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ASYMPTOTICALLY HYPERBOLIC METRICS ON A UNIT BALL ADMITTING MULTIPLE HORIZONS

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Abstract. In this paper, we construct an asymptotically hyperbolic metric with scalar curvature -6 on the unit ball $\mathbf{D}^3$, which contains multiple horizons.

1. Introduction

In general relativity, the initial data set of the Cauchy problem for Einstein equations which is denoted by (M, g_{ij}, p_{ij}) is of great importance. Here (M, g_{ij}) is a complete Riemannian 3-manifold and p_{ij} is a symmetric 2-tensor on M satisfying constrain equations (see [7]). Among all of the initial data sets, those with asymptotically flat (AF) (see [7]) and asymptotically hyperbolic (AH) (see Definition 2.1) metrics are of most interest so far.

On the other hand, a horizon, which is defined by a surface $\Sigma \subset M$ satisfying $H_\Sigma = tr_\Sigma(p)$ (see [2]), is a very interesting geometric object. When $p = 0$ (i.e. time symmetric case), the horizon is nothing but a minimal surface. The Schwarzschild and anti-de Sitter–Schwarzschild space are the simplest examples for AF and AH manifolds with horizon, respectively. But they both have nontrivial topology. However, for physical and mathematical reasons, people intend to construct topologically trivial manifolds with horizons. In [1], R. Beig and N. Ó Murchadha show that there exists an AF metric which contains a horizon, with a scalar flat on $\mathbb{R}^3$. Also, Miao in [6] constructs the same kind of AF manifolds by making use of the Schwarzschild metric and the conformal deformation. These results offer examples of globally regular and AF initial data for the Einstein vacuum equations with minimal surfaces. Combining the method of Miao with that of Chruściel and Delay [3], the author of [4] gives an example of a scalar flat AF metric on $\mathbb{R}^3$ admitting multiple horizons.

In recent years, AH manifolds have drawn more and more attention of both mathematicians and physicists. They arise when considering solutions to the Einstein field equations with a negative cosmological constant or when considering "hyperboloidal hypersurfaces" in space-times which are asymptotically flat in isotropic directions. The horizons in AH manifolds are more than minimal surfaces, since

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AH manifolds can be realized as asymptotically null spacelike hypersurfaces in asymptotically flat space-time. Therefore, in an AH manifold, horizons refer not only to boundaries of domains which are minimal surfaces (in the case of considering a negative cosmological constant) but also to boundaries satisfying $H = \pm 2$ (in the case of an asymptotically null spacelike hypersurface in AF spacetimes). Recently, in [8], the authors provide an example of an AH manifold with a constant scalar curvature -6 and horizons (see also Theorem 2.2). Their main idea is to glue the anti-de Sitter–Schwarzchild space with a ball and deform it conformally several times; then they get the desired manifold. Furthermore, they can prove that the mass of their example can be arbitrarily large or small. So, it is natural to consider constructing such an AH manifold admitting multiple horizons. More precisely, in this paper we show that there exists an AH metric on the unit ball $\mathbf{D}^3$ with constant scalar curvature $R = -6$ and multiple horizons (see Theorem 2.4). First, we will construct a metric on a unit 3-ball with multiple horizons using the cut-and-glue method. Secondly, we will conformally deform the metric to an AH metric with constant scalar curvature $R = -6$ by solving a nonlinear PDE; then, as in [8], the existence of horizons follows from the implicit function theorem. We'd like to remark that it seems that our method should work for the construction of an AF manifold with multiple horizons, as has been done in [4].

The outline of this paper is as follows. In Section 2 we cut the parts containing horizons from some examples of AH manifolds given by [8] and glue them together smoothly. Consequently, we obtain a new AH metric with multiple horizons. In Section 3, we perturb the new metric conformally to an AH metric with constant scalar curvature -6. Then by keeping the location of horizons far enough from each other, we show that the existence of the multiple horizons is guaranteed by a lemma in [8].

2. Construction of an asymptotically hyperbolic metric on a unit ball with multiple horizons

In this section, we will complete the first step of the proof of the main result; namely, we will construct an asymptotically hyperbolic metric on $\mathbf{D}^3$ admitting multiple horizons by gluing arguments, but the scalar curvature may not be equal to -6. First of all, let us recall some basic definitions and facts.

Definition 2.1. A complete non-compact Riemannian manifold (X^3, g) is said to be asymptotically hyperbolic if there is a compact manifold $(\overline{X}, \overline{g})$ with boundary ∂X and a smooth function t on $\overline{X}$ such that the following are true:

(i) $X = \overline{X} \setminus \partial X$.
(ii) $t = 0$ on ∂X, and $t > 0$ on X.
(iii) $\overline{g} = t^2 g$ extends to be C^3 up to the boundary.
(iv) $|dt|_{\overline{g}} = 1$ at ∂X.
(v) Each component Σ of ∂X is the standard two sphere $(\mathbb{S}^2, g_0)$, and there is a collar neighborhood of Σ where

$$g = \sinh^{-2} t (dt^2 + g_t)$$

with

$$g_t = g_0 + \frac{t^3}{3}h + O(t^4),$$

where h is a C^2 symmetric two tensor on $\mathbb{S}^2$.

It is proved in [9] that for an AH manifold (X^3, g) with scalar curvature $R_g \geq -6$, the mass of an end of X corresponding to a boundary component Σ of ∂X is well-defined and given by

$$M = \frac{1}{16\pi}[(\int_{\mathbb{S}^2} tr_{g_0}(h)dv_{g_0})^2 - |\int_{\mathbb{S}^2} tr_{g_0}(h)(x)xdv_{g_0}|^2]^{\frac{1}{2}},$$

where x is the standard coordinates of a point on $\mathbb{S}^2$ in $\mathbb{R}^3$.

We denote the standard hyperbolic space by $\mathbb{H}^3$ and introduce the ball model for $\mathbb{H}^3$ which is denoted by $(\mathbf{D^3}, ds^2_{\mathbb{H}^3})$. Here, $\mathbf{D^3}$ is the unit ball in $\mathbb{R}^3$, and $ds^2_{\mathbb{H}^3}$ is the standard hyperbolic metric which is defined as follows:

$$ds^2_{\mathbb{H}^3} = \frac{4}{(1-|x|^2)^2}\sum_{i=1}^{3}(dx^i)^2,$$

where $\sum_{i=1}^{3}(dx^i)^2$ is the Euclidean metric.

In [8], the authors construct a family of asymptotically hyperbolic metrics on $\mathbf{D}^3$ as follows:

Theorem 2.2 ([8])**.** *Let $\mathbf{D}^3$ be the unit ball in $\mathbb{R}^3$. For any $M > 0$ and δ, there is a smooth complete metric g on $\mathbf{D}^3$ with constant scalar curvature -6 such that the following are true:*

(i) *$(\mathbf{D}^3, g)$ is asymptotically hyperbolic with mass M_g satisfying $|M_g - M| < \delta$.*
(ii) *There exist surfaces S_1, S_2 and S_3 which are topological spheres with constant mean curvature $-2, 0, 2$ respectively such that S_1 is in the interior of S_2 and S_2 is in the interior of S_3.*
(iii) *Outside a compact set U, the metric g is conformal to the standard hyperbolic metric of $\mathbf{D}^3$, and S_1, S_2 and S_3 are contained in $\mathbf{D}^3 \setminus U$.*

Let us fix some notation for our paper. We will denote the origin by o. Let $x \in \mathbf{D}^3$ and $B_x(\rho)$ be the geodesic ball centered at x with radius ρ under the standard hyperbolic metric on $\mathbf{D}^3$. The hyperbolic distance starting from x to y will be denoted by $\rho_x(y)$ (simply by $\rho(y)$ if x is the origin). Without loss of generality, we may reformulate Theorem 2.2 in the following way:

Theorem 2.3. *Let S_1, S_2, S_3 be the surfaces and g be the AH metric which are given by Theorem 2.2. Then there exists a geodesic ball $B_o(\delta)$ under the hyperbolic metric such that S_1, S_2, S_3 are contained in $B_o(\delta)\setminus B_o(\frac{\delta}{2})$ and g is conformal to the standard hyperbolic metric of $\mathbf{D}^3$ on $\mathbf{D}^3 \setminus B_o(\frac{\delta}{4})$.*

Using the gluing method and conformal deformation again, we are able to prove our main result:

Theorem 2.4. *Let $\mathbf{D}^3$ be the unit open ball in $\mathbb{R}^3$. For any $K > 0$, there is an AH metric g on $\mathbf{D}^3$ with constant scalar curvature -6 such that there are $\{x_k\}_{k=1}^K \subset \mathbf{D}^3$ and surfaces S_1^i, S_2^i and S_3^i, $1 \leq i \leq K$, which are topological spheres with constant mean curvature $-2, 0, 2$ respectively and are contained in $B_{x_i}(\delta)$ and which do not* *intersect each other. Moreover, S_1^i is in the interior of S_2^i, and S_2^i is in the interior of S_3^i; and outside a compact set, the metric g is conformal to the standard hyperbolic metric of $\mathbf{D}^3$.*

We consider only the case for $K = 2$, since the other cases are essentially the same. By gluing arguments, we will show

Proposition 2.5. *There is a smooth AH metric $\widetilde{g}$ on $\mathbf{D}^3$ with the following properties:*

1. *$B_o(\delta_2) \setminus B_o(\frac{\delta_2}{2})$ and $B_p(\delta_1) \setminus B_p(\frac{\delta_1}{2})$ each contain the surfaces with mean curvature $2, 0, -2$. Here, o, p are two points in $\mathbf{D}^3$, with hyperbolic distance being $2\tau \triangleq \rho(p) > 10(\delta_1 + \delta_2)$, so that $B_o(\delta_2)$ and $B_p(\delta_1)$ do not intersect each other. Without loss of generality, we may assume $\tau \geq 100$.*
2. *The scalar curvature $R_{\widetilde{g}}$ of $\widetilde{g}$ satisfies that*
$$R_{\widetilde{g}}(x) = -6$$
for $x \in \mathbf{D}^3 \setminus (B_p(\tau+2) \setminus B_p(\tau+1))$ and
$$|R_{\widetilde{g}}(x) + 6| \leq Ce^{-3\tau}$$
for $x \in B_p(\tau+2) \setminus B_p(\tau+1)$. Here, C is a positive constant which is independent of τ.
3. *$\widetilde{g}$ is conformal to the hyperbolic metric outside $B_o(\frac{\delta_2}{4}) \cup B_p(\frac{\delta_1}{4})$.*

Proof. Choose two AH metrics g_1, g_2 as given by Theorem 2.2 (not necessarily having the same mass). Let $B_o(\delta_1)$ and $B_o(\delta_2)$ be the sets described in Theorem 2.3 for $(\mathbf{D}^3, g_1)$ and $(\mathbf{D}^3, g_2)$ respectively such that for $i = 1, 2$,
$$g_i(y) = \phi_i^4(y) ds^2_{\mathbb{H}^3},$$
where $y \in \mathbf{D}^3 \setminus B_o(\frac{\delta_i}{4})$. Also, by Theorem 4.1 in [8], ϕ_i satisfies
$$\| \phi_i(y) - 1 \|_{\mathbf{C}^3} \leq Ce^{-3\rho(y)} \tag{2.1}$$
for $y \in \mathbf{D}^3 \setminus B_o(\frac{\delta_i}{4})$, and C is a positive constant that is independent of y.

Now, we will glue g_1 and g_2 together as follows.

Let us introduce the upper halfspace model $\mathbb{R}^3_+$ for $\mathbb{H}^3$ and label the point $x \in \mathbb{H}^3$ by $(\mathbf{x}, y)$ with $\mathbf{x} \in \mathbb{R}^2$ and $y \in \mathbb{R}_+$. Under this coordinate system, the standard metric for hyperbolic space can be expressed as
$$ds^2_{\mathbb{H}^3} = \frac{(dx^1)^2 + (dx^2)^2 + dy^2}{y^2};$$
here, $\mathbf{x} = (x^1, x^2) \in \mathbb{R}^2$. Suppose $o = (\mathbf{0}, 1)$ and $p = (\mathbf{x}_p, y_p)$. Then there is a hyperbolic translation F which maps o to p with $\rho(p) \triangleq 2\tau > 10(\delta_1 + \delta_2)$,
$$\begin{aligned} F : B_o(\tau+3) &\longrightarrow B_p(\tau+3), \\ (\mathbf{x}, y) &\longmapsto F(\mathbf{x}, y) = (\mathbf{x}_p + y_p\mathbf{x}, y_p y). \end{aligned}$$
Then it is easy to see that F induces a natural isometry between the standard hyperbolic metric in $B_p(\tau+3)$ and that in $B_o(\tau+3)$:
$$(B_p(\tau+3), (F^{-1})^* ds^2_{\mathbb{H}^3}|_{B_0(\tau+3)}) \cong (B_p(\tau+3), ds^2_{\mathbb{H}^3}|_{B_p(\tau+3)}).$$
Indeed, $(F^{-1})^*(ds^2_{\mathbb{H}^3}|_{B_0(\tau+3)}) = ds^2_{\mathbb{H}^3}|_{B_p(\tau+3)}$ in the sense that the metrics on both sides have the same components under the standard upper halfspace coordinates.

Therefore, we can pull g_1 on $B_o(\tau+3)$ to $B_p(\tau+3)$ by the diffeomorphism F, which gives an isometry:

$$(B_p(\tau+3),(F^{-1})^*g_1)\cong(B_o(\tau+3),g_1).$$

Because of (2.2), we can identify $B_o(\tau+3)$ and $B_p(\tau+3)$ both equipped with hyperbolic metric via F, and

$$(F^{-1})^*g_1=(\phi_1\circ F^{-1})^4(F^{-1})^*(ds^2_{\mathbb{H}^3}|_{B_0(\tau+3)})=(\phi_1\circ F^{-1})^4ds^2_{\mathbb{H}^3}|_{B_p(\tau+3)}$$

for $x\in B_p(\tau+3)\setminus B_p(\frac{\delta_1}{4})$. Also, the inequality (2.1) for ϕ_1 can be described as

$$\|\ \phi_1\circ F^{-1}(y)-1\ \|_{\mathbf{C}^3}\leq Ce^{-3\rho_p(y)}$$

for $y\in B_p(\tau+3)\setminus B_p(\frac{\delta_1}{4})$. For simplicity, $\phi_1\circ F^{-1}(y)$ will still be denoted by $\phi_1(y)$ in the sequel.

Let η be a smooth cut-off function such that $0\leq\eta\leq1$ and

$$\eta(x)=\begin{cases}1 & x\in B_p(\tau+1),\\ \\ 0 & x\in\mathbf{D}^3\setminus B_p(\tau+2).\end{cases}$$

Hence $||\eta||_{C^2}$ is uniformly bounded. Next, we define a new metric $\widetilde{g}$ on $\mathbf{D}^3$, which is given by

$$\widetilde{g}(x)=\begin{cases}(F^{-1})^*g_1, & x\in B_p(\tau+1);\\ (\eta\phi_1+(1-\eta)\phi_2)^4ds^2_{\mathbb{H}^3}|_{B_p(\tau+2)\setminus B_p(\tau+1)}, & x\in B_p(\tau+2)\setminus B_p(\tau+1);\\ g_2, & x\in\mathbf{D}^3\setminus B_p(\tau+2).\end{cases}$$

By its definition, we see $\widetilde{g}$ satisfies (1) in Proposition 2.5.

Again by the definition of $\widetilde{g}$ and (2.1), we can calculate that the scalar curvature $R_{\widetilde{g}}$ of $\widetilde{g}$ satisfies that

$$R_{\widetilde{g}}(x)=-6\quad\text{for}\quad x\in\mathbf{D}^3\setminus(B_p(\tau+2)\setminus B_p(\tau+1))$$

and for $x\in B_p(\tau+2)\setminus B_p(\tau+1)$,

$$|R_{\widetilde{g}}(x)+6|\leq Ce^{-3\tau}.$$

Thus, we verified (2) in Proposition 2.5, and by Theorem 2.3, we see that (3) in Proposition 2.5 is also true; therefore, we finish to prove the proposition. □

Remark 2.6. One can see from the construction of $\widetilde{g}$ that it depends on τ. To emphasize this, we will denote $\widetilde{g}$ by $\widetilde{g}_\tau$ in the next section.

3. Proof of the main result by conformal deformation

In this section, we will prove our main result of Theorem 2.4. Namely, we perturb $\widetilde{g}_\tau$ constructed in the last section by conformal deformation and show that the resulting metric is an AH metric with scalar curvature equal to -6 and containing multiple horizons. For this purpose, we need

Lemma 3.1. *Let $\widetilde{g}_\tau$ be constructed as in Proposition* 2.5*. Then there is $u_\tau>0$ such that $g_\tau=u_\tau^4\widetilde{g}_\tau$ is an AH metric with scalar curvature $R=-6$, and*

$$\lim_{\tau\to\infty}\sup_{\mathbf{D}^3}|u_\tau-1|=0.$$

Moreover, outside a compact set, the metric g_τ is conformal to the standard hyperbolic metric on $\mathbf{D}^3$.

Proof. It is sufficient to solve the following equation:

$$\begin{cases} \triangle_{\widetilde{g}_\tau} u - \frac{3}{4}u^5 - \frac{1}{8}R_{\widetilde{g}_\tau} u = 0, \\ \\ \lim\limits_{\rho(x)\to+\infty} u(x) = 1. \end{cases} \tag{3.1}$$

To do this, we will use exhausting domain arguments. Let us choose a sequence $\{\rho_k\}_{k=1}^\infty$ with $\rho_1 \geq 3\tau$ such that $\rho_k < \rho_{k+1}$ for $k \in \mathbf{N}$ and $\rho_k \to +\infty$ as $k \to +\infty$. Consider the following Dirichlet problem:

$$\begin{cases} \triangle_{\widetilde{g}_\tau} u_i - \frac{3}{4}u_i^5 - \frac{1}{8}R_{\widetilde{g}_\tau} u_i = 0, \quad \text{on} \quad B_o(\rho_i), \\ \\ u_i|_{\partial B_o(\rho_i)} = \phi_2^{-1}. \end{cases} \tag{3.2}$$

By the standard variational method, we see (3.2) has a smooth nonnegative solution. Also, by the maximal principle, u_i must be positive. By assumption for ϕ_2, we know that $1 - Ce^{-3\rho_i} \leq u_i|_{\partial B_o(\rho_i)} \leq 1 + Ce^{-3\rho_i}$. We claim that

$$\sup_{B_o(\rho_i)} |u_i - 1| \leq Ce^{-3\tau}; \tag{3.3}$$

here, C is independent of i and τ. Let us prove that the lower bounded estimate is true. Indeed, suppose u attains its minimum at x_0. If $x_0 \in \partial B_o(\rho_i)$, then the claim of the lower bound follows; otherwise, at an interior point x_0, one has

$$0 \leq u_i^{-1} \triangle_{\widetilde{g}_\tau} u_i \quad = \quad \frac{3}{4}u_i^4 + \frac{1}{8}R_{\widetilde{g}_\tau}.$$

By (2) of Proposition 2.5, we get the claim for the lower bound. By similar arguments, we will get the upper bounded estimate. Thus the claim is true. Then, by exhausting domain, we get the solution u_τ of (3.1), and

$$\sup_{\mathbf{D}^3} |u_\tau - 1| \leq Ce^{-3\tau};$$

here, C is a constant which is independent of τ.

Next, we will construct a barrier of the equation at infinity of the manifold, and by using this we show that u approaches 1 at a desired rate.

Let $v_i = u_i\phi_2$. Since g_2 is conformal to the standard hyperbolic metric outside $B_o(3\tau)$, we have

$$\begin{cases} \mathbf{L}(v_i) := \triangle_{\mathbb{H}^3} v_i - \frac{3}{4}v_i(v_i^4 - 1) = 0, \quad \text{for} \quad x \in B_o(\rho_i) \setminus B_o(\overline{\rho}), \\ \\ v_i|_{\partial B_o(\rho_i)} = 1 \end{cases}$$

for some $\overline{\rho} \geq 3\tau$, where $\triangle_{\mathbb{H}^3}$ is the Laplacian operator for hyperbolic metric. Set $f_-(x) = 1 - \lambda e^{-3\rho(x)}$, for $\lambda \geq 0$. Hence,

$$\begin{aligned} \mathbf{L}(f_-) \quad &= \quad -9\lambda e^{-3\rho(x)} + 6\lambda e^{-3\rho(x)} \coth\rho \\ &\quad -\frac{3}{4}(1 - \lambda e^{-3\rho(x)})[(1 - \lambda e^{-3\rho(x)})^4 - 1] \\ &= \quad 6\lambda(\coth\rho - 1)e^{-3\rho(x)} + O(e^{-6\rho(x)}). \end{aligned}$$

Since $(\coth\rho - 1)e^{-3\rho(x)} > 2e^{-5\rho(x)}$, for sufficiently large $\overline{\rho}$ we have $\mathbf{L}(f_-) > 0$ in $B_o(\rho_i) \setminus B_o(\overline{\rho})$. So let us choose $\lambda = e^{3\overline{\rho}}$; then $\mathbf{L}(f_-) > 0$ whenever $\rho \geq \overline{\rho}$, $v_i = 1 \geq f_-$ at $\partial B_o(\rho_i)$, and $v_i \geq f_- = 0$ at $\partial B_o(\overline{\rho})$. Due to (2.1) and (3.3), by choosing τ and $\bar{\rho}$ sufficiently large, we may assume $v_i \geq 2^{-\frac{1}{2}} > 5^{-\frac{1}{4}}$ on $B_o(\rho_i) \setminus B_o(\overline{\rho})$. Now, we claim that $v_i \geq f_-$ on $B_o(\rho_i) \setminus B_o(\overline{\rho})$. For any point of the boundary, the claim is obviously true. Suppose the claim fails. Then for any $p \in B_o(\rho_i) \setminus B_o(\overline{\rho})$ with $(v_i - f_-)(p) = \inf_{B_o(\rho_i)\setminus B_o,\ (\overline{\rho})}(v_i - f_-) < 0$ is an interior point at which we have

$$5^{-\frac{1}{4}} < 2^{-\frac{1}{2}} \leq v_i(p) < f_-(p).$$

Thus, at p, we have

$$0 \leq \triangle_{\mathbb{H}^3}(v_i - f_-) \leq \frac{3}{4}v_i(v_i^4 - 1) - \frac{3}{4}f_-(f_-^4 - 1) < 0,$$

which is a contradiction. Therefore, we obtain

$$v_i - 1 \geq -\lambda e^{-3\rho} \quad \text{in} \quad B_o(\rho_i) \setminus B_o(\overline{\rho}).$$

On the other hand, by noting the super solution $f_+(x) = 1 + \lambda e^{-3\rho(x)}$ and using similar arguments, we can prove that $v_i - 1 \leq \lambda e^{-3\rho}$ in $B_o(\rho_i) \setminus B_o(\overline{\rho})$. Hence, we have

$$|v_i - 1| \leq \lambda e^{-3\rho(x)} \quad \text{in} \quad B_o(\rho_i) \setminus B_o(\overline{\rho}).$$

Consequently, u_τ satisfies that

$$|u_\tau - 1| \leq \lambda e^{-3\rho} \quad \text{in} \quad \mathbf{D}^3 \setminus B_o(\overline{\rho}), \tag{3.4}$$

and also by (3.3)

$$\sup_{\mathbf{D}^3} |u_\tau - 1| \leq Ce^{-3\tau}; \tag{3.5}$$

as mentioned above here C is a constant that is independent of τ.

Now applying Lemma 4.2 in [8], we conclude from (3.4) that the manifold $(\mathbf{D^3}, g_\tau)$ with $g_\tau = u_\tau^4 \widetilde{g}_\tau$ is an AH manifold. Also, it follows from Proposition 2.5 that g_τ is conformal to the hyperbolic metric outside some compact subset of $\mathbf{D}^3$. Thus, we finish the proof of the lemma. □

Next, we have the following:

Lemma 3.2. *Let g_τ be as in Lemma* 3.1*, which depends on τ. Then for sufficiently large τ, $(\mathbf{D}^3, g_\tau)$ contains surfaces S_1^i, S_2^i and S_3^i, $1 \leq i \leq 2$, which are topological spheres with constant mean curvature $-2, 0, 2$ and are contained in $B_o(\delta)$ and $B_p(\delta)$ respectively which do not intersect each other. Moreover, S_1^i is in the interior of S_2^i, and S_2^i is in the interior of S_3^i.*

Proof. Let us show that horizons are in $(B_o(2\delta), g_\tau)$ when τ is large enough. In fact, by Lemma 4.4 in [8], it is sufficient to show that

$$\|u_\tau - 1\|_{C^{2,\alpha}(B_o(2\delta)\setminus B_o(\frac{\delta}{4}))} \leq \epsilon. \tag{3.6}$$

Here ϵ and u_τ are given in Lemma 4.4 in [8] and Lemma 3.1 respectively. Note in $(B_o(2\delta), \widetilde{g}_\tau)$ that sectional curvature is bounded and the injective radius has a uniform positive lower bound. Then $B_o(\frac{3}{2}\delta)$ can be covered by a finite number of harmonic coordinates which have uniform size (for the existence of harmonic coordinates and estimates of their size, see [5]). The number and size of these harmonic coordinates are independent of τ. For any $x \in B_o(\frac{3}{2}\delta)$, without loss of generality, we may assume $B_x(1)$ has already been covered by such harmonic coordinates; then the components of the metric g_τ under the harmonic coordinates satisfy

$$\| (\widetilde{g}_\tau)_{ij} \|_{C^{1,\alpha}} \leq C.$$

Here C is independent of τ and x. Now in $B_x(1)$, we have the equation

$$\triangle_{\tilde{g}_\tau} u_\tau = \frac{3}{4} u_\tau (u_\tau^4 - 1),$$

by (3.1). Then combining (3.5) with the standard estimate of PDE, we get

$$\|u_\tau - 1\|_{C^{2,\alpha}(B_x(1))} \leq Ce^{-3\tau},$$

where C is independent of τ. Since x is arbitrary in $B_o(2\delta)$ and τ can be arbitrarily large, we get (3.6), which implies there are horizons in $B_o(2\delta)$. By the same arguments, we can show there are also horizons in $B_p(2\delta)$. Thus we complete the proof of the lemma. □

Combining the above lemmas, we get the proof of Theorem 2.4, which proves our main result.

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