CONVERGENT MARTINGALES OF OPERATORS
AND THE RADON NIKODÝM PROPERTY
IN BANACH SPACES

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Abstract. We extend Troitsky’s ideas on measure-free martingales on Banach lattices to martingales of operators acting between a Banach lattice and a Banach space. We prove that each norm bounded martingale of cone absolutely summing (c.a.s.) operators (also known as 1-concave operators), from a Banach lattice $E$ to a Banach space $Y$, can be generated by a single c.a.s. operator. As a consequence, we obtain a characterization of Banach spaces with the Radon Nikodým property in terms of convergence of norm bounded martingales defined on the Chaney-Schaefer $l\otimes\Delta_p Y$. This extends a classical martingale characterization of the Radon Nikodým property, formulated in the Lebesgue-Bochner spaces $L^p(\mu, Y)$ ($1 < p < \infty$).

1. Introduction

Let $(\Omega, \Sigma, \mu)$ be a finite measure space and let $Y$ be a Banach space. For $1 \leq p < \infty$, let $L^p(\mu, Y)$ denote the space of (classes of a.e. equal) Bochner $p$-integrable functions $f : \Omega \rightarrow Y$ and denote the Bochner norm on $L^p(\mu, Y)$ by $\Delta_p$, i.e.

$$\Delta_p(f) = \left( \int_\Omega \|f\|_Y^p \, d\mu \right)^{1/p}.$$ 

In the Lebesgue-Bochner spaces, martingale theory provides an important link to the geometric properties of the Banach space $Y$. For example, $Y$ has the Radon Nikodým property if and only if every martingale $(f_i) \subset L^p(\mu, Y)$, which is uniformly $\|\cdot\|_p$-bounded, converges in the $\|\cdot\|_p$-norm for all finite measure spaces $(\Omega, \Sigma, \mu)$ and $1 < p < \infty$ (cf. [9]).

It is well known that $L^p(\mu, Y)$ is isometrically isomorphic to the norm completion $L^p(\mu)\bar{\otimes}\Delta_p Y$ of $L^p(\mu) \otimes\Delta_p Y$, where $\Delta_p$ denotes the induced Bochner norm (cf. [4, 5]). Chaney [1] characterized the Radon Nikodým property on $Y$ in terms of equality of $L^p(\mu)\bar{\otimes}\Delta_p Y$ and the space of Dinculeanu’s operators from $L^q(\mu)$ to $Y$ ($1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$). These are the operators $T : L^q(\mu) \rightarrow Y$ for which $\infty > \|T\|_q := \sup\{\sum_{i=1}^n \|\alpha_i T(\chi_{E_i})\|_Y\}$, where the supremum is taken over all...
simple functions $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ with $\chi_{E_1}, \ldots, \chi_{E_n} \in L^q(\mu)$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $\|f\|_q \leq 1$ (cf. [6, 7, 8]).

The aim of this paper is to extend Chaney’s characterization by using Troitsky’s [18] generalized notion of a martingale in a Banach lattice.

Chaney and Schaefer extended the Bochner norm to the tensor product of a Banach lattice $E$ and a Banach space $Y$; the norm $\|\cdot\|_l$, given by

$$\|u\|_l = \inf \left\{ \left\| \sum_{i=1}^{n} y_i \cdot |x_i| \right\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

for all $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes Y$, coincides with the Bochner norm on $L^p(\mu) \otimes Y$ for all $\sigma$-finite measure spaces $(\Omega, \Sigma, \mu)$ and $1 \leq p < \infty$ (see [1, 17] and [11]).

Let $E$ be a Banach lattice and let $Y$ be a Banach space. Recall from [17] Chapter IV, §3 that a linear map $T : E \to Y$ is called cone absolutely summing if for every positive summable sequence $(x_i)$ in $E$, the sequence $(Tx_i)$ is absolutely summable in $Y$. Such operators are known, in the terminology of Krivine, as 1-concave operators (cf. [15, p. 45] and [13]). The space

$$\mathcal{L}^{\text{cas}}(E, Y) = \{ T : E \to Y : T \text{ is cone absolutely summing} \}$$

is a Banach space with respect to the norm defined by

$$\|T\|_{\text{cas}} = \sup \left\{ \left\| \sum_{i=1}^{n} T x_i \right\| : x_1, \ldots, x_n \in E_+, \left\| \sum_{i=1}^{n} x_i \right\| = 1, n \in \mathbb{N} \right\}$$

for all $T \in \mathcal{L}^{\text{cas}}(E, Y)$. Cone absolutely summing operators generalize Dinculeanu’s operators, mentioned above.

Cone absolutely summing operators extend the Chaney-Schaefer $l$-tensor product in the following sense: The canonical map $E^* \otimes l Y \to \mathcal{L}^{\text{cas}}(E, Y)$ given by

$$\sum_{i=1}^{n} x_i^* \otimes y_i := u \mapsto L_u, \text{ where } L_u x = \sum_{i=1}^{n} \langle x, x_i^* \rangle y_i \text{ for all } x \in E,$$

is an isometry (cf. [17] Chapter IV, §7] and [11, 14]). Here, $E^*$ denotes the continuous dual of $E$.

2. Preliminaries

We recall the abstract notions in [18] for a filtration and a martingale in a Banach space.

**Definition 2.1.** Let $E$ be a Banach lattice and $Y$ a Banach space.

(a) Let $(T_i)$ be a sequence of contractive projections on $Y$. If $T_{i,j} = T_i T_j$ for each $i, j \in \mathbb{N}$, then $(T_i)$ is called a filtration on $Y$. In the case where $(T_i)$ is a filtration on a Banach lattice, we will always assume each $T_i$ to be positive.

(b) If $(T_i)$ is a filtration on $E$ with the property that each $R(T_i)$ is a (closed) Riesz subspace of $E$, then $(T_i)$ will be called a BL-filtration on $E$ (this terminology is also used in [2]).

(c) If $(T_i)$ is a filtration on $E$ with each $T_i$ strictly positive (i.e., $T_i$ is positive and $\{ f \in E : T_i(|f|) = 0 \} = \{0\}$), then $(T_i)$ is called a strictly positive filtration.

(d) If $(T_i)$ is a filtration on $Y$ and $(f_i) \subset Y$, then the pair $(f_i, T_i)$ is called a martingale in $Y$ if $T_i f_j = f_i$ for all $i \leq j$. 
(e) If \((f_i, T_i)\) is a martingale in \(Y\), then \((f_i, T_i)\) is called fixed if there exists \(f \in Y\) such that \(f_i = T_i f\) for all \(i \in \mathbb{N}\). In this case, \((f_i, T_i)\) is said to be fixed on \(f\).

(f) If \((T_i)\) is a filtration on \(E\), then \((f_i, T_i)\) is called a submartingale in \(E\) if \(T_if_j \geq f_i\) for all \(i \leq j\).

(g) Let \((T_i)\) be a filtration on \(Y\). We say that \((T_i)\) is complemented in \(Y\) if there exists a contractive projection \(T_\infty : Y \to Y\) with \(\mathcal{R}(T_\infty) = \bigcup_{i=1}^\infty \mathcal{R}(T_i)\) and \(T_i T_\infty = T_\infty T_i = T_i\) for all \(i \in \mathbb{N}\).

If \((\Omega, \Sigma, \mu)\) is a finite measure space, \(1 \leq p < \infty\) and \((\Sigma_i)\) an increasing sequence of \(\sigma\)-subalgebras of \(\Sigma\), it follows that the sequence of conditional expectations \((\mathbb{E} \cdot | \Sigma_i))\) on \(L^p(\mu)\) satisfies the above definition of a BL-filtration. Moreover, \((\mathbb{E} \cdot | \Sigma_i))\) is complemented by \(\mathbb{E} \cdot | \bigvee_{i=1}^\infty \Sigma_i) : L^p(\mu) \to L^p(\mu)\), where \(\bigvee_{i=1}^\infty \Sigma_i\) denotes the \(\sigma\)-algebra generated by \(\bigcup_{i=1}^\infty \Sigma_i\).

In [17] Chapter III, §11, Proposition 11.5 it is shown that if \(T : E \to E\) is a strictly positive projection on a Banach lattice \(E\), then \(\mathcal{R}(T)\) is a Banach sublattice of \(E\). Consequently, every strictly positive filtration on \(E\) is also a BL-filtration.

It is easily verified that if \((T_i)\) is a filtration on \(E\), then the sequence of adjoint operators \((T_i^*)\) is a filtration on \(E^*\). However, it is not clear that \((T_i^*)\) is a BL-filtration whenever \((T_i)\) is a BL-filtration. We need an additional definition.

**Definition 2.2.** Let \(E\) be a Banach lattice with a non-empty quasi-interior \(Q_+\). A filtration \((T_i)\) on \(E\) is said to be quasi-interior preserving if \(T_iQ_+ \subset Q_+\) for each \(i \in \mathbb{N}\).

For background reading on quasi-interior points in Banach lattices, we refer the reader to [17] Chapter II, §6.

**Lemma 2.3.** Let \(E\) be a Banach lattice with a non-empty quasi-interior \(Q_+\). If \(T : E \to E\) is a positive projection, then \(TQ_+ \subset Q_+\) if and only if there exists a quasi-interior point \(0 < e \in E_+\) such that \(Te = e\).

**Proof.** Note that \(T \geq 0\) implies that \(T\) is bounded. Suppose there exists \(e \in Q_+\) such that \(Te = e\) and let \(q \in Q_+\). By [17] Chapter II, §6, Theorem 6.3, we have \(\lim_{n \to \infty} T(nq \wedge e) = T(e) = e\). Also, \(0 \leq T(nq \wedge e) \leq (nTq) \wedge e \leq e\) with \(T(nq \wedge e) \uparrow\). Hence, \(\lim_{n \to \infty} (nTq) \wedge e = e\). Now let \(p \in E_+\). Using [17] Chapter II, §6, Theorem 6.3] again, we obtain

\[
p = \lim_{m \to \infty} (me) \wedge p = \lim_{m \to \infty} (m \lim_{n \to \infty} (nTq) \wedge e) \wedge p = \lim_{m \to \infty} \lim_{n \to \infty} (m(nTq) \wedge e) \wedge p.
\]

As \((m(nTq) \wedge e) \wedge p \leq (mnTq) \wedge p \leq p\) for each \(n, m \in \mathbb{N}\), it follows that

\[
\lim_{m \to \infty} \lim_{n \to \infty} (mnTq) \wedge p = p.
\]

Thus, \(Tq \in Q_+\) by [17] Chapter II, §6, Theorem 6.3], i.e. \(TQ_+ \subset Q_+\). The converse is trivial. \(\square\)

The spaces \(L^p(\mu), 1 \leq p < \infty\), have non-empty quasi-interior (e.g., \(1\) is a quasi-interior point of \(L^p(\mu)\)) and if \((\Sigma_i)\) is an increasing sequence of \(\sigma\)-subalgebras of \(\Sigma\), then the sequence of conditional expectations \((\mathbb{E} \cdot | \Sigma_i))\) is a filtration with \(\mathbb{E} \cdot | \Sigma_i) = 1\) for each \(i \in \mathbb{N}\). By the above lemma, \((\mathbb{E} \cdot | \Sigma_i))\) is a filtration that is quasi-interior preserving.
Proposition 2.4. Suppose that $E$ is a Banach lattice possessing a non-empty quasi-interior $Q_+$ and $T : E \to E$ is a bounded linear operator. Then $TQ_+ \subset Q_+$ if and only if $T^* : E^* \to E^*$ is strictly positive.

Proof. By [17, Chapter II, §6, Theorem 6.3], $q \in Q_+$ if and only if $\langle q, f^* \rangle > 0$ for all $0 < f^* \in E^*$. With this in mind, assume that $TQ_+ \subset Q_+$. Since $Q_+$ is dense in $E_+$, it follows that $T \geq 0$, which implies $T^* \geq 0$. To show strict positivity, suppose that $T^*f^* = 0$ for some $f^* \in E^*_+$. Then $0 = \langle q, T^*f^* \rangle = \langle Tq, f^* \rangle$ for $q \in Q_+$. Since $Tq \in Q_+$, it follows that $f^* = 0$. Conversely, if $T^*$ is strictly positive, then for all $0 < f^* \in E^*$ we have $T^*f^* > 0$. Thus, for $q \in Q_+$, it follows that $\langle Tq, f^* \rangle = \langle q, T^*f^* \rangle > 0$ for all $f^* \in E^*$. Consequently, $Tq \in Q_+$.

Corollary 2.5. Suppose that $E$ is a Banach lattice possessing non-empty quasi-interior. Then for any quasi-interior preserving filtration $(T_i)$ on $E$, we have that $(T^*_i)$ is a BL-filtration on $E^*$.

The following convergence result was shown in [2].

Proposition 2.6. Let $Y$ be a Banach space and $(T_i)$ a filtration on $Y$. Then the following statements hold:

(a) $f \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ if and only if $\|T_i f - f\| \to 0$ as $i \to \infty$.

(b) If $(f_i)$ is a martingale relative to $(T_i)$, then $(f_i)$ converges to $f$ if and only if $f \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ and $f_i = T_i f$ for all $i \in \mathbb{N}$.

Consider the case where a filtration $(T_i)$ on a Banach lattice $E$ is complemented in $E$ by a contractive projection $T_\infty : E \to E$. Since each $T_i$ is assumed to be positive, it follows that $T_\infty$ is also positive. Indeed, if $f \in E_+$, then $T_i f \in E_+$ for each $i \in \mathbb{N}$. Thus, Proposition 2.6 implies $\lim_{i \to \infty} T_i f = \lim_{i \to \infty} T_\infty f = T_\infty f \geq 0$.

Definition 2.7. Let $Y$ be a Banach space and $(T_i)$ a filtration on $Y$.

(a) Define the space of norm bounded martingales as

$$
\mathcal{M}(Y, T_i) = \{(f_i, T_i) \text{ a martingale in } Y : \sup_i \|f_i\| < \infty\},
$$

with the norm defined by $\|(f_i, T_i)\| = \sup_i \|f_i\|$ for all $(f_i, T_i) \in \mathcal{M}(Y, T_i)$.

(b) Define the space of norm convergent martingales by

$$
\mathcal{M}_{nc}(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : (f_i) \text{ is norm convergent in } Y\}.
$$

(c) Define the space of fixed martingales by

$$
\mathcal{M}_f(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : \exists f \in Y \text{ so that } T_i f = f, \forall i \in \mathbb{N}\}.
$$

It is easily shown that $\mathcal{M}(Y, T_i)$ and $\mathcal{M}_{nc}(Y, T_i)$ are Banach spaces. By Proposition 2.6(b), we have the inclusions $\mathcal{M}_{nc}(Y, T_i) \subset \mathcal{M}_f(Y, T_i) \subset \mathcal{M}(Y, T_i)$. Note that if $(T_i)$ is a complemented filtration on $Y$, then $\mathcal{M}_{nc}(Y, T_i) = \mathcal{M}_f(Y, T_i)$. If $(f_i, T_i)$ is fixed, then it is also fixed on an element in $\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ and thus convergent by Proposition 2.6.

In the case where $Y$ is reflexive, Troitsky showed in [18, Corollary 18] that $\mathcal{M}_f(Y, T_i) = \mathcal{M}(Y, T_i)$. If $(T_i)$ is also complemented in $Y$, then $\mathcal{M}_{nc}(Y, T_i) = \mathcal{M}_f(Y, T_i) = \mathcal{M}(Y, T_i)$.

For a BL-filtration $(T_i)$ on a Banach lattice $E$, [2] Proposition 3.7 implies that $\mathcal{M}_{nc}(E, T_i)$ is a Banach lattice with respect to the ordering defined by $(f_i, T_i) \geq (g_i, T_i)$ if and only if $\forall i \in \mathbb{N}, f_i \geq g_i$.
Proposition 3.1. Let $0 \Leftrightarrow f_i \geq 0$ for all $i \in \mathbb{N}$. Moreover, it is shown that $\mathcal{M}_{nc}(E,T_i)$ is Riesz isometric to $\bigcup_{i=1}^{\infty} R(T_i)$.

In the case where $E$ is a KB-space and $(T_i)$ is a filtration on $E$, Troitsky showed in [18, Theorem 7] that $\mathcal{M}(E,T_i)$ is also a Banach lattice under this ordering. Moreover, Troitsky showed the following, which is key to our main result (cf. [18, p. 446]).

**Proposition 2.8.** Let $E$ be a KB-space and $(T_i)$ a filtration on $E$. If $(s_i,T_i)$ is a norm bounded submartingale in $E$, then there exists a unique least martingale $(f_i,T_i) \in \mathcal{M}(E,T_i)$ such that $s_i \leq f_i$ for each $i \in \mathbb{N}$. Moreover, $\|(f_i,T_i)\| \leq \|(s_i,T_i)\|$.

The notion of a filtration $(T_i)$ on a Banach lattice $E$ can be extended to the $l$-tensor product $E \otimes Y$ for any Banach space $Y$. Indeed, it follows from the proof of [2, Lemma 4.2] that $(T_i \otimes \text{id}_Y)$ is a sequence of commuting contractive projections on $E \otimes Y$ with increasing range. This extension is consistent with a classical filtration on the Lebesgue-Bochner spaces.

3. Cone absolutely summing martingales

To characterize the Radon Nikodým property, we require a fair amount of preparation. We first consider a canonical filtration on the space of cone absolutely summing operators from a Banach lattice $E$ to a Banach space $Y$.

**Proposition 3.1.** Let $E$ be a Banach lattice and $Y$ a Banach space. Suppose that $(T_i)$ is a BL-filtration on $E$. Then the sequence $(\hat{T}_i)$ of maps $\hat{T}_i : \mathcal{L}^{cas}(E,Y) \to \mathcal{L}^{cas}(E,Y)$, defined by $\hat{T}_i F = F \circ T_i$ for each $F \in \mathcal{L}^{cas}(E,Y)$ and $i \in \mathbb{N}$, is a sequence of contractive projections on $\mathcal{L}^{cas}(E,Y)$ with $\hat{T}_i \wedge j = \hat{T}_i \hat{T}_j$.

**Proof.** Since $(T_i)$ is a filtration, $F \circ T_i \in \mathcal{L}^{cas}(E,Y)$ and $\hat{T}_i$ is a well-defined, linear projection for each $i \in \mathbb{N}$. It also follows from

$$
\|\hat{T}_i F\|_{\text{cas}} = \sup \left\{ \sum_{j=1}^{n} \|F T_i x_j\| : (x_j)_{i=1}^{n} \subset E_+, \left\| \sum_{j=1}^{n} x_j \right\| \leq 1 \right\}
$$

$$
= \sup \left\{ \sum_{j=1}^{n} \| F x_j \| : (x_j)_{i=1}^{n} \subset \mathcal{R}(T_i)_+, \left\| \sum_{j=1}^{n} x_j \right\| \leq 1 \right\}
$$

$$
\leq \sup \left\{ \sum_{j=1}^{n} \| F x_j \| : (x_j)_{i=1}^{n} \subset E_+, \left\| \sum_{j=1}^{n} x_j \right\| \leq 1 \right\}
$$

$$
= \|F\|_{\text{cas}}
$$

that each $\hat{T}_i$ is bounded and $\sup_{i \in \mathbb{N}} \|\hat{T}_i\| = 1$. Moreover,

$$
\hat{T}_i \hat{T}_j F = F \circ T_j \circ T_i = F \circ T_i \wedge j = \hat{T}_i \wedge j F
$$

for each $F \in \mathcal{L}^{cas}(E,Y)$ and $i,j \in \mathbb{N}$. This completes the proof. \qed

In view of the above proposition, we are justified in making the following definition.
Thus, it follows by \cite[Chapter II, representation theorem for bounded operator \(T\)]{17} is called the filtration on \(L^{\text{cas}}(E,Y)\) induced by \((T_i)\).

We exhibit a known characterization of cone absolutely summing operators.

**Lemma 3.3.** Let \(E\) be a Banach lattice, \(Y\) a Banach space and \(l > 0\). For any bounded operator \(T : E \to Y\) the following statements are equivalent:

(a) \(T\) is cone absolutely summing with \(\|T\|_{\text{cas}} \leq l\).

(b) There exists \(x^{\ast}_T \in E^{\ast}_+\) so that \(\|x^{\ast}_T\| \leq l\) and \(\|Tx\| \leq \langle |x|, x^{\ast}_T \rangle\) for all \(x \in E\).

(c) There exist an \(AL\)-space \(L\), \(0 \leq T_1 \in \mathcal{L}(E,L)\) and \(T_2 \in \mathcal{L}(L,Y)\) such that \(T = T_2 \circ T_1\), where \(\|T_1\| \leq l\) and \(\|T_2\| \leq 1\).

In the case where \(E\) is separable, we may take \(L = L^1(\mu)\) in (c), where \((\Omega, \Sigma, \mu)\) is a finite measure space.

The proof of the equivalence of (a), (b) and (c) in the above lemma may be found in \cite[Chapter IV, §3, Proposition 3.3]{17}. However, the last part of the lemma requires a proof: Assume \(E\) is separable. Then, by \cite[Chapter II, §6, Proposition 6.2]{17}, there exists a quasi-interior point \(0 \leq e \in E\). By construction, the map \(T_1 : E \to L\) is a Riesz homomorphism with dense range (see \cite[p. 243, §3]{17}). Thus, it follows by \cite[Chapter II, §6, Proposition 6.4]{17} that \(T_1 e\) is a quasi-interior point of \(L\). Hence, \(T_1 e\) is also a weak order unit of \(L\). It follows by Kakutani’s representation theorem for \(AL\)-spaces (cf. \cite[Theorem 1.b.2]{12} or \cite[Theorem 1.b.2]{13}) that \(L\) is Riesz and isometrically isomorphic to \(L^1(\mu)\), where \((\Omega, \Sigma, \mu)\) may be chosen to be finite.

For our next result, we recall that the functional \(x^{\ast}_T \in E^{\ast}_+\) in Lemma 3.3(b) is the extension of the additive map \(\rho_T : E_+ \to E_+\), defined by

\[
\rho_T(x) = \sup \left\{ \sum_{i=1}^{\infty} \|Tx_i\| : (x_i) \in (\ell^1 \circ \iota) E_+^+, \sum_{i=1}^{\infty} x_i = x \right\}
\]

for each \(x \in E_+\) (cf. \cite[Chapter IV, §2, Theorem 2.7]{17}). Here, \(\ell^1 \circ \iota) E\) denotes the space of unconditionally summable sequences in \(E\).

**Proposition 3.4.** Let \(E\) be a Banach lattice with order continuous dual and \(Y\) a Banach space. Suppose that \((T_i)\) is a BL-filtration on \(E\) and \((\hat{T}_i)\) is the filtration on \(L^{\text{cas}}(E,Y)\) induced by \((T_i)\). If \((F_i, \hat{T}_i) \in \mathcal{M}(L^{\text{cas}}(E,Y), \hat{T}_i)\), then there exists \(0 \leq (f^{\ast}_i, T^{\ast}_i) \in \mathcal{M}(E^{\ast}, \hat{E}^{\ast}_+)\) such that \(\|F_i x\| \leq \langle |x|, f^{\ast}_i \rangle\) for each \(x \in E\) and \(i \in \mathbb{N}\).

Moreover, \(\sup_{i \in \mathbb{N}} \|f^{\ast}_i\| \leq \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}}\).

**Proof.** By Lemma 3.3 there exists, for each \(F_i\), a positive functional \(x^{\ast}_{F_i} \in E^{\ast}\) with \(\|x^{\ast}_{F_i}\| \leq \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}} := l\) and \(\|F_i x\| \leq \langle |x|, x^{\ast}_{F_i} \rangle\) for each \(x \in E\) and \(i \in \mathbb{N}\). Define \(s^{\ast}_i \in \hat{E}_+\) by \(\langle x, s^{\ast}_i \rangle = \langle T_i x, x^{\ast}_{F_i} \rangle\) for each \(x \in E\) and \(i \in \mathbb{N}\). Then, \(\sup_{i \in \mathbb{N}} \|s^{\ast}_i\| \leq l\) and, since \(x^{\ast}_{F_i} \geq 0\), we get

\[
\|F_i x\| = \|\hat{T}_i F_i x\| = \|F_i T_i x\| \leq \langle |T_i x|, x^{\ast}_{F_i} \rangle \leq \langle T_i |x|, x^{\ast}_{F_i} \rangle = \langle |x|, s^{\ast}_i \rangle
\]
for all $x \in E$ and $i \in \mathbb{N}$. We now show that $(s_i^*, T_i^*)$ is a submartingale. Let $i \leq j$ and $x \in E_+$. Then,

$$
\langle x, T_i^* s_j^* \rangle = \langle T_i x, s_j^* \rangle = \langle T_j T_i x, x_{F_j}^* \rangle = \langle T_i x, x_{F_j}^* \rangle
$$

$$
= \sup \left\{ \sum_{n=1}^{\infty} \|F_j n\| : (x_n) \in (l^1 \odot E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
$$

$$
\geq \sup \left\{ \sum_{n=1}^{\infty} \|F_j n\| : (x_n) \in (l^1 \odot E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
$$

as $(l^1 \odot \mathcal{R}(T_i))_+ \subset (l^1 \odot E)_+$. Since $T_i$ is a projection, it follows for $x \in E$ that $x \in \mathcal{R}(T_i)$ if and only if $x = T_i x$. In addition, the positivity of $T_i$ implies $(T_i x_n) \in (l^1 \odot \mathcal{R}(T_i))_+$ for all $(x_n) \in (l^1 \odot E)_+$. Consequently, for $(x_n) \in (l^1 \odot E)_+$, we have $(x_n) \in (l^1 \odot \mathcal{R}(T_i))_+$ if and only if $(x_n) = (T_i x_n)$. Thus,

$$
\sup \left\{ \sum_{n=1}^{\infty} \|F_j n\| : (x_n) \in (l^1 \odot \mathcal{R}(T_i))_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
$$

$$
= \sup \left\{ \sum_{n=1}^{\infty} \|F_j n\| : (x_n) \in (l^1 \odot E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
$$

$$
= \sup \left\{ \sum_{n=1}^{\infty} \|F_j n\| : (x_n) \in (l^1 \odot E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\}
$$

$$
= \langle T_i x, x_{F_i}^* \rangle = \langle x, s_i^* \rangle.
$$

Since $s_i^*(x) \leq T_i^* s_j^*(x)$ for all $x \in E_+$, it follows that $s_i^* \leq T_i^* s_j^*$. Consequently, $(s_i^*, T_i^*)$ is a submartingale. Since $E^*$ is order continuous, it follows that $E^*$ is a KB-space (cf. [3], Theorem 2.4.14). Thus, by Proposition 3.2 there exists a unique least martingale $0 \leq (f_i^*, T_i^*) \in \mathcal{M}(E^*, T_i^*)$ that dominates the submartingale $(s_i^*, T_i^*)$, with $\sup_{i \in \mathbb{N}} \|f_i^*\| \leq \sup_{i \in \mathbb{N}} \|s_i^*\| \leq l = \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}}$. Hence,

$$
\|F_i x\| \leq \langle |x|, s_i^* \rangle \leq \langle |x|, f_i^* \rangle
$$

for all $x \in E$, and the proof is complete. 

**Theorem 3.5.** Let $E$ be a Banach lattice with order continuous dual and $Y$ a Banach space. Suppose that $(T_i)$ is a BL-filtration on $E$ and $(\mathcal{T}_i)$ is the filtration on $\mathcal{L}^\text{cas}(E, Y)$ induced by $(T_i)$. Then $\mathcal{M}(\mathcal{L}^\text{cas}(E, Y), \mathcal{T}_i) = \mathcal{M}(\mathcal{L}^\text{cas}(E, Y), \mathcal{T}_i)$.

**Proof.** The inclusion $\mathcal{M}(\mathcal{L}^\text{cas}(E, Y), \mathcal{T}_i) \subset \mathcal{M}(\mathcal{L}^\text{cas}(E, Y), \mathcal{T}_i)$ is obvious. For the reverse inclusion, let $(F_i, \mathcal{T}_i) \in \mathcal{M}(\mathcal{L}^\text{cas}(E, Y), \mathcal{T}_i)$. By Proposition 3.3 there exists $0 \leq (f_i^*, T_i^*) \in \mathcal{M}(E^*, T_i^*)$ such that $\sup_{i \in \mathbb{N}} \|f_i^*\| \leq \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}}$ and $\|F_i x\| \leq \langle |x|, f_i^* \rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Let $\sup_{i \in \mathbb{N}} \|f_i^*\| := K$ and define $f^* : \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \to \mathbb{R}$ by $\langle x, f^* \rangle = \lim_{i \to \infty} \langle x, f_i^* \rangle$ for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. Observe that $f^*$ is well defined, as for $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ there exists $i \in \mathbb{N}$ such that $x \in \mathcal{R}(T_i)$. Consequently, $i \leq j$ implies $(x, f_i^*) = (x, T_i^* f_j^*) = (T_i x, f_j^*) = (x, f_j^*)$. Thus,

$$
\langle x, f^* \rangle = \lim_{i \to \infty} \langle x, f_i^* \rangle \text{ exists for each } x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i).
$$

Evidently, $f^*$ is positive, linear and the inequality $\langle x, f^* \rangle = \lim_{i \to \infty} \|x, f_i^*\| \leq \lim_{i \to \infty} \|f_i^* \| \|x\| = K \|x\|$ shows that $f^*$ is also bounded with norm $\|f^*\| \leq K$. 

Now define a map $F : \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \to Y$ by $Fx = \lim_{i \to \infty} F_i x$ for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. The map $F$ is well defined because, for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$, there is some $i \in \mathbb{N}$ for which $x \in \mathcal{R}(T_i)$. Thus, $i \leq j$ implies $F_i x = \tilde{T}_i F_j x = F_i T_i x = F_j x$ so that $Fx = \lim_{i \to \infty} F_i x$ exists for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. It is now evident that $F$ is linear. Moreover, since $\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ is a Riesz subspace of $E$, we have
\[
\|Fx\| = \lim_{i \to \infty} \|F_i x\| \leq \lim_{i \to \infty} \langle |x|, f_i^* \rangle = \langle |x|, f^* \rangle \leq K \|x\|
\]
for all $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. Thus, $F$ is bounded. Let $\overline{F}$ and $\hat{F}$ denote the unique continuous extensions of $f^*$ and $F$ respectively to the Banach sublattice $\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ of $E$. Then we have $\|\overline{F}x\| \leq \langle |x|, \overline{F} \rangle$ for all $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. Consequently, $(\overline{F}, \hat{T}_i)$ possesses an extension $\hat{F}_\infty \in \mathcal{L}^{\text{cas}}(E,Y)$ with $\|\hat{F}_\infty\|_{\text{cas}} = \|\hat{F}_\infty\|_{\text{cas}}$. Finally, $\hat{T}_i \hat{F}_\infty x = F_T i x = \lim_{i \to \infty} F_i T_i x = \lim_{j \to \infty} F_j T_j x = F_j x$ for all $x \in E$ and $i \in \mathbb{N}$. Thus, $(F_i, \hat{T}_i) \in \mathcal{M}_1(\mathcal{L}^{\text{cas}}(E,Y), \hat{T}_i)$. \hfill \Box

We continue our preparations with the next lemma, which is a simple restatement of well-known facts about order continuity of the norm in dual Banach lattices.

**Lemma 3.6.** Let $E$ be a Banach lattice such that $E^*$ has order continuous norm. If $T : E \to \ell^1$ is a positive linear operator, then $T$ is compact.

**Proof.** Let $T : E \to \ell^1$ be a positive operator. Denote the restriction of $T^*$ to $c_0$ by $T^*|_{c_0}$. Then $T^*|_{c_0} : c_0 \to E^*$ is positive. But $E^*$ is a KB-space by [19, Theorem 2.4.14]; thus, $T^*|_{c_0}$ is weakly compact (cf. [17, Chapter II, §3, Proposition 5.15]). Consequently, $(T^*|_{c_0})^* : E^{**} \to \ell^1$ is compact because $\ell^1$ has the Schur property. Hence, $T = (T^*|_{c_0})^*|_E$ is compact. \hfill \Box

Lastly, we need the following characterization of the $l$-tensor product, which is shown in [13, Theorem 5.2].

**Theorem 3.7.** Let $E$ be a Banach lattice, $Y$ a Banach space and $T \in \mathcal{L}(E,Y)$. Then $T \in E^* \hat{\otimes} Y$ if and only if there exist $0 \leq S \in \mathcal{L}(E,\ell^1)$ and $R \in \mathcal{L}(\ell^1,Y)$ such that $S$ is compact and $T = R \circ S$. Further, $\|T\|_{\text{cas}} = \inf \|R\| \|S\|_r$, where the infimum is taken over all such factorizations of $T$.

We are now prepared to characterize the Radon Nikodym property:

**Theorem 3.8.** Let $Y$ be a Banach space. Then the following statements are equivalent:

(a) $Y$ has the Radon Nikodym property.

(b) $E^* \hat{\otimes} Y = \mathcal{L}^{\text{cas}}(E,Y)$ for all separable Banach lattices $E$ with order continuous dual.

(c) $\mathcal{M}(E^* \hat{\otimes} Y, T_i^* \hatotimes \text{id}_Y) = \mathcal{M}(E^* \hat{\otimes} Y, T_i^* \hatotimes \text{id}_Y)$ for all separable Banach lattices $E$ with order continuous dual and all BL-filtrations $(T_i)$ on $E$.

(d) $\mathcal{M}(E \hat{\otimes} Y, T_i \hatotimes \text{id}_Y) = \mathcal{M}(E \hat{\otimes} Y, T_i \hatotimes \text{id}_Y)$ for all separable reflexive Banach lattices $E$ and all complemented, quasi-interior preserving BL-filtrations $(T_i)$ on $E$.

(e) $\mathcal{M}(E \hat{\otimes} Y, T_i \hatotimes \text{id}_Y) = \mathcal{M}(E, T_i) \hat{\otimes} Y$ for all separable reflexive Banach lattices $E$ and all complemented, quasi-interior preserving BL-filtrations $(T_i)$ on $E$. 

proof: (a)$\Rightarrow$ (b) Let $E$ be a separable Banach lattice with order continuous dual. Let $T \in \mathcal{L}^\text{cos}(E, Y)$. By Lemma 3.3 there exist a finite measure space $(\Omega, \Sigma, \mu)$ and operators $0 \leq T_1 \in \mathcal{L}(E, L^1(\mu))$ and $T_2 \in \mathcal{L}(L^1(\mu), Y)$ such that $T = T_2 \circ T_1$ where $\|T_1\| \leq \|T\|_\text{cos}$ and $\|T_2\| \leq 1$. Since $Y$ has the Radon Nikodým property, the Lewis-Stegall Theorem (cf. [11], Chapter III, §1, Theorem 8) guarantees the existence of operators $0 \leq S_1 \in \mathcal{L}(L^1(\mu), \ell^1)$ and $S_2 \in \mathcal{L}(\ell^1, Y)$ such that $T_2 = S_2 \circ S_1$. Since $E^*$ is order continuous, the positive operator $S_1 \circ T_1 : E \to \ell^1$ is compact by Lemma 3.4. Hence, $T \in E^* \hat{\otimes} Y$ by Theorem 3.7.

(b)$\Rightarrow$(c) Suppose $E$ is a separable Banach lattice with order continuous dual and $(T_i)$ is a filtration on $E$. Let $(f_i, T_i^* \otimes \text{id}_Y) \in \mathcal{M}(E^* \hat{\otimes} Y, T_i^* \otimes \text{id}_Y)$. By (b), $E^* \hat{\otimes} Y$ is isometric to $\mathcal{L}^\text{cos}(E, Y)$ under the continuous extension of the canonical isometry $E^* \otimes_1 Y \to \mathcal{L}^\text{cos}(E, Y)$, given by $u \mapsto L_u$, where $L_u = \sum_{i=1}^n f_i^* \otimes y_i$ for $u = \sum_{i=1}^n x_i^* \otimes y_i$. Let $(F_i) \subset \mathcal{L}^\text{cos}(E, Y)$ be the sequence corresponding to the martingale $(f_i, T_i^* \otimes \text{id}_Y)$. Observe that, for $u = \sum_{k=1}^n x_k \otimes y_k \in E^* \otimes Y$, we have

$$(T_i^* \otimes \text{id}_Y)u \mapsto \sum_{i=1}^n \langle \cdot, T_i^* x_i^* \rangle y_i = L_u \circ T_i = \hat{T}_i L_u$$

for all $i \in \mathbb{N}$. It follows that $(F_i, \hat{T}_i) \in \mathcal{M}(\mathcal{L}^\text{cos}(E, Y), \hat{T}_i)$. By Theorem 3.5 there exists $F_\infty \in \mathcal{L}^\text{cos}(E, Y)$ such that $\hat{T}_i F_\infty = F_i$ for each $i \in \mathbb{N}$. Consequently, $(f_i, T_i^* \otimes \text{id}_Y) \in \mathcal{M}(E^* \hat{\otimes} Y, T_i^* \otimes \text{id}_Y)$.

(c)$\Rightarrow$(d) Since $E$ is a separable reflexive Banach lattice, $E$ has non-empty quasi-interior, $E^{**}$ is order continuous and $E^*$ is separable (cf. [17], Chapter II, §5, Theorem 5.16]). Moreover, by Corollary 2.5 $(T_i^*)$ is a BL-filtration on $E^*$. Consequently, by (c) and [13, Theorem 6.1], we have

$$\mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y) = \mathcal{M}(E^{**} \hat{\otimes} Y, T_i^{**} \otimes \text{id}_Y) = \mathcal{M}(E^{**} \hat{\otimes} Y, T_i^{**} \otimes \text{id}_Y)$$

Since the BL-filtration $(T_i)$ is complemented by a (positive) contractive projection $T_\infty : E \to E$, we have by [2] Lemma 4.2 and Lemma 5.1 that $\mathcal{R}(T_\infty \otimes \text{id}_Y) = \mathcal{R}(T_\infty) \hat{\otimes} Y = \bigcup_{i=1}^\infty \mathcal{R}(T_i) \hat{\otimes} Y = \bigcup_{i=1}^\infty \mathcal{R}(T_i \otimes \text{id}_Y)$. It now follows by a continuity argument that $(T_i \otimes \text{id}_Y)$ is a filtration on $E \hat{\otimes} Y$ complemented by $T_\infty \otimes \text{id}_Y$. Thus, $\mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y) = \mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y)$, as required.

(d)$\Rightarrow$(a) For all finite measure spaces $(\Omega, \Sigma, \mu)$ and $1 < p < \infty$, the Banach lattice $L^p(\mu)$ is separable and reflexive. By (d), it follows that $\mathcal{M}(L^p(\mu), \Sigma_i) = \mathcal{M}(L^p(\mu), \Sigma_i)$ for every filtration $(\Sigma_i)$. Thus, $Y$ has the Radon Nikodym property by [11, Theorem II.2.2].

(d)$\Rightarrow$(e) Suppose $\mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y) = \mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y)$. Since $(T_i)$ is a BL-filtration, it follows that $\mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y) = \mathcal{M}(E, T_i) \hat{\otimes} Y$ by [2, Corollary 5.2]. Since $E$ is reflexive, $\mathcal{M}(E, T_i)$ is Riesz and isometrically isomorphic to $\mathcal{M}(E, T_j)$. Thus, $\mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y) = \mathcal{M}(E, T_i) \hat{\otimes} Y = \mathcal{M}(E, T_j) \hat{\otimes} Y$ by [14, Theorem 6.1]. Conversely, using [2, Corollary 5.2] and [14, Theorem 6.1] again, we obtain $\mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y) = \mathcal{M}(E, T_i) \hat{\otimes} Y = \mathcal{M}(E, T_j) \hat{\otimes} Y = \mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y)$, as required.

$\square$
The above theorem allows us to generalize [5 Chapter IV, §1, Theorem 1], which characterizes the “Asplund spaces”. A Banach space $Y$ is called an Asplund space if $Y^*$ has the Radon Nikodým property.

**Corollary 3.9.** Let $Y$ be a Banach space. Then $Y$ is an Asplund space if and only if $E^{*} \hat{\otimes} Y^* = (E \hat{\otimes} Y)^*$ for all separable Banach lattices $E$ with order continuous dual.

**Proof.** By Theorem 3.8, $Y^*$ has the Radon Nikodým property if and only if $E^{*} \hat{\otimes} Y^* = \mathcal{L}^{\text{cas}}(E, Y^*)$ for all separable Banach lattices $E$ with order continuous dual. But, by a theorem of Jacobs, we have $\mathcal{L}^{\text{cas}}(E, Y^*) = (E \hat{\otimes} Y)^*$ (cf. [10, 17 Chapter IV, §7, Theorem 7.4] and [3]). Thus, $E^{*} \hat{\otimes} Y^* = \mathcal{L}^{\text{cas}}(E, Y^*) = (E \hat{\otimes} Y)^*$, which completes the proof.

It is important to note that the above theorem does not include the case $E = L^1(\mu)$. However, by [5, Chapter IV, §1, Theorem 1], $Y$ is an Asplund space if and only if $L^1(\mu, Y)^* = L^\infty(\mu, Y^*)$ for all finite measure spaces $(\Omega, \Sigma, \mu)$.

Combining Theorem 3.8 with [2, Theorem 5.3] yields another corollary:

**Corollary 3.10.** Let $Y$ be a Banach space. Then the following conditions are equivalent:

(a) $Y$ has the Radon Nikodým property.

(b) For every separable reflexive Banach lattice $E$ and every complemented, quasi-interior preserving BL-filtration $(T_i)$ on $E$, we have $(f_n) \in \mathcal{M}(E \hat{\otimes} Y, T_i \otimes \text{id}_Y)$ if and only if for each $i \in \mathbb{N}$, there exist $(x_i^{(n)}, T_n)_{n=1}^{\infty} \in \mathcal{M}_{\text{nc}}(E, T_i)$ and $y_i \in Y$ such that, for each $n \in \mathbb{N}$, we have $f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i$, where $\| \sum_{i=1}^{\infty} \lim_{n \to \infty} x_i^{(n)} \| < \infty$ and $\lim_{i \to \infty} \| y_i \| = 0$.

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