A BANACH-STONE THEOREM FOR RIESZ ISOMORPHISMS OF BANACH LATTICES

JIN XI CHEN, ZI LI CHEN, AND NGAI-CHING WONG

(Communicated by N. Tomczak-Jaegermann)

Abstract. Let $X$ and $Y$ be compact Hausdorff spaces, and $E$, $F$ be Banach lattices. Let $C(X,E)$ denote the Banach lattice of all continuous $E$-valued functions on $X$ equipped with the pointwise ordering and the sup norm. We prove that if there exists a Riesz isomorphism $\Phi : C(X,E) \to C(Y,F)$ such that $\Phi f$ is non-vanishing on $Y$ if and only if $f$ is non-vanishing on $X$, then $X$ is homeomorphic to $Y$, and $E$ is Riesz isomorphic to $F$. In this case, $\Phi$ can be written as a weighted composition operator: $\Phi f(y) = \Pi(y)(\varphi(y)f(\varphi(y)))$, where $\varphi$ is a homeomorphism from $Y$ onto $X$, and $\Pi(y)$ is a Riesz isomorphism from $E$ onto $F$ for every $y$ in $Y$. This generalizes some known results obtained recently.

1. Introduction

Let $X$ and $Y$ be compact Hausdorff spaces, and $C(X)$, $C(Y)$ denote the spaces of real-valued continuous functions defined on $X$, $Y$ respectively. There are three versions of the Banach-Stone theorem. That is to say, surjective linear isometries, ring isomorphisms and lattice isomorphisms from $C(X)$ onto $C(Y)$ yield homeomorphisms between $X$ and $Y$, respectively (cf. [1, 6, 14]).

Jerison [13] got the first vector-valued version of the Banach-Stone theorem. He proved that if the Banach space $E$ is strictly convex, then every surjective linear isometry $\Phi : C(X,E) \to C(Y,E)$ can be written as a weighted composition operator

$$\Phi f(y) = \Pi(y)(f(\varphi(y))), \quad \forall f \in C(X,E), \forall y \in Y.$$  

Here $\varphi$ is a homeomorphism from $Y$ onto $X$, and $\Pi$ is a continuous map from $Y$ into the space $(L(E,E), SOT)$ of bounded linear operators on $E$ equipped with the strong operator topology (SOT). Furthermore, $\Pi(y)$ is a surjective linear isometry on $E$ for every $y$ in $Y$. After Jerison [13], many vector-valued versions of the Banach-Stone theorem have been obtained in different ways (see, e.g., [3, 4, 5, 7, 9, 10, 12, 15]).

Let $E$, $F$ be non-zero real Banach lattices, and $C(X,E)$ be the Banach lattice of all continuous $E$-valued functions on $X$ equipped with the pointwise ordering and the sup norm. Note that, in general, a Riesz isomorphism (i.e., lattice isomorphism) from $C(X,E)$ onto $C(Y,F)$ does not necessarily induce a topological
homeomorphism from \( X \) onto \( Y \) (cf. [16] Example 3.5)). To consider the Banach-Stone theorems for continuous Banach lattice-valued functions, we would like to mention the papers [3, 7, 16]. In particular, when \( E, F \) are both Banach lattices and Riesz algebras, Miao, Cao and Xiong [16] recently proved that if \( F \) has no zero-divisor and there exists a Riesz algebraic isomorphism \( \Phi : C(X, E) \rightarrow C(Y, F) \) such that \( \Phi f \) is non-vanishing on \( Y \) if \( f \) is non-vanishing on \( X \), then \( X \) is homeomorphic to \( Y \), and \( E \) is Riesz algebraically isomorphic to \( F \). By saying \( f \) in \( C(X, E) \) is non-vanishing, we mean that \( 0 \notin f(X) \). Indeed, under these conditions they obtained that \( \Phi^{-1}g \) is non-vanishing on \( X \) if \( g \in C(Y, F) \) is non-vanishing on \( Y \). Note that every Riesz algebraic isomorphism must be a Riesz isomorphism.

Let \( E \) and \( F \) be Banach lattices. More recently, Ercan and Önal [7] have established that if \( F \) is an AM-space with unit, i.e., a \( C(K) \)-space, and there exists a Riesz isomorphism \( \Phi : C(X, E) \rightarrow C(Y, F) \) such that \( \Phi f \) is non-vanishing on \( Y \) if and only if \( f \) is non-vanishing on \( X \), then \( X \) is homeomorphic to \( Y \) and \( E \) is Riesz isomorphic to \( F \).

Inspired by [5, 7, 16], one can ask a natural question:

**Question 1.** Is \( X \) homeomorphic to \( Y \) if \( E, F \) are Banach lattices and there exists a Riesz isomorphism \( \Phi : C(X, E) \rightarrow C(Y, F) \) such that both \( \Phi \) and \( \Phi^{-1} \) are non-vanishing preserving?

In this paper we show the answer to the above question is affirmative. Moreover, in this case \( \Phi \) can be written as a weighted composition operator:

\[
\Phi f(y) = \Pi(y)(f(\varphi(y))), \quad \forall f \in C(X, E), \forall y \in Y,
\]

where \( \varphi \) is a homeomorphism from \( X \) onto \( Y \), and \( \Pi(y) \) is a Riesz isomorphism from \( E \) onto \( F \) for every \( y \) in \( Y \). This generalizes the results obtained by Cao, Reilly and Xiong [9], Miao, Cao, and Xiong [16], and Ercan and Önal [7].

Our notions are standard. For the undefined notions and basic facts concerning Banach lattices we refer the reader to the monographs [1, 2, 14].

## 2. A Banach-Stone Theorem for Riesz Isomorphisms

In the following we always assume \( X \) and \( Y \) are compact Hausdorff spaces, \( E \) and \( F \) are non-zero Banach lattices, and \( L(E, F) \) is the space of bounded linear operators from \( E \) into \( F \) equipped with SOT. For \( x \) in \( X \) and \( y \) in \( Y \), let \( M_x \) and \( N_y \) be defined as

\[
M_x = \{ f \in C(X, E) : f(x) = 0 \}, \quad N_y = \{ g \in C(Y, F) : g(y) = 0 \}.
\]

Clearly, \( M_x \) and \( N_y \) are closed (order) ideals in \( C(X, E) \) and \( C(Y, F) \), respectively.

**Lemma 2.** Let \( \Phi : C(X, E) \rightarrow C(Y, F) \) be a Riesz isomorphism such that \( \Phi(f) \) is non-vanishing on \( Y \) if and only if \( f \) is non-vanishing on \( X \). Then for each \( x \) in \( X \) there exists a unique \( y \) in \( Y \) such that

\[
\Phi M_x = N_y.
\]

In particular, this defines a bijection \( \varphi \) from \( Y \) onto \( X \) by \( \varphi(y) = x \).

**Proof.** For each \( x \) in \( X \), let

\[
Z(\Phi M_x) = \{ y \in Y : \Phi f(y) = 0 \text{ for all } f \in M_x \}.
\]

We first claim that \( Z(\Phi M_x) \) is non-empty. Suppose, on the contrary, that \( Z(\Phi M_x) \) is empty. Then for each \( y \) in \( Y \) there would exist an \( f_y \) in \( M_x \) such that \( \Phi f_y(y) \neq 0 \),
and thus $\Phi f_y$ is non-vanishing in an open neighborhood of $y$. Note that $|f_y| \in M_x$, and $\Phi |f_y| = |\Phi f_y|$ since $\Phi$ is a Riesz isomorphism. Therefore, we can assume further that both $f_y$ and $\Phi f_y$ are positive by replacing them by their absolute values if necessary. By the compactness of $Y$, we can choose finitely many $f_1, \ldots, f_n$ from $M_x^+$ such that the positive functions $\Phi f_1, \ldots, \Phi f_n$ have no common zero in $Y$. Hence $\Phi (f_1 + \cdots + f_n)$ is strictly positive; that is, $\Phi (f_1 + \cdots + f_n)(y) > 0$ for each $y$ in $Y$. This contradicts the fact that $f_1 + \cdots + f_n$ vanishes at $x$. We thus prove that $Z(\Phi M_x) \neq \phi$.

Next, we claim that $Z(\Phi M_x)$ is a singleton. Indeed, if $y_1, y_2 \in Z(\Phi M_x)$, then we would have $\Phi M_x \subseteq N_{y_1}$, $i = 1, 2$. Applying the above argument to $\Phi^{-1}$, we shall have $\Phi^{-1} N_{y_i} \subseteq M_{x_i}$, for some $x_i$ in $X$, $i = 1, 2$. It follows that $\Phi M_x \subseteq N_{y_i} \subseteq \Phi M_{x_i}$, $i = 1, 2$. Then $x = x_1 = x_2$ since $\Phi$ is bijective and $X$ is Hausdorff. Thus,

$$y_1 = y_2 \quad \text{and} \quad \Phi M_x = N_{y_1} = N_{y_2}.$$

Now, we can define a bijective map $\varphi : Y \rightarrow X$ such that $\Phi M_{\varphi(y)} = N_y$, $\forall y \in Y$. \hfill $\square$

The following main result answers affirmatively the question mentioned in the introduction and solves the conjecture of Ercan and "Onal in [7].

**Theorem 3.** Let $\Phi : C(X, E) \rightarrow C(Y, F)$ be a Riesz isomorphism such that $\Phi f$ is non-vanishing on $Y$ if and only if $f$ is non-vanishing on $X$. Then $Y$ is homeomorphic to $X$, and $\Phi$ can be written as a weighted composition operator

$$\Phi f(y) = \Pi(y)(f(\varphi(y))), \quad \forall f \in C(X, E), \forall y \in Y.$$

Here $\varphi$ is a homeomorphism from $Y$ onto $X$, and $\Pi(y)$ is a Riesz isomorphism from $E$ onto $F$ for every $y$ in $Y$. Moreover, $\Pi : Y \rightarrow (\mathcal{L}(E, F), SOT)$ is continuous, and $\|\Phi\| = \sup_{y \in Y} \|\Pi(y)\|$.

**Proof.** First, we show that the bijection $\varphi$ given in Lemma 2 is a homeomorphism from $Y$ onto $X$. It suffices to verify the continuity of $\varphi$ since $Y$ is compact and $X$ is Hausdorff. To this end, suppose, to the contrary, that there would exist a net $\{y_\lambda\}$ in $Y$ converging to $y_0$ in $Y$, but $\varphi(y_\lambda)$ converges to $x_0 \neq \varphi(y_0)$ in $X$.

Let $U_{x_0}$ and $U_{\varphi(y_0)}$ be disjoint open neighborhoods of $x_0$ and $\varphi(y_0)$, respectively. First, for any $f$ in $C(X, E)$ vanishing outside $U_{\varphi(y_0)}$, we claim that $\Phi f(y_0) = 0$. Indeed, since $\varphi(y_\lambda)$ belongs to $U_{x_0}$ for $\lambda$ large enough and $f(x) = 0$ for any $x$ in $U_{x_0}$, we have that $f \in M_{\varphi(y_\lambda)}$. It follows from Lemma 2 that $\Phi f \in N_{y_\lambda}$; that is, $\Phi f(y_\lambda) = 0$ when $\lambda$ is large enough. Thus, $\Phi f(y_0) = 0$ since $y_\lambda \rightarrow y_0$ and $\Phi f$ is continuous.

Let $\chi \in C(X)$ such that $\chi$ vanishes outside $U_{\varphi(y_0)}$ and $\chi(\varphi(y_0)) = 1$. Then, for any $h$ in $C(X, E)$, we have $h = \chi h + (1 - \chi)h$. Since $\chi h$ vanishes outside $U_{\varphi(y_0)}$, by the above argument, we can see that $\Phi(\chi h)(y_0) = 0$. Clearly, $\Phi((1 - \chi)h)$ vanishes at $y_0$ since $(1 - \chi)h \in M_{\varphi(y_0)}$. Thus, $\Phi h(y_0) = \Phi(\chi h)(y_0) + \Phi((1 - \chi)h)(y_0) = 0$ for any $h$ in $C(X, E)$. This leads to a contradiction since $\Phi$ is surjective. So $\varphi$ is continuous and thus a homeomorphism from $Y$ onto $X$ satisfying $\Phi M_{\varphi(y)} = N_y$ for each $y$ in $Y$. 


Next, note that ker $\delta_{\varphi}(y) = \ker \delta_{y} \circ \Phi$, where $\delta_{y}$ is the Dirac functional. Hence, there is a linear operator $\Pi(y) : E \to F$ such that $\delta_{y} \circ \Phi = \Pi(y) \circ \delta_{\varphi(y)}$. In other words,

$$\Phi f(y) = \Pi(y)(f(\varphi(y))), \quad \forall f \in C(X,E), \forall y \in Y.$$  

See, e.g., [8, p. 67].

It is routine to verify the other assertions in the statement of this theorem. For the convenience of the reader, we give a sketch of the rest of the proof. For $e$ in $E$, let $1_{X} \otimes e \in C(X,E)$ be defined by $(1_{X} \otimes e)(x) = e$ for each $x$ in $X$. Let $y$ in $Y$ be fixed. If $e \neq 0$, then $\Pi(y)e = \Pi(y)((1_{X} \otimes e)(\varphi(y))) = \Phi(1_{X} \otimes e)(y) \neq 0$ since $1_{X} \otimes e$ is non-vanishing. Hence, $\Pi(y)$ is one-to-one. On the other hand, for $u$ in $F$ we can find a function $f$ in $C(X,E)$ such that $\Phi f = 1_{Y} \otimes u$ by the surjectivity of $\Phi$. Let $e = f(\varphi(y))$. Then $\Pi(y)e = \Pi(y)(f(\varphi(y))) = \Phi f(y) = u$. That is, $\Pi(y)$ is surjective. To see that $\Pi(y)$ is a Riesz isomorphism, let $e_{1}, e_{2} \in E$. Then $\Pi(y)(e_{1} \vee e_{2}) = \Phi(1_{X} \otimes (e_{1} \vee e_{2})')(y) = \Phi(1_{X} \otimes e_{1})(y) \vee \Phi(1_{X} \otimes e_{2})(y) = \Pi(y)e_{1} \vee \Pi(y)e_{2}$, since $\Phi$ is a Riesz isomorphism.

Recall that every positive operator between Banach lattices is continuous. Let $e \in E$. Since $\|\Pi(y)e\| = \|\Phi(1_{X} \otimes e)(y)\| \leq \|\Phi(1_{X} \otimes e)\| \leq \|\Phi\|\|e\|$, we have $\|\Pi(y)\| \leq \|\Phi\|$ for all $y$ in $Y$. On the other hand, for any $f$ in $C(X,E)$ and any $y$ in $Y$, we can see that $\|\Phi f(y)\| = \|\Pi(y)(f(\varphi(y)))\| \leq \|\Pi(y)\||f||$. Consequently,

$$\|\Pi(y)\| \leq \sup_{y \in Y} \|\Pi(y)\|.$$

Finally, we prove that $\Pi : Y \to (C(E,F), SOT)$ is continuous. To this end, let $\{y_{\lambda}\}$ be a net such that $y_{\lambda} \to y$ in $Y$. Then, for any $e$ in $E$, $\|\Pi(y_{\lambda})e - \Pi(y)e\| = \|\Phi(1_{X} \otimes e)(y_{\lambda}) - \Phi(1_{X} \otimes e)(y)\| \to 0$, since $\Phi(1_{X} \otimes e)$ is continuous on $Y$. 

In the above results, we have to assume that both $\Phi$ and $\Phi^{-1}$ are non-vanishing preserving. In the following example, we can see that the inverse of a non-vanishing preserving Riesz isomorphism is not necessarily non-vanishing preserving.

**Example 4.** Let $X = \{1,2\}$ be equipped with the discrete topology, let $E = \mathbb{R}$ have its usual ordering and norm, and let $Y = \{0\}$ and $F = \mathbb{R}^{2}$ with the pointwise ordering and the sup norm. Define $\Phi : C(X,E) \to C(Y,F)$ by $\Phi f(0) = (f(1), f(2))$. Clearly, the Riesz isometric isomorphism $\Phi$ is non-vanishing preserving, but its inverse $\Phi^{-1}$ is not.

Let $E, F$ be both Banach lattices and Riesz algebras. Miao, Cao and Xiong [16] recently proved that if $F$ has no zero-divisor and there exists a Riesz algebraic isomorphism $\Phi : C(X,E) \to C(Y,F)$ such that $\Phi f$ is non-vanishing on $Y$ if $f$ is non-vanishing on $X$, then $X$ is homeomorphic to $Y$ and $E$ is Riesz algebraically isomorphic to $F$. In fact, from their proof we can see that $\Phi f$ is non-vanishing on $Y$ if and only if $f$ is non-vanishing on $X$; that is, both $\Phi$ and $\Phi^{-1}$ are non-vanishing preserving. Therefore, the result of Miao, Cao and Xiong can be restated as follows.

**Corollary 5 (16).** Let $E, F$ be both Banach lattices and Riesz algebras. If $F$ has no zero-divisor and $\Phi : C(X,E) \to C(Y,F)$ is a Riesz algebraic isomorphism such that $\Phi f$ is non-vanishing on $Y$ if $f$ is non-vanishing on $X$, then $\Phi$ is a weighted composition operator

$$\Phi f(y) = \Pi(y)(f(\varphi(y))), \quad \forall f \in C(X,E), \forall y \in Y.$$  

Here $\varphi$ is a homeomorphism from $Y$ onto $X$, and $\Pi(y)$ is a Riesz algebraic isomorphism from $E$ onto $F$ for every $y$ in $Y$. 

In Theorem 3 when \( X, Y \) are compact Hausdorff spaces and \( E = F = \mathbb{R} \), the lattice hypothesis about \( \Phi \) can be dropped.

**Example 6.** Let \( X, Y \) be compact Hausdorff spaces, and \( C(X), C(Y) \) be the Banach spaces of continuous real-valued functions defined on \( X, Y \), respectively. Assume \( \Phi : C(X) \to C(Y) \) is a linear map such that \( \Phi f \) is non-vanishing on \( Y \) if and only if \( f \) is non-vanishing on \( X \).

Note that \( (\Phi f_1 X)^{-1} \Phi \) is a unital linear map preserving non-vanishing. Let \( \lambda \) be in the range of \( f \). Then \( f - \lambda \mathbf{1}_X \) is not invertible, and thus neither is \( (\Phi f_1 X)^{-1} \Phi f - \lambda \mathbf{1}_Y \). It follows that \( \lambda \) is in the range of \( (\Phi f_1 X)^{-1} \Phi f \). The converse also holds. Therefore, the range of \( (\Phi f_1 X)^{-1} \Phi f \) coincides with the range of \( f \) for each \( f \) in \( C(X) \). In particular, \( (\Phi f_1 X)^{-1} \Phi \) is a unital linear isometry from \( C(X) \) into \( C(Y) \). By the Holsztyński Theorem [11], there is a compact subset \( Y_0 \) of \( Y \) and a quotient map \( \varphi : Y_0 \to X \) such that

\[
(\Phi f_1 X)^{-1} \Phi f \big|_{Y_0} = f \circ \varphi, \quad \forall f \in C(X).
\]

In case \( \Phi \) is surjective, the classical Banach-Stone Theorem ensures that \( \varphi \) is a homeomorphism from \( Y = Y_0 \) onto \( X \). Moreover, if \( \Phi f_1 X \) is strictly positive on \( Y \), then \( \Phi \) is a Riesz isomorphism. However, when \( \Phi \) is not surjective the situation is a bit uncontrollable. For example, consider \( \Phi : C[0, 1] \to C([0, \frac{1}{2}] \cup [1, \frac{3}{2}]) \) defined by

\[
\Phi f(y) = \begin{cases} 
  f(2y), & \text{if } 0 \leq y \leq 1/2; \\
  (2y - 2)f(0) + (3 - 2y)f(1), & \text{if } 1 \leq y \leq \frac{3}{2}.
\end{cases}
\]

Clearly, the thus defined \( \Phi \) is a non-surjective linear isometry preserving non-vanishing in two ways, but \( [0, 1] \) is not homeomorphic to \( [0, \frac{1}{2}] \cup [1, \frac{3}{2}] \).

Finally, we borrow an example from [15] which shows that the surjectivity cannot be guaranteed by many other properties we usually consider.

**Example 7.** Let \( \omega \) and \( \omega_1 \) be the first infinite and the first uncountable ordinal numbers, respectively. Let \( [0, \omega_1] \) be the compact Hausdorff space consisting of all ordinal numbers \( x \) not greater than \( \omega_1 \) and equipped with the topology generated by order intervals. Note that every continuous function \( f \) in \( C[0, \omega_1] \) is eventually constant. More precisely, there is a non-limit ordinal \( x_f \) such that \( \omega < x_f < \omega_1 \) and \( f(x) = f(\omega_1) \) for all \( x \geq x_f \).

Define \( \phi : [0, \omega_1] \to [0, \omega_1] \) by setting

\[
\phi(0) = \omega_1, \quad \phi(n) = n - 1 \text{ for all } n = 1, 2, \ldots, \quad \text{and } \phi(x) = x \text{ for all } x \geq \omega.
\]

Let \( \Phi : C[0, \omega_1] \to C[0, \omega_1] \) be the non-surjective composition operator defined by \( \Phi f = f \circ \phi \). It is plain that \( \Phi \) is an isometric unital algebraic and lattice isomorphism from \( C[0, \omega_1] \) onto its range. In fact, one can see in [15, Example 3.3] that the map \( \Phi \) is a non-surjective linear \( n \)-local automorphism of \( C[0, \omega_1] \), where \( n = 1, 2, \ldots, \omega \); i.e., the action of \( \Phi \) on any set \( S \) of cardinality not greater than \( n \) agrees with an automorphism \( \Phi_S \).

**Acknowledgment**

The authors would like to thank the referee for comments which have improved this paper.
References


Department of Mathematics, Southwest Jiaotong University, Chengdu 610031, People’s Republic of China

E-mail address: jinxichen@home.swjtu.edu.cn

Department of Mathematics, Southwest Jiaotong University, Chengdu 610031, People’s Republic of China

E-mail address: zlchen@home.swjtu.edu.cn

Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan

E-mail address: wong@math.nsysu.edu.tw