_n$ instead of $X \times \cdots \times X$.

4. Inclusion theorem for polynomials

If $P \in \mathcal{P}(^nX;Y)$, we denote by $\overset{\vee}{P}$ the symmetric n-linear mapping associated to P. The following simple result will be useful in this section.

Proposition 2. *Let $n \in \mathbb{N}$ and $s \geq 1$. If $P \in \mathcal{P}_{as,s}(^nX;Y)$, then $\overset{\vee}{P} \in \mathcal{L}_{as,s}(^nX;Y)$ and* $\left\|\overset{\vee}{P}\right\|_{as,s} \leq e^n \|P\|_{as,s}$.

Proof. If $(x_j^{(1)})_{j=1}^{\infty}, ..., (x_j^{(n)})_{j=1}^{\infty} \in B_{l_s^u(X)}$, then, by invoking the Polarization Formula, for every j, we obtain

$$\overset{\vee}{P}(x_j^{(1)}, ..., x_j^{(n)}) = \frac{1}{2^n n!} \sum_{\varepsilon_i=-1,1} \varepsilon_1 \cdots \varepsilon_n P(\varepsilon_1 x_j^{(1)} + \cdots + \varepsilon_n x_j^{(n)}).$$

So, we can infer that

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty} \left\|\overset{\vee}{P}(x_j^{(1)}, ..., x_j^{(n)})\right\|^s\right)^{\frac{1}{s}} \\
&= \frac{1}{2^n n!} \left(\sum_{j=1}^{\infty} \left\|\sum_{\varepsilon_i=-1,1} \varepsilon_1 \cdots \varepsilon_n P(\varepsilon_1 x_j^{(1)} + \cdots + \varepsilon_n x_j^{(n)})\right\|^s\right)^{\frac{1}{s}} \\
&\leq \frac{1}{2^n n!} \sum_{\varepsilon_i=-1,1} \left(\sum_{j=1}^{\infty} \left\|P(\varepsilon_1 x_j^{(1)} + \cdots + \varepsilon_n x_j^{(n)})\right\|^s\right)^{\frac{1}{s}} \\
&\leq \frac{1}{2^n n!} \|P\|_{as,s} \sum_{\varepsilon_i=-1,1} \left\|(\varepsilon_1 x_j^{(1)} + \cdots + \varepsilon_n x_j^{(n)})_{j=1}^{\infty}\right\|_{w,s}^n \\
&\leq \frac{1}{2^n n!} \|P\|_{as,s} \, 2^n \left(\left\|(x_j^{(1)})_{j=1}^{\infty}\right\|_{w,s} + \cdots + \left\|(x_j^{(n)})_{j=1}^{\infty}\right\|_{w,s}\right)^n \\
&\leq \frac{1}{n!} \|P\|_{as,s} \, n^n \leq e^n \|P\|_{as,s}.
\end{aligned}$$

□

Theorem 4 (Inclusion Theorem)**.** *If X has cotype 2 and $n \geq 2$, then*

$$\mathcal{P}_{as,q}(^nX;Y) \subseteqq \mathcal{P}_{as,r}(^nX;Y)$$

holds true for $1 \leq r \leq q \leq 2$ and all Banach spaces Y. Besides,

$$\|.\|_{as,r} \leq (16e)^n \, C_2(X)^{(\frac{7}{2}-\theta)n} \, \|.\|^{1-\theta} \, \|.\|_{as,q}^{\theta},$$

where θ is given by

$$\frac{1}{r} = (1-\theta) + \frac{\theta}{q}.$$

Proof. Let $P \in \mathcal{P}_{as,q}(^nX;Y)$. From Proposition 2 we know that $\overset{\vee}{P} \in \mathcal{L}_{as,q}(^nX;Y)$, and from Theorem 3 we deduce that

$$\overset{\vee}{P} \in \mathcal{L}_{as,r}(^nX;Y)$$

and

$$\left\|\overset{\vee}{P}\right\|_{as,r} \leq 16^n C_2(X)^{\left(\frac{7}{2}-\theta\right)n} \left\|\overset{\vee}{P}\right\|^{1-\theta} \left\|\overset{\vee}{P}\right\|_{as,q}^{\theta},$$

with

$$\frac{1}{r} = (1-\theta) + \frac{\theta}{q}.$$

So,

$$\begin{aligned}\|P\|_{as,r} \leq \left\|\overset{\vee}{P}\right\|_{as,r} &\leq 16^n C_2(X)^{\left(\frac{7}{2}-\theta\right)n} \left\|\overset{\vee}{P}\right\|^{1-\theta} \left\|\overset{\vee}{P}\right\|_{as,q}^{\theta} \\ &\leq 16^n C_2(X)^{\left(\frac{7}{2}-\theta\right)n} \left(e^n \|P\|\right)^{1-\theta} \left(e^n \|P\|_{as,q}\right)^{\theta} \\ &= (16e)^n C_2(X)^{\left(\frac{7}{2}-\theta\right)n} \|P\|^{1-\theta} \|P\|_{as,q}^{\theta}.\end{aligned}$$

□

5. Inclusion theorem for holomorphic mappings

The next theorem extends the previous results for holomorphic mappings.

Theorem 5. *If $1 \leq r \leq q \leq 2$ and X has cotype 2, then*

$$\mathcal{H}_{as,q}^{(a)}(X;Y) \subseteqq \mathcal{H}_{as,r}^{(a)}(X;Y)$$

for all Banach spaces Y.

Proof. If $f \in \mathcal{H}_{as,q}^{(a)}(X;Y)$, from Theorem 2 we know that there are positive constants k_1 and K_1 such that

$$\left\|\frac{1}{m!}\widehat{d}^m f(a)\right\|_{as,q} \leq K_1 k_1^m$$

for every $m \in \mathbb{N}$. Hence, using Theorem 4 we get, for $m \geq 2$,

$$\left\|\frac{1}{m!}\widehat{d}^m f(a)\right\|_{as,r} \leq (16e)^m C_2(X)^{\left(\frac{7}{2}-\theta\right)m} \left\|\frac{1}{m!}\widehat{d}^m f(a)\right\|^{1-\theta} \left\|\frac{1}{m!}\widehat{d}^m f(a)\right\|_{as,q}^{\theta} \leq K_2 k_2^m.$$

From Remark 1, we know that

$$df(a) \in \mathcal{L}_{as,q}(X;Y) = \mathcal{L}_{as,r}(X;Y),$$

and we can easily find adequate constants k_3 and K_3 such that

$$\left\|\frac{1}{m!}\widehat{d}^m f(a)\right\|_{as,r} \leq K_3 k_3^m$$

for every $m \in \mathbb{N}$. Hence Theorem 2 asserts that $f \in \mathcal{H}_{as,r}^{(a)}(X;Y)$. □

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