NUMERICAL PEAK POINTS
AND NUMERICAL ŠILOV BOUNDARY
FOR HOLOMORPHIC FUNCTIONS

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Abstract. In this paper, we characterize the numerical and numerical strong-
peak points for $A_\infty(B_E : E)$ when $E$ is the complex space $l_1$ or $C(K)$. We
also prove that \{$(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1$ for all $n \in \mathbb{N}$\} is the numerical
Šilov boundary for $A_\infty(B_{l_1} : l_1)$.

1. Introduction

Throughout this paper we will just consider complex Banach spaces. For a
Banach space $E$, $S_E$ and $B_E$ will be the unit sphere and the closed unit ball of $E$,
respectively. If $E$ and $F$ are Banach spaces, $C_0(B_E : F)$ denotes the Banach space
of the bounded continuous functions $f : B_E \to F$, endowed with the supremum
norm. In the case that $F = \mathbb{C}$, we write $C_0(B_E)$. An $N$-homogeneous polynomial $P$
from $E$ to $F$ is a mapping such that there is an $N$-linear (and bounded) mapping
$L$ from $E$ to $F$ satisfying

$$P(x) = L(x, \ldots, x), \quad \forall x \in E.$$ 

The set of all $N$-homogenous polynomials from $E$ to $F$ is denoted by $P^N(E : F)$.

We denote by $A_\infty(B_E : F)$ the Banach space of the bounded continuous function
$f : B_E \to F$ such that $f$ is holomorphic on the open unit ball, endowed with the supremum
norm. If $F = \mathbb{C}$, we write $A_\infty(B_E)$.

A result of Šilov asserts that if $\mathcal{A}$ is a unital separating subalgebra of $C(K)$ ($K$
is a compact Hausdorff topological space), there is a smallest closed subset $S \subset K$
such that every function of $\mathcal{A}$ attains its norm at some point of $S$ ([7], Theorem
I.4.2). Bishop [5] proved that if $K$ is metrizable, in fact, there is a minimal subset
of $K$ satisfying the above condition for every separating subalgebra of $C(K)$. That
subset is the set of peak points for $\mathcal{A}$. Globevnik [8] introduced the corresponding
concepts of the boundary of a subalgebra $\mathcal{A}$ of $C_0(\Omega)$, the set of bounded and
continuous functions on a topological space $\Omega$ not necessarily compact, and studied

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the boundaries for \( \Omega = B_{c_0} \) and \( A \) a certain space of holomorphic functions. A subset \( B \subset B_E \) is a boundary for a subspace \( A \subset C_b(B_E) \) if
\[
\|f\| = \sup_{z \in B} |f(z)|, \quad \forall f \in A.
\]
An element \( x \in S_E \) is called a peak point for \( A \) if there exists \( h \in A \) such that \( |h(x)| = \|h\|_{B_E} \) and \( |h(z)| < \|h\|_{B_E} \) for every \( z \in B_E \setminus \{x\} \). In this case we say that \( h \) peaks at \( x \). A peak point \( x \) is called a strong peak point for \( A \) if there exists \( h \in A \) such that \( |h(x)| = \|h\|_{B_E} \) and, given \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for every \( z \in B_E \) with \( \|z - x\| > \epsilon \), we have \( |h(z)| < \|h\|_{B_E} - \delta \). In this case we say that \( h \) peaks strongly at \( x \).

In 1971, Harris [3] introduced the definition of a spatial numerical range for a bounded and holomorphic function defined on a Banach space and the corresponding concept of numerical radius. The spatial numerical range of a bounded function \( f \) from \( B_E \) to \( E \) is given by
\[
W(f) := \{x^*(f(x)) : (x, x^*) \in \Pi(E)\},
\]
where we denoted by \( \Pi(E) \) the following subset:
\[
\Pi(E) := \{(x, x^*) \in S_E \times S_{E^*} : x^*(x) = 1\}.
\]
The numerical radius \( v(f) \) is just the number
\[
v(f) := \sup\{\lambda : \lambda \in W(f)\}.
\]
For more background and information about numerical ranges and radii, we refer the reader to [6]. For a linear space \( A \subset C_b(B_E : E) \), M. Acosta and the author [3] gave the corresponding definition of numerical boundary for \( A \). We say that \( B \subset \Pi(E) \) is called a numerical boundary for \( A \) if
\[
\sup_{(x, x^*) \in B} |x^*(f(x))| = v(f), \quad \forall f \in A.
\]
in the case that \( B \) is a \((\|\| \times w^*)\)-closed numerical boundary for \( A \) that is minimal under the previous conditions, \( B \) is said to be the numerical \( \check{S}i\text{lo}v \) boundary. In [3], the authors studied numerical boundaries for holomorphic functions on some classical Banach spaces. Parallel to the concepts of peak and strong peak points, we introduce the corresponding definition of numerical and numerical strong-peak points. An element \( (x, x^*) \in \Pi(E) \) is called a numerical peak point for \( A \) if there exists \( h \in A \) such that \( |x^*(h(x))| = v(h) \) and \( |z^*(h(z))| < v(h) \) for every \( (z, z^*) \in \Pi(E) \setminus \{(x, x^*)\} \). In this case we say that \( h \) peaks numerically at \( (x, x^*) \). Also, \( (x, x^*) \in \Pi(E) \) is called a numerical strong-peak point for \( A \) if there exists a function \( h \in A \) such that \( |x^*(h(x))| = v(h) \), and for any \( (x_n, x_n^*) \in \Pi(E) \) with \( \lim_{n \to \infty} |x_n^*(h(x_n))| = v(h) \) we have that \( x_n \to x \) in norm and \( z_n^* \to x^* \) in \( w^* \)-topology. In this case we say that \( h \) peaks strongly-numerically at \( (x, x^*) \). It is clear that every numerical strong-peak point is a numerical peak point. Note that if \( E \) is a finite dimensional space, then every numerical peak point is also a numerical strong-peak point. It is immediate from definition that every \((\|\| \times w^*)\)-closed numerical boundary for \( A_\infty(B_E : E) \) contains all numerical strong-peak points.

In Section 2, we characterize the numerical peak points for \( A_\infty(B_{l_1} : l_1) \) and prove that every numerical peak point for \( A_\infty(B_{l_1} : l_1) \) is also a numerical strong-peak point. We prove that \( \{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\} \) is the numerical \( \check{S}i\text{lo}v \) boundary for \( A_\infty(B_{l_1} : l_1) \).
In Section 3, we characterize the numerical peak points for \( A_\infty(B_{C(K)} : C(K)) \) when \( K \) is a compact metrizable space. If \( K \) is an infinite compact Hausdorff topological space, we prove that there are no numerical strong-peak points for \( A_\infty(B_{C(K)} : C(K)) \).

2. Numerical peak points and numerical Šilov boundary for holomorphic functions on \( l_1 \)

Lemma 2.1. If \((x, x^*) \in \Pi(l_1)\) is a numerical peak point for \( A_\infty(B_{l_1} : l_1)\), then \(|x^*(e_n)| = 1 \) for all \( n \in \mathbb{N} \).

Proof. Let \( x := (v_n)_n \in S_{l_1}, x^* := (w_n)_n \in S_{l_\infty} \). Assume that there exists a positive integer \( n_0 \) such that \(|w_{n_0}| < 1\). Since the subset of peak points in \( S_E \) for \( A_\infty(B_{l_1}) \) is invariant under surjective linear isometries on \( l_1 \), we may assume that \( n_0 = 1 \), so \(|w_1| < 1\). Let \( h \in A_\infty(B_{l_1} : l_1) \) peak numerically at \((x, x^*)\) with \( h := (h_n)_n \) for some \( h_n \in A_\infty(B_{l_1}) \). Let \( x_\lambda^* := \lambda v_1 + \sum_{n>1} w_n e_n \) for \(|\lambda| \leq 1\). We claim that \( v_1 \neq 0 \). If \( v_1 = 0 \), then \((x, x_\lambda^*) \in \Pi(l_1)\) for \(|\lambda| \leq 1\). We define the polynomial \( \psi : \mathcal{T}(0, 1) \to \mathbb{C} \) by

\[
\psi(\lambda) := \lambda h_1(x) + \sum_{n>1} w_nh_n(x) (|\lambda| \leq 1).
\]

Since \( \max_{|\lambda| \leq 1} |\psi(\lambda)| = v(h) = |x^*(h(x))| = |\psi(w_1)| \), by the Maximum Modulus Theorem, \( \psi(\lambda) = v(h) \) for all \(|\lambda| \leq 1\). Choose any complex number \( \beta \) with \(|\beta| = 1\). Then \((x, x_\beta^*) \neq (x, x^*) \in \Pi(l_1)\) and \( |\psi(\beta)| = |x_\beta^*(h(x))| = v(h) \). Since \( h \) peaks numerically at \((x, x^*)\), we have a contradiction. Thus \( v_1 \neq 0 \). We define the 1-degree polynomial \( u : \mathcal{T}(0, 1) \to \mathbb{C} \) by

\[
u(\lambda) := \lambda v_1 + \sum_{n>1} w_nv_n (|\lambda| \leq 1).
\]

Since \( 1 = x^*(x) = |u(w_1)| \), by the Maximum Modulus Theorem, \( u(\lambda) = 1 \) for all \(|\lambda| \leq 1\). By the same reason as in the above argument, \((x, x_\beta^*) \in \Pi(l_1)\) and \( |\psi(\beta)| = |x_\beta^*(h(x))| = v(h) \) for any \( \beta \in \mathbb{C} \) with \(|\beta| = 1\). Thus we have a contradiction. Therefore \(|w_1| = 1\). \( \square \)

Theorem 2.2. \( S := \{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \) for all \( n \in \mathbb{N} \}\) is the set of all numerical strong-peak points for the space of 2-degree polynomials in \( A_\infty(B_{l_1} : l_1) \) and every numerical peak point for the space of 2-degree polynomials in \( A_\infty(B_{l_1} : l_1) \) is a numerical strong-peak point.

Proof. By Lemma 2.1, it is enough to show the first statement of the theorem. Let \((y_0, y_0^*) \in S\). We will prove that \((y_0, y_0^*) \) is a numerical strong-peak point for the space of 2-degree polynomials in \( A_\infty(B_{l_1} : l_1) \). Let \( J = \text{supp}(y_0) \). Since the subset of peak points in \( S_E \) for \( A_\infty(B_{l_1}) \) is invariant under surjective linear isometries on \( l_1 \), we can assume that \( y_0(k) > 0 \) for all \( k \in J \). Let \( a_n := y_0^*(e_n) \) for all \( n \in \mathbb{N} \). Then \( a_k = 1 \) for all \( k \in J \). By Theorem 2.6 in [3] there exists a 2-degree polynomial \( f \in A_\infty(B_{l_1}) \) which peaks strongly at \( y_0 \) with \( \|f\| = 1 \). Let \( \lambda_n := \text{sign}(a_n) \) and \( z_0 := \sum_{n=1} a_n b_ne_n \in S_{l_1} \) with \( b_n > 0 \) for all \( n \in \mathbb{N} \). We define a 2-degree polynomial \( h \in A_\infty(B_{l_1} : l_1) \) by \( h(x) := f(x)z_0 \) \( (x \in l_1) \). Clearly \( y_0^*(h(y_0)) = 1 = \|h\| = v(h) \). We claim that \( h \) peaks strongly-numerically at \((y_0, y_0^*)\). Consider a sequence \( \{(x_n, x_n^*)\} \) in \( \Pi(l_1) \) such that \( \lim_{n \to \infty} |x_n^*(h(x_n))| = v(h) \). We
will prove that $x_n \to y_0$ in norm and $x_n^* \to y_0^*$ in $w^*$-topology. Since

$$1 = \lim_{n \to \infty} |x_n^*(h(x_n))| \leq \lim_{n \to \infty} |f(x_n)| \leq 1$$

and $f$ peaks strongly at $y_0$, $x_n \to y_0$ in norm. Write $x_n^* := (a_j^{(n)})_j \in S_{l_1}$ and $x_n := (c_j^{(n)})_j \in S_{l_1}$ for all $n, j \in \mathbb{N}$. Let $x := (d_j) \in l_1$. First we claim that

$$\lim_{n \to \infty} a_k^{(n)} = 1 \text{ for all } k \in J. \text{ Let } i_0 \in J \text{ be fixed. Let } \{a^{(n_i)}_{i_0}\} \text{ be any subsequence of } \{a^{(n)}_{i_0}\}. \text{ As the set } \{a^{(n)}_{i_0}\} \text{ is a bounded subset of } \mathbb{C}, \text{ there exist a subsequence } \{a^{(n_i)}_{i_0}\} \text{ and a complex number } r \text{ with } |r| \leq 1 \text{ such that } \lim_{i \to \infty} a^{(n_i)}_{i_0} = r. \text{ Since } 1 = x_n^*(x_n) \text{ for all } n \in \mathbb{N}, \text{ we have}$$

$$1 = \sum_{j=1}^{\infty} a_j^{(n_i)} c_j^{(n_i)} \text{ for all } l \in \mathbb{N}.$$ 

Since $x_n \to y_0$ in norm, we have $\lim_{n \to \infty} c^{(n)}_{i_0} = y_0(i_0) > 0$. For a sufficiently large $l$, $c^{(n_i)}_{i_0} > 0$. It follows that, for a sufficiently large $l$,

$$1 = a^{(n_i)}_{i_0} c^{(n_i)}_{i_0} + \sum_{j \neq i_0} a_j^{(n_i)} c_j^{(n_i)}$$

$$\leq |a^{(n_i)}_{i_0}| c^{(n_i)}_{i_0} + \sum_{j \neq i_0} |a_j^{(n_i)}| |c_j^{(n_i)}|$$

$$\leq \sum_{j=1}^{\infty} |c_j^{(n_i)}| = 1.$$

Thus $1 = a^{(n_i)}_{i_0} = |a^{(n_i)}_{i_0}|$ for a sufficiently large $l$, so $r = 1$. Therefore $\lim_{n \to \infty} a^{(n)}_{i_0} = 1$. We claim that $\lim_{n \to \infty} a^{(n)}_{j_0} = a_j$ for all $j \in \mathbb{N} \setminus J$. Let $j_0 \in \mathbb{N} \setminus J$ be fixed. Let $\{a^{(n_i)}_{j_0}\}$ be any subsequence of $\{a^{(n)}_{j_0}\}$. As the set $\{a^{(n_i)}_{j_0}\}$ is a bounded subset of $\mathbb{C}$, there exist a subsequence $\{a^{(n_j)}_{j_0}\}$ and a complex number $\rho$ with $|\rho| \leq 1$ such that $\lim_{i \to \infty} a^{(n_j)}_{j_0} = \rho$. Since

$$1 = \lim_{n \to \infty} |x_n^*(h(x_n))| \leq \lim_{n \to \infty} |f(x_n)| |x_n^*(z_0)| \leq 1,$$

we have

$$1 = \lim_{n \to \infty} |x_n^*(z_0)| = \lim_{n \to \infty} |\sum_{j=1}^{\infty} a_j^{(n)} \lambda_j b_j|.$$

It follows that for fixed $i_0 \in J$,

$$1 = \lim_{n \to \infty} |\sum_{j=1}^{\infty} a_j^{(n_j)} \lambda_j b_j|$$

$$\leq \lim_{n \to \infty} |a^{(n_j)}_{j_0} \lambda_j b_{j_0} + a^{(n)}_{i_0} b_{i_0}| + \sum_{j \in \mathbb{N} \setminus \{i_0, j_0\}} b_j$$

$$\leq b_{j_0} + b_{i_0} + \sum_{j \in \mathbb{N} \setminus \{i_0, j_0\}} b_j = 1.$$
Thus
\[
\lim_{l \to \infty} |a_{jn}^{(n)}| \lambda_j b_{j_0} + a_{i_0}^{(n)} b_{i_0} = |\rho \lambda_j b_{j_0} + b_{i_0}| = b_{j_0} + b_{i_0},
\]
showing \(\rho \lambda_j = 1\). We have \(\rho = a_{j_0}\). Therefore \(\lim_{n \to \infty} a_{j_0}^{(n)} = a_{j_0}\). Since the \(w^*\)-topology in \(l_\infty\) is the convergence in each coordinate, we complete the proof. \(\square\)

By Theorem 2.3 in [3], \(\Pi(l_p)\) is the numerical Šilov boundary for \(A_\infty(B_{l_p} : l_p)\) for each \(1 < p < \infty\). The following is an application of Theorem 2.2 to the numerical Šilov boundary.

**Theorem 2.3.** \(S := \{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\}\) is the numerical Šilov boundary for \(A_\infty(B_{l_1} : l_1)\).

**Proof.** It is easy to show that \(S\) is \((\| \cdot \| \times w^*)\)-closed. By Theorem 2.2 and the fact that every \((\| \cdot \| \times w^*)\)-closed numerical boundary for \(A_\infty(B_{l_1} : l_1)\) contains all numerical strong-peak points, it suffices to prove that \(S\) is a numerical boundary for \(A_\infty(B_{l_1} : l_1)\). Let \(h \in A_\infty(B_{l_1} : l_1)\) and let \(\epsilon > 0\). Choose \((x_0, x_0^*) \in \Pi(l_1)\) such that \(|v(h) - \epsilon < |x_0^*(h(x_0))|\). Let \(x_0 := (w_0)_n, x_0^* := (w_0^*_n)\) and \(h := (h_n)_n\) for some \(h_n \in A_\infty(B_{l_1})\). We claim that there exists a complex sequence \(\{\lambda_n\}\) in the unit disk such that if
\[
z_n^* := \sum_{1 \leq j \leq n} \lambda_j e_j + \sum_{j > n} w_j e_j
\]
for all \(n \in \mathbb{N}\), then \((x_0, z_n^*) \in \Pi(l_1)\) and \(|z_{n+1}^*(h(x_0))| \geq |z_n^*(h(x_0))| > |v(h) - \epsilon|\) for all \(n \in \mathbb{N}\). If \(|w_1| = 1\), let \(\lambda_1 := w_1\). Otherwise \(w_1 = 0\); hence
\[
(x_0, \lambda_1 e_1 + \sum_{n>1} w_n e_n) \in \Pi(l_1) \text{ for all } |\lambda| \leq 1.
\]
If \(h_1(x_0) = 0\), let \(\lambda_1 := 1\). Assume that \(h_1(x_0) \neq 0\). We define a nonconstant 1-degree polynomial \(\psi_1 : D(0, 1) \to \mathbb{C}\) by
\[
\psi_1(\lambda) := \lambda h_1(x_0) + \sum_{n>1} w_n h_n(x_0) (|\lambda| \leq 1).
\]
Since \(\max_{|\lambda| = 1} |\psi_1(\lambda)| \geq |\psi_1(w_1)| = |x_0^*(h(x_0))|\), by the Maximum Modulus Theorem, there exists a complex number \(\lambda_1\) with \(|\lambda_1| = 1\) such that
\[
|\psi_1(\lambda_1)| \geq |z_1^*(h(x_0))| > v(h) - \epsilon.
\]
Let \(z_1^* := \lambda_1 e_1 + \sum_{n>1} w_n e_n\). Then \((x_0, z_1^*) \in \Pi(l_1)\) and \(|z_1^*(h(x_0))| \geq |x_0^*(h(x_0))| > v(h) - \epsilon|\). If \(|w_2| = 1\), let \(\lambda_2 := w_2\). Otherwise \(w_2 = 0\); hence
\[
(x_0, \lambda_1 e_1 + \lambda_2 e_2 + \sum_{n>2} w_n e_n) \in \Pi(l_1) \text{ for all } |\lambda| \leq 1.
\]
If \(h_2(x_0) = 0\), let \(\lambda_2 := 1\). Assume that \(h_2(x_0) \neq 0\). We define a nonconstant 1-degree polynomial \(\psi_2 : D(0, 1) \to \mathbb{C}\) by
\[
\psi_2(\lambda) := \lambda_1 h_1(x_0) + \lambda_2 h_2(x_0) + \sum_{n>2} w_n h_n(x_0) (|\lambda| \leq 1).
\]
Since \(\max_{|\lambda| = 1} |\psi_2(\lambda)| \geq |\psi_2(b_2)| = |\psi_1(\lambda_1)| > v(h_0) - \epsilon\), by the Maximum Modulus Theorem, there exists a complex number \(\lambda_2\) with \(|\lambda_2| = 1\) such that
\[
|\psi_2(\lambda_2)| \geq |\psi_1(\lambda_1)| > v(h) - \epsilon.
\]
Let \(z_2^* := \sum_{1 \leq j \leq 2} \lambda_j e_j + \sum_{n>2} w_n e_n\). Then \((x_0, z_2^*) \in \Pi(l_1)\) and \(|z_2^*(h(x_0))| \geq |z_1^*(h(x_0))| > v(h) - \epsilon\). Continuing this process, we can get a complex sequence
\{\lambda_n\} in the unit disk satisfying the claim. Let \( z^* := (\lambda_n)_{n=1}^\infty \in \ell_\infty \). We will show that \((x_0, z^*) \in S\). Indeed, it follows that for each \( n \in \mathbb{N} \),
\[
|z^*(x_0) - 1| = |z^*(x_0) - z_n^*(x_0)| \leq \sum_{j>n} |\lambda_j - w_j| |v_j| \leq 2 \sum_{j>n} |v_j| \to 0,
\]
as \( n \to \infty \). Thus \( z^*(x_0) = 1 \). We will show that
\[
\lim_{n \to \infty} |z_n^*(h(x_0))| = |z^*(h(x_0))|.
\]
Let \( h(x_0) := (\beta_n)_{n=1}^\infty \in l_1 \). It follows that
\[
|z^*(h(x_0)) - z_n^*(h(x_0))| \leq \sum_{j>n} |\lambda_j - w_j| |\beta_j| \leq 2 \sum_{j>n} |\beta_j| \to 0,
\]
as \( n \to \infty \). Thus we have
\[
v(h) - \epsilon < \limsup_{n \to \infty} |z_n^*(h(x_0))| = |z^*(h(x_0))| \leq \sup_{(x,x^*) \in S} |x^*(h(x))| \leq v(h),
\]
which shows that \( \sup_{(x,x^*) \in S} |x^*(h(x))| = v(h) \). Since \( h \in A_\infty(B_{l_1} : l_1) \) is arbitrary, we complete the proof. \( \square \)

3. Numerical Peak Points and Numerical Strong-Peak Points on \( C(K) \)

**Theorem 3.1.** Let \( K \) be a compact Hausdorff topological space with at least two points. If \((x_0, x_0^*) \in \Pi(C(K))\) is a numerical peak point for \( A_\infty(B_{C(K)} : C(K)) \), then:

(a) There exists a unique \( t_0 \in K \) such that
\[
x_0(t) = 1, \forall t \in K \text{ and } x_0^* = \text{sign}(x_0(t_0))\delta_{t_0}.
\]
Hence \( x_0 \) is an extreme point of \( B_{C(K)} \).

(b) \( x_0 \) is a peak point for \( A_\infty(B_{C(K)}) \).

**Proof.** Let \( h \in S_{A_\infty(B_{C(K)} : C(K))} \) peak numerically at \((x_0, x_0^*)\). Then \( v(h) = |x_0^*(h(x_0))| \). By Theorem 2.7 in \([2]\), \( 1 = v(h) = \|h\| \). Choose an element \( t_0 \in K \) such that \( 1 = \|h(x_0)|| = \|\delta_{t_0} \circ h(x_0)\| = \|\delta_{t_0} \circ h\| \). We claim that \( x_0(t) = 1, \forall t \in K \). For every complex number \( \lambda \) in the unit disk, the function \( x_0 + \lambda(1 - |x_0|) \in C(K) \), and for every \( t \in K \), it is satisfied that \( |x_0(t) + \lambda(1 - |x_0(t)|)| \leq 1 \). Define the continuous function \( \phi : \overline{D}(0,1) \to \mathbb{C} \) by
\[
\phi(\lambda) := \delta_{t_0}(h(x_0 + \lambda(1 - |x_0|)))(|\lambda| \leq 1).
\]
Note that \( \phi \) is holomorphic on \( D(0,1) \) and \( |\phi(\lambda)| \leq 1 \) for every \( \lambda \) in the unit disk. Also \( |\phi(0)| = |\delta_{t_0} \circ h(x_0)| = 1 \). Since \( \phi \) attains its maximum modulus at \( 0 \), \( \phi \) is constant. We choose a complex number \( \lambda_0 \) satisfying the facts that \( |\lambda_0| = 1 \) and \( |x_0(t_0) + \lambda_0(1 - |x_0(t_0)|)| = 1 \). So \( \phi(0) = \phi(\lambda_0) \). The element \( z_0 := x_0 + \lambda_0(1 - |x_0|) \) is in the unit ball of \( C(K) \) and \( |z_0(t_0)| = 1 \). Then \((z_0, z_0^* \delta_{t_0}) \in \Pi(C(K)) \). Since \( |\phi(\lambda_0)| = |z_0(t_0) \delta_{t_0}(h(z_0))| = v(h) = 1 \), we have \( z_0 = x_0 \) and \( x_0^* = x_0^* \delta_{t_0} \). Thus \( \lambda_0(1 - |x_0(t)|)| = 0 \) for all \( t \in K \), so \( x_0(t) = 1 \) for all \( t \in K \). So \( \delta_{t_0} \). The uniqueness of \( t_0 \) follows from Urysohn’s Lemma. Therefore we have proved assertion (a).
We will show that $\delta_n \circ h$ peaks at $x_0$. Let $y \in S_{C(K)}$ such that $1 = |\delta_n \circ h(y)| = \|\delta_n \circ h\|$. By the same argument as in the proof of assertion (a), we have $|y(t_0)| = 1$. Note that $(y, \text{sign}(y(t_0))\delta_n) \in \Pi(C(K))$. Since

$$1 = |\text{sign}(y(t_0))\delta_n(h(y))| = \|\delta_n \circ h\| = \|h\| = v(h),$$

we have $y = x_0$ and $x_0^* = \text{sign}(y(t_0))\delta_n$, which show assertion (b). \hfill $\square$

**Theorem 3.2.** (1) Let $K$ be a compact metrizable space.

Then $M := \{(x, \text{sign}(x(t))\delta_t) : x \in \text{ext}B_{C(K)}, t \in K\}$ is the set of all numerical peak points for the space of 1-degree polynomials in $A_\infty(B_{C(K)} : C(K))$.

(2) If $K$ is an infinite compact topological space, then there are no numerical strong-peak points for $A_\infty(B_{C(K)} : C(K))$.

**Proof.** (1): By (a) of Theorem 3.1, it suffices to show that if $(x_0, \text{sign}(x_0(t))\delta_{t_0}) \in M$, then it is a numerical peak point for the space of 1-degree polynomials in $A_\infty(B_{C(K)} : C(K))$. Since the subset of peak points in $S_{C(K)}$ for $A_\infty(B_{C(K)} : C(K))$ is invariant under surjective linear isometries on $C(K)$, we can assume that $x_0(t) = 1$ for all $t \in K$. Since $K$ is a metrizable space, there exists a dense subset $\{l_n\}$ in $K$. Choose $(\alpha_n) \in S_1$ with $\alpha_n > 0$ for all $n \in \mathbb{N}$. We claim that there exists a function $y_0 \in S_{C(K)}$ such that $y_0(t_0) = 1$ and $0 \leq y_0(t) < 1$ for all $t \in K \setminus \{t_0\}$. Indeed, let $d$ be a metric in $K$. Let

$$A_n := \{t \in K : d(t, t_0) \geq \frac{1}{n} \} \ (n \in \mathbb{N}).$$

Clearly $A_n$ is a closed subset of $K$ with $t_0 \notin A_n$ for all $n \in \mathbb{N}$. By the Tietze Extension Theorem, for each $n \in \mathbb{N}$, there exists a sequence $\{z_n\}$ in $C(K)$ such that $0 \leq z_n \leq 1$, $z_n(t_0) = 1$ and $z_n(A_n) = \{0\}$. We define the function

$$y_0(t) := \sum_{n=1}^{\infty} \frac{1}{2^n} z_n(t) \ (t \in K).$$

We define a 1-degree polynomial $h \in A_\infty(B_{C(K)} : C(K))$ by

$$h(x) := \sum_{n=1}^{\infty} \alpha_n (1 + x(l_n)) y_0 \ (x \in C(K)).$$

We claim that $h$ peaks numerically at $(x_0, \delta_{t_0})$. Let $(z_0, z_0^*) \in \Pi(C(K))$ be such that $v(h) = |z_0^*(h(z_0))|$. Since $2 \geq \|h\| \geq v(h) \geq \delta_{t_0}(h(x_0)) = 2$, we have $2 = v(h) = \|h\|$. Since $2 = |z_0^*(h(z_0))| = \|h(z_0)\|$, we have $z_0(l_n) = 1$ for all $n \in \mathbb{N}$. Since $\{l_n\}$ is a dense subset of $K$, $z_0(t) = 1 = x_0(t)$ for all $t \in K$. Thus $z_0 = x_0$. By the Riesz Representation Theorem on $C(K)^*$, there exists a unique regular complex Baire measure $\mu = v + iw$ on $K$ ($v$ and $w$ are positive measures) satisfying

$$z_0^*(x) = \int_K x(t) \ d\mu = \int_K x(t) \ dv + i \int_K x(t) \ dw \ (x \in C(K))$$

with $\|z_0^*\| = |\mu| = 1$. Since

$$1 = z_0^*(x_0) = \int_K x_0 \ dv + i \int_K x_0 \ dw = v(K) + iw(K),$$

$v(K) = 1$, $w(K) = 0$. Thus $w = 0$ and

$$z_0^*(x) = \int_K x(t) \ dv \ (x \in C(K)).$$
It follows that
\[ 2 = |z_0^*(h(x_0))| = \left| \int_K h(x_0) dv \right| = 2 \int_K \left| y_0(t) dv \right| \leq 2 \int_K |y_0(t)| dv \leq 2. \]

Thus \( 1 = \int_K |y_0(t)| dv. \) We claim that \( v(\{t_0\}) = 1. \) Otherwise, by the regularity of \( v \) and the choice of \( y_0, \) there exists an open subset \( \theta_0 \) of \( K \) containing \( t_0 \) such that \( v(K \setminus \theta_0) > 0. \) Let \( \delta_0 := \max_{t \in K \setminus \theta_0} |y_0(t)| < 1. \) It follows that
\[ 1 = \int_K |y_0(t)| dv \leq v(\theta_0) + \delta_0 \ v(K \setminus \theta_0) < v(K) = 1, \]
which is impossible. Thus \( v(\{t_0\}) = 1 \) and \( v(K \setminus \{t_0\}) = 0. \) Therefore, we have
\[ z_0^*(x) = \int_K x(t) \ dv = x(t_0) \ v(\{t_0\}) = x(t_0) = \delta_{t_0}(x) \]
for all \( x \in C(K), \) showing \( z_0^* = \delta_{t_0}. \)

(2): By (a) of Theorem 3.1, it suffices to show that if \( (x_0, x_0^*) \in \Pi(C(K)) \) with \( x_0 \in extB_{C(K)} \), then \( (x_0, x_0^*) \) is not a numerical strong-peak point. Let \( \{t_n\} \) in \( K \) be the sequence such that there is a sequence \( \{x_n\} \) in \( B_{C(K)} \) such that \( 0 \leq x_n \leq 1, x_n(t_n) = 1, \) for all \( n \) and \( \text{supp}(x_n) \cap \text{supp}(x_m) = \emptyset \) \((n \neq m). \) Let \( h \in \mathcal{A}_\infty(B_{C(K)} : C(K)) \) such that \( 1 = v(h) = |x_0^*(h(x_0))|. \) We will show that \( h \) cannot peak strongly numerically at \( (x_0, x_0^*). \) By Theorem 2.7 in [2], we have \( v(h) = \|h\| = 1. \) There exists a \( t_0 \in K \) such that \( \text{sign}(x_0(t_0)) \delta_{t_0} \circ h \in \mathcal{A}_\infty(B_{C(K)} \) and
\[ 1 = \|h(x_0(t_0))\| = |\delta_{t_0}(x_0(t_0))| = |\text{sign}(x_0(t_0)) \delta_{t_0}(h(x_0))| = \|\delta_{t_0} \circ h\|. \]
Let \( z_n := x_0(1 - x_n) \) for all \( n \in \mathbb{N}. \) Since the support of \( x_n \) are pairwise disjoint, there is a positive integer \( N \) such that \( x_n(t_0) = 0 \) for all \( n > N. \) Let \( \lambda_n := \text{sign}(x_n(t_0)) \) for all \( n > N. \) Thus \( (z_n, \lambda_n \delta_{t_0}) \in \Pi(C(K)) \) for all \( n > N. \) Since \( \{x_n\} \) is equivalent to a \( c_0 \)-basis, then it converges weakly to \( 0. \) By the Rainwater theorem, the sequence \( \{z_n\} \) is in the unit ball of \( C(K) \) and converges weakly to \( x_0. \) Since \( C(K) \) has the Dunford-Pettis property, then it has also the polynomial Dunford-Pettis property \([10], \) and so, if we follow the argument in the proof Proposition 4.1 in \([1], \) then
\[ |\lambda_n \delta_{t_0} \circ h(z_n)| \to 1. \]
It follows that
\[ \|z_n - x_0\| = \|x_0 x_n\| \geq |x_0(t_n) x_n(t_n)| = 1 \]
for all \( n \in \mathbb{N}. \) Therefore, we have proved that \( (x_0, x_0^*) \) is not a numerical strong-peak point. \( \square \)

It is known in [3], Theorem 5.2, that there is no numerical Šilov boundary for \( \mathcal{A}_\infty(B_{C(K)} : C(K)) \) if \( K \) is an infinite compact Hausdorff topological space.

**Corollary 3.3.** Let \( n \in \mathbb{N}. \) Then \( M := \{(x, \text{sign}(x(t)) \delta_t) \in \Pi(l_\infty^n) : |x(k)| = 1 \text{ for all } k = 1, 2, \ldots, n, \text{ for some } t = 1, 2, \ldots, n\} \) is the numerical Šilov boundary for \( \mathcal{A}_\infty(B_{l_\infty^n} : l_\infty^n). \)

**Proof.** \((\subseteq): \) By Proposition 5.1 in [3], it follows.
\((\supseteq): \) Clearly \( M \) is \((\| \| \times w^*)\)-closed. By Theorem 3.2 (1), \( M \) is the set of all numerical peak points for \( \mathcal{A}_\infty(B_{l_\infty^n} : l_\infty^n). \) Since \( l_\infty^n \) is finite dimensional, \( M \) is the set of all numerical strong-peak points. \( \square \)
References


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