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ON A THEOREM OF BRAUER AND WIELANDT

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Abstract. A classical theorem of Brauer and Wielandt which states the result under the assumption $G = [G, G]$ is extended to all finite groups.

Let G be a finite group and C_1, C_2, …, C_s be the conjugacy classes of G. For a subset S of G, define

$$\bar{S} = \sum_{s \in S} s \in \mathbb{C}G,$$

where $\mathbb{C}G$ is the group ring of G over the complex number field $\mathbb{C}$. On page 37 of W. Feit [Characters of Finite Groups, Benjamin, New York, 1967]), one finds a result. Feit writes on it: "The following striking consequence of (6.9)[1] was discovered by Brauer[2] and Wielandt. However, it is a result that seems to be difficult to use."

Theorem (Brauer and Wielandt)**.** *$\bar{C}_1\bar{C}_2 \cdots \bar{C}_s = c\bar{G}$ for some $c \in \mathbb{C}$ if and only if $G = [G, G] = G'$.*

Let me add a corollary to this theorem.

Corollary. *If $G = G'$, then $C_1C_2 \cdots C_s = G$.*

For the proof of it, just observe that the c in the theorem cannot be equal to 0 and every element of G has an expression in the set $C_1C_2 \cdots C_s$ at least once. The converse of this corollary also holds (see Corollary 5) but it does not seem to follow immediately from the theorem of Brauer and Wielandt (see "Acknowledgements" for the referee's comment).

I do not intend to argue with Feit on the latter part of the quotation given above, but I will show that a more general result can be proved. That is:

Corollary 3. *For every finite group G, there exists $c \in \mathbb{C}$ such that $\bar{C}_1\bar{C}_2 \cdots \bar{C}_s = c\,\overline{\alpha G'}$.*

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[1] A theorem of Burnside quoted later in this note.

[2] Proc. International Congress 1954, Vol. 1, pp. 1-9. The name of Wielandt is mentioned only in a footnote.

Here $x_j \in C_j$ and $\alpha = x_1 x_2 \cdots x_s$. First of all, the constant c in the theorem of Brauer and Wielandt can easily be computed and is[3]

$$c = \frac{|C_1||C_2| \cdots |C_s|}{|G|}.$$

The reason for this is that the product $\bar{C}_1 \bar{C}_2 \cdots \bar{C}_s$ is a positive linear combination of elements of G and the sum of its coefficients is $|C_1||C_2| \cdots |C_s|$. If it is a constant multiple of $\bar{G}$, then the c must be equal to the value given above.

With this c in mind, we can rewrite the relation of Brauer and Wielandt as follows:

$$\frac{\bar{C}_1}{|C_1|} \frac{\bar{C}_2}{|C_2|} \cdots \frac{\bar{C}_s}{|C_s|} = \frac{\bar{G}}{|G|}.$$

Therefore, in this form, the constant c is reduced to 1. Moreover, if we define, for a subset S of G, the *normalized* sum:

$$\hat{S} = \frac{\bar{S}}{|S|},$$

then the result of Brauer and Wielandt can be restated as

$$\hat{C}_1 \hat{C}_2 \cdots \hat{C}_s = \hat{G} \Longleftrightarrow G = G'.$$

For an arbitrary finite group G, define:

$$\hat{\Omega}_G = \hat{C}_1 \hat{C}_2 \cdots \hat{C}_s,$$
$$\alpha = x_1 x_2 \cdots x_s,$$

where x_i is an (arbitrary but fixed once chosen) representative from the conjugacy class C_i for $i = 1, 2, \ldots, s$. Note that there are many choices for the element α, but the coset $\alpha G'$ is uniquely determined. We will show in Theorem 2 (the main result of this paper) that $\hat{\Omega}_G = \alpha \hat{G}'$ for every finite group G. In the original notation of the result of Brauer and Wielandt, this implies that $\bar{C}_1 \bar{C}_2 \cdots \bar{C}_s$ is proportional to $\overline{\alpha G'}$ for every finite group G (Corollary 3).

Examples. (1). Suppose G is abelian. Then every conjugacy class C_i is a singleton. Therefore

$$\hat{\Omega}_G = \prod_{x \in G} x.$$

Rearranging the product by combining $\{x, x^{-1}\}$, we obtain

$$\hat{\Omega}_G = \prod_{t \in \mathrm{Inv}(G)} t,$$

where $\mathrm{Inv}(G)$ is the set of all involutions of G. As is easily seen in the case of a four group and by induction in the general cases, $\prod_{t \in \mathrm{Inv}(G)} t = 1$ if the Sylow 2-subgroup of G is not cyclic and $\prod_{t \in \mathrm{Inv}(G)} t = t_0$, where t_0 is the unique involution of G if the Sylow 2-subgroup of G is (nontrivial) cyclic.

(2). $\hat{\Omega}_{S_4} = \hat{A}_4$, $\hat{\Omega}_{S_5} = \widehat{S_5 \backslash A_5}$ and $\hat{\Omega}_{S_8} = \hat{A}_8$. In general, $\hat{\Omega}_{S_n} = \hat{A}_n$ if the number of odd partitions of n is even and $\hat{\Omega}_{S_n} = \widehat{S_n \backslash A_n}$ in the contrary cases. This is proved in the Appendix.

[3] Brauer's paper in footnote 2 actually has this constant, although Feit did not mention it in his book.

Define $\mathrm{Irr}(G) = \{\chi_1 = 1, \chi_2, \ldots, \chi_s\}$, the set of all irreducible characters of G over $\mathbb{C}$ and let us compute the quantity $\hat{\Omega}_G$ for an arbitrary finite group G. Note that the element $\hat{G}$ is an idempotent of the center $Z(R)$ of the group ring $R = \mathbb{C}G$. $\hat{\Omega}_G$ is an element in $Z(R)$ also. Recall that $\dim(Z(R)) = s$ and there are exactly s algebra homomorphisms from $Z(R)$ to $\mathbb{C}$, each corresponding to an irreducible character of G. Let us denote those homomorphisms by $\omega_1, \omega_2, \ldots, \omega_s$ with ω_i corresponding to the irreducible character χ_i of G for $i = 1, 2, \ldots, s$. It is well known that

$$\omega_i(\bar{C}_j) = \frac{|C_j|\chi_i(x_j)}{\chi_i(1)},$$

and so

$$\omega_i(\hat{C}_j) = \frac{\chi_i(x_j)}{\chi_i(1)}.$$

Let $\{e_i \mid i = 1, 2, \ldots, s\}$ be the set of all primitive idemponents of $Z(R)$. Then

$$Z(R) = \bigoplus_{i=1}^{s} \mathbb{C}e_i.$$

Moreover, we have

$$\omega_i(e_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Write $\hat{\Omega}_G$ using a basis $\{e_1, e_2, \ldots, e_s\}$ of $Z(R)$:

$$\hat{\Omega}_G = \sum_{i=1}^{s} a_i e_i, \ a_i \in \mathbb{C}.$$

Applying ω_i to both sides of the equality above, we obtain

$$\omega_i(\hat{\Omega}_G) = \prod_{j=1}^{s} \frac{\chi_i(x_j)}{\chi_i(1)} = a_i.$$

A theorem of Burnside states that if $\deg(\chi_i) \geq 2$, then there exists j such that $\chi_i(x_j) = 0$. Therefore $\omega_i(\hat{\Omega}_G) = a_i = 0$ in this case. On the other hand, if $\deg(\chi_i) = 1$, then

$$a_i = \chi_i\left(\prod_{j=1}^{s} x_j\right) = \chi_i(\alpha).$$

This implies $a_i \neq 0$ if and only if $\deg(\chi_i) = 1$. Rearrange the indices i so that $\{\chi_1 = 1, \chi_2, \ldots, \chi_r\}$ is the set of all irreducible characters of degree 1.

Theorem 1. *If $\alpha \in G'$, then $\hat{\Omega}_G$ is the sum of the primitive idempotents of $Z(R)$ corresponding to the irreducible characters of degree* 1:

$$\hat{\Omega}_G = \sum_{i=1}^{r} e_i.$$

Proof. If $\alpha \in G'$, then all coefficients a_i, for $i = 1, 2, \ldots, r$, are equal to 1. Hence the result follows. □

The formula for the idempotent e_i is well known and given by

$$e_i = \frac{\chi_i(1)}{|G|} \sum_{x \in G} \chi_i(x^{-1})x.$$

If χ_i is taken to be the trivial character $\chi_1 = 1$, then $a_1 = 1$ and

$$e_1 = \frac{1}{|G|} \sum_{x \in G} x = \hat{G}.$$

Therefore,

$$\hat{\Omega}_G = \hat{G} + \sum\nolimits_{i=2}^{r} a_i e_i.$$

The result of Brauer and Wielandt can now be obtained easily. If $G = G'$, then G possesses no nontrivial characters of degree 1 and so $\hat{\Omega}_G = \hat{G}$. On the other hand, if G possesses a nontrivial character χ_i of degree 1, then as shown above $a_i \neq 0$ and so $\hat{\Omega}_G \neq \hat{G}$. This proves the result of Brauer and Wielandt.

Let us study $\hat{\Omega}_G$ a little more. We have the following result.

Theorem 2. $\hat{\Omega}_G = \alpha \hat{G}'$.

Proof. As stated above, we have in general

$$\hat{\Omega}_G = \sum\nolimits_{i=1}^{r} a_i e_i, \ a_i = \chi_i(\alpha).$$

Moreover,

$$e_i = \frac{1}{|G|} \sum_{x \in G} \chi_i(x^{-1})x.$$

Therefore,

$$\hat{\Omega}_G = \frac{1}{|G|} \sum\nolimits_{i=1}^{r} \left(\sum_{x \in G} \chi_i(\alpha x^{-1}) \right) x.$$

Note again that χ_i is of degree 1 for $i = 1, 2, \ldots, r$ and so $\chi_i(\alpha)\chi_i(x^{-1}) = \chi_i(\alpha x^{-1})$. Replacing αx^{-1} by y^{-1}, this can be rewritten as

$$\hat{\Omega}_G = \frac{1}{|G|} \sum_{y \in G} \left(\sum\nolimits_{i=1}^{r} \chi_i(y^{-1}) \right) y\alpha.$$

Note that the characters of G of degree 1 can be identified with the characters of the abelian group G/G'. Therefore, the orthogonality relations of characters of G/G' implies: if $y \notin G'$, then $\sum_{i=1}^{r} \chi_i(y^{-1}) = 0$ and $\sum_{i=1}^{r} \chi_i(y^{-1}) = [G : G']$ if $y \in G'$. We now conclude

$$\hat{\Omega}_G = \frac{1}{|G|} \sum_{y \in G'} [G : G'] y\alpha = \frac{1}{|G|} [G : G'] \left(\sum_{y \in G'} y \right) \alpha = \alpha \hat{G}'.$$

This completes the proof of Theorem 2. □

Theorem 2 (the following corollary also) is a generalization of the result of Brauer and Wielandt. In fact, if $G = G'$, then $\alpha \in G'$ and so $\hat{\Omega}_G = \alpha \hat{G}' = \hat{G}$. On the other hand, if $\hat{\Omega}_G = \hat{G}$, then $\alpha \hat{G}' = \hat{G}$. Hence $\alpha G' = G$ as sets. This implies $G = G'$.

Corollary 3. *For every finite group G, there exists $c \in \mathbb{C}$ such that $\bar{C}_1 \bar{C}_2 \cdots \bar{C}_s = c\,\overline{\alpha G'}$. ($c = |C_1||C_2| \cdots |C_s|/|G'|$.)*

Proof. This is a trivial consequence of Theorem 2 since $\alpha \bar{G}' = \overline{\alpha G'}$. □

Corollary 4. *For every finite group G, it follows that $C_1C_2\cdots C_s = \alpha G'$. In particular, if $\alpha \in G'$, then $C_1C_2\cdots C_s = G'$.*

Proof. Considering subsets of G in lieu of sums of group elements in $\mathbb{C}G$, we see that this is an easy consequence of Corollary 3. □

We can now prove the converse of the corollary to the result of Brauer and Wielandt.

Corollary 5. *If $C_1C_2\cdots C_s = G$, then $G = G'$.*

Proof. Noting that $\alpha G'$ is a single coset of G', we see that $\alpha G' = G$ implies $G = G'$. □

Results such as these last two corollaries will not be easy to prove by a pure group-theoretical argument.

Lemma 6. $\alpha^2 \in G'$.

Proof. By definition $\alpha = x_1x_2\cdots x_s$. Write $\bar{G} = G/G'$ (by abuse of notation). So $\bar{\alpha} = \bar{x}_1\bar{x}_2\cdots\bar{x}_s$. Note that the coset $\bar{x}_j = x_jG'$ is a union of conjugacy classes of G. If $\bar{x}_j \neq \bar{x}_j^{-1}$, then the cosets x_jG' and $x_j^{-1}G'$ are disjoint. Therefore, the corresponding products in $\bar{\alpha}$ reduce to $\bar{1}$, and so in the expression of $\bar{\alpha}$ only involutions of G/G' can actually appear nontrivially. Hence the lemma follows. □

Corollary 7. *If G/G' is of odd order, then $\alpha \in G'$ and $\hat{\Omega}_G = \hat{G}'$.*

Proof. Since $\alpha^2 \in G'$ by Lemma 6, we have $\alpha \in G'$ under the condition of this lemma. The corollary now follows from Theorem 2. □

Corollary 8. *The order of G is odd if and only if $\hat{\Omega}_H = \hat{H}'$ for every subgroup H of G.*

Proof. If $|G|$ is odd, the assertion follows from Corollary 7. If $|G|$ is even, then G contains a subgroup H of order 2. It is trivial that $\hat{\Omega}_H = h \neq \hat{H}'$ where h is the unique involution of H. This proves the corollary. □

Appendix. Earlier we stated $\hat{\Omega}_{S_n} = \hat{A}_n$ if there is an even number of odd partitions of n and $\hat{\Omega}_{S_n} = \widehat{S_n \backslash A_n}$ in the contrary cases. To prove this result, we only need to determine whether $\alpha \in A_n$. Hence the result.

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