NUMERICAL PEAK POINTS AND NUMERICAL ŠILOV BOUNDARY FOR HOLOMORPHIC FUNCTIONS

SUNG GUEN KIM

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Abstract. In this paper, we characterize the numerical and numerical strong-peak points for $A_\infty(B_E : F)$ when $E$ is the complex space $l_1$ or $C(K)$. We also prove that $\{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\}$ is the numerical Šilov boundary for $A_\infty(l_1 : l_1)$.

1. Introduction

Throughout this paper we will just consider complex Banach spaces. For a Banach space $E$, $S_E$ and $B_E$ will be the unit sphere and the closed unit ball of $E$, respectively. If $E$ and $F$ are Banach spaces, $C_b(B_E : F)$ denotes the Banach space of the bounded continuous functions $f : B_E \to F$, endowed with the supremum norm. In the case that $F = \mathbb{C}$, we write $C_b(B_E)$. An $N$-homogeneous polynomial $P$ from $E$ to $F$ is a mapping such that there is an $N$-linear (and bounded) mapping $L$ from $E$ to $F$ satisfying

$$P(x) = L(x, \ldots, x), \quad \forall x \in E.$$ 

The set of all $N$-homogenous polynomials from $E$ to $F$ is denoted by $P^N(E : F)$. We denote by $A_\infty(B_E : F)$ the Banach space of the bounded continuous function $f : B_E \to F$ such that $f$ is holomorphic on the open unit ball, endowed with the supremum norm. If $F = \mathbb{C}$, we write $A_\infty(B_E)$. A result of Šilov asserts that if $A$ is a unital separating subalgebra of $C(K)$ ($K$ is a compact Hausdorff topological space), there is a smallest closed subset $S \subset K$ such that every function of $A$ attains its norm at some point of $S$ ([7], Theorem I.4.2). Bishop [5] proved that if $K$ is metrizable, in fact, there is a minimal subset of $K$ satisfying the above condition for every separating subalgebra of $C(K)$. That subset is the set of peak points for $A$. Globevnik [8] introduced the corresponding concepts of the boundary of a subalgebra $A$ of $C_0(\Omega)$, the set of bounded and continuous functions on a topological space $\Omega$ not necessarily compact, and studied


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4339
A \subset C

from a bounded and holomorphic function defined on a Banach space and the corresponding boundaries for \( \Omega = B_{c_0} \) and \( A \) a certain space of holomorphic functions. A subset \( B \subset E \) is a boundary for a subspace \( A \subset C_b(B_E) \) if

\[
\|f\| = \sup_{z \in B} |f(z)|, \quad \forall f \in A.
\]

An element \( x \in S_E \) is called a peak point for \( A \) if there exists \( h \in A \) such that \( |h(x)| = \|h\|_{B_E} \) and \( |h(z)| < \|h\|_{B_E} \) for every \( z \in B_E \setminus \{x\} \). In this case we say that \( h \) peaks at \( x \). A peak point \( x \) is called a strong peak point for \( A \) if there exists \( h \in A \) such that \( |h(x)| = \|h\|_{B_E} \) and, given \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for every \( z \in B_E \) with \( \|z - x\| > \epsilon \), we have \( |h(z)| < \|h\|_{B_E} - \delta \). In this case we say that \( h \) peaks strongly at \( x \).

In 1971, Harris [3] introduced the definition of a spatial numerical range for a bounded and holomorphic function defined on a Banach space and the corresponding concept of numerical radius. The spatial numerical range of a bounded function \( f \) from \( B_E \) to \( E \) is given by

\[
W(f) := \{ x^* (f(x)) : (x, x^*) \in \Pi(E) \},
\]

where we denoted by \( \Pi(E) \) the following subset:

\[
\Pi(E) := \{ (x, x^*) \in S_E \times S_E^* : x^*(x) = 1 \}.
\]

The numerical radius \( v(f) \) is just the number

\[
v(f) := \sup \{ |\lambda| : \lambda \in W(f) \}.
\]

For more background and information about numerical ranges and radii, we refer the reader to [6]. For a linear space \( A \subset C_b(B_E : E) \), M. Acosta and the author [3] gave the corresponding definition of numerical boundary for \( A \). We say that \( B \subset \Pi(E) \) is called a numerical boundary for \( A \) if

\[
\sup_{(x, x^*) \in B} |x^* (f(x))| = v(f), \quad \forall f \in A.
\]

In the case that \( B \) is a \( (\| \| \times \| \cdot \|) \)-closed numerical boundary for \( A \) that is minimal under the previous conditions, \( B \) is said to be the numerical \( \check{S} \)ilov boundary. In [3], the authors studied numerical boundaries for holomorphic functions on some classical Banach spaces. Parallel to the concepts of peak and strong peak-points, we introduce the corresponding definition of numerical and numerical strong-peak points. An element \( (x, x^*) \in \Pi(E) \) is called a numerical peak point for \( A \) if there exists \( h \in A \) such that \( |x^* (h(x))| = v(h) \) and \( |z^* (h(z))| < v(h) \) for every \( (z, z^*) \in \Pi(E) \setminus \{(x, x^*)\} \). In this case we say that \( h \) peaks numerically at \( (x, x^*) \). Also, \( (x, x^*) \in \Pi(E) \) is called a numerical strong-peak point for \( A \) if there exists a function \( h \in A \) such that \( |x^* (h(x))| = v(h) \), and for any \( (x_n, x_n^*) \in \Pi(E) \) with \( \lim_{n \to \infty} x_n^* (h(x_n)) = v(h) \) we have that \( x_n \to x \) in norm and \( x_n^* \to x^* \) in \( w^* \)-topology. In this case we say that \( h \) peaks strongly numerically at \( (x, x^*) \). It is clear that every numerical strong-peak point is a numerical peak point. Note that if \( E \) is a finite dimensional space, then every numerical peak point is also a numerical strong-peak point. It is immediate from definition that every \( (\| \| \times \| \cdot \|) \)-closed numerical boundary for \( A_{\infty}(B_E : E) \) contains all numerical strong-peak points.

In Section 2, we characterize the numerical peak points for \( A_{\infty}(B_{l_1} : l_1) \) and prove that every numerical peak point for \( A_{\infty}(B_{l_1} : l_1) \) is also a numerical strong-peak point. We prove that \( \{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\} \) is the numerical \( \check{S} \)ilov boundary for \( A_{\infty}(B_{l_1} : l_1) \).
In Section 3, we characterize the numerical peak points for \( A_\infty(B_{C(K)} : C(K)) \) when \( K \) is a compact metrizable space. If \( K \) is an infinite compact Hausdorff topological space, we prove that there are no numerical strong-peak points for \( A_\infty(B_{C(K)} : C(K)) \).

2. Numerical peak points and numerical Šilov boundary for holomorphic functions on \( l_1 \)

Lemma 2.1. If \((x, x^*) \in \Pi(l_1)\) is a numerical peak point for \( A_\infty(B_{l_1} : l_1) \), then \( |x^*(e_n)| = 1 \) for all \( n \in \mathbb{N} \).

Proof. Let \( x := (v_n)_n \in S_{l_1}, x^* := (w_n)_n \in S_{l_\infty} \). Assume that there exists a positive integer \( n_0 \) such that \( |w_{n_0}| < 1 \). Since the subset of peak points in \( S_E \) for \( A_\infty(B_{l_1}) \) is invariant under surjective linear isometries on \( l_1 \), we may assume that \( n_0 = 1 \), so \( |w_1| < 1 \). Let \( h \in A_\infty(B_{l_1} : l_1) \) peak numerically at \((x, x^*)\) with \( h := (h_n)_n \) for some \( h_n \in A_\infty(B_{l_1}) \). Let \( x^*_1 := \lambda v_1 + \sum_{n \geq 1} w_nh_n \) for \( |\lambda| \leq 1 \). We claim that \( v_1 \neq 0 \). If \( v_1 = 0 \), then \((x, x^*_1) \in \Pi(l_1)\) for \( |\lambda| \leq 1 \). We define the polynomial \( \psi : \overline{D}(0, 1) \to \mathbb{C} \) by

\[
\psi(\lambda) := \lambda h_1(x) + \sum_{n > 1} w_nh_n(x) \quad (|\lambda| \leq 1).
\]

Since \( \max_{|\lambda| \leq 1} |\psi(\lambda)| = v(h) = |x^*(h(x))| = |\psi(w_1)| \), by the Maximum Modulus Theorem, \( \psi(\lambda) = v(h) \) for all \( |\lambda| \leq 1 \). Choose any complex number \( \beta \) with \( |\beta| = 1 \). Then \((x, x^*_\beta) \neq (x, x^*) \in \Pi(l_1) \) and \( |\psi(\beta)| = |x^*_\beta(h(x))| = v(h) \). Since \( h \) peaks numerically at \((x, x^*)\), we have a contradiction. Thus \( v_1 \neq 0 \). We define the 1-degree polynomial \( u : \overline{D}(0, 1) \to \mathbb{C} \) by

\[
u(\lambda) := \lambda v_1 + \sum_{n > 1} w_nv_n \quad (|\lambda| \leq 1).
\]

Since \( 1 = x^*(x) = |u(w_1)| \), by the Maximum Modulus Theorem, \( u(\lambda) = 1 \) for all \( |\lambda| \leq 1 \). By the same reason as in the above argument, \((x, x^*_\beta) \in \Pi(l_1) \) and \( |\psi(\beta)| = |x^*_\beta(h(x))| = v(h) \) for any \( \beta \in \mathbb{C} \) with \( |\beta| = 1 \). Thus we have a contradiction. Therefore \( |w_1| = 1 \). \( \square \)

Theorem 2.2. \( S := \{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\} \) is the set of all numerical strong-peak points for the space of 2-degree polynomials in \( A_\infty(B_{l_1} : l_1) \) and every numerical peak point for the space of 2-degree polynomials in \( A_\infty(B_{l_1} : l_1) \) is a numerical strong-peak point.

Proof. By Lemma 2.1, it is enough to show the first statement of the theorem. Let \((y_0, y_0^*) \in S \). We will prove that \((y_0, y_0^*) \) is a numerical strong-peak point for the space of 2-degree polynomials in \( A_\infty(B_{l_1} : l_1) \). Let \( J = \text{supp}(y_0) \). The subset of peak points in \( S_E \) for \( A_\infty(B_{l_1}) \) is invariant under surjective linear isometries on \( l_1 \), so we can assume that \( y_0(k) > 0 \) for all \( k \in J \). Let \( a_n := y_0^*(e_n) \) for all \( n \in \mathbb{N} \). Then \( a_k = 1 \) for all \( k \in J \). By Theorem 2.6 in [4] there exists a 2-degree polynomial \( f \in A_\infty(B_{l_1}) \) which peaks strongly at \( y_0 \) with \( \|f\| = 1 \). Let \( \lambda_n := \text{sgn}(a_n) \) and \( z_0 := \sum_{n=1}^\infty \lambda_n b_ne_n \in S_{l_1} \) with \( b_n > 0 \) for all \( n \in \mathbb{N} \). We define a 2-degree polynomial \( h \in A_\infty(B_{l_1} : l_1) \) by \( h(x) := f(x)z_0 \). Clearly \( y_0^*(h(y_0)) = 1 = \|h\| = v(h) \). We claim that \( h \) peaks strongly at \( (y_0, y_0^*) \).

Consider a sequence \( \{(x_n, x_n^*)\} \in \Pi(l_1) \) such that \( \lim_{n \to \infty} |x_n^*(h(x_n))| = v(h) \). We
will prove that \( x_n \to y_0 \) in norm and \( x^*_n \to y^*_0 \) in \( w^* \)-topology. Since
\[
1 = \lim_{n \to \infty} |x^*_n(h(x_n))| \leq \lim_{n \to \infty} |f(x_n)| \leq 1
\]
and \( f \) peaks strongly at \( y_0 \), \( x_n \to y_0 \) in norm. Write \( x^*_n := (a^{(n)}_j)_j \in S_{l^*} \) and \( x_n := (c^{(n)}_j)_j \in S_l \) for all \( n, j \in \mathbb{N} \). Let \( x := (d_j) \in l_1 \). First we claim that \( \lim_{n \to \infty} a^{(n)}_k = 1 \) for all \( k \in J \). Let \( i_0 \in J \) be fixed. Let \( \{a^{(n)}_{i_0}\} \) be any subsequence of \( \{a^{(n)}_{i_0}\} \). As the set \( \{a^{(n)}_{i_0}\} \) is a bounded subset of \( \mathbb{C} \), there exist a subsequence \( \{a^{(n_{i_0})}_{i_0}\} \) and a complex number \( r \) with \( |r| \leq 1 \) such that \( \lim_{n \to \infty} a^{(n_{i_0})}_{i_0} = r \). Since \( 1 = x^*_n(x_n) \) for all \( n \in \mathbb{N} \), we have
\[
1 = \sum_{j=1}^{\infty} a^{(n_{i_0})}_{i_0} c^{(n_{i_0})}_j \quad \text{for all } l \in \mathbb{N}.
\]
Since \( x_n \to y_0 \) in norm, we have \( \lim_{n \to \infty} c^{(n_{i_0})}_{i_0} = y_0(i_0) > 0 \). For a sufficiently large \( l \), \( c^{(n_{i_0})}_{i_0} > 0 \). It follows that, for a sufficiently large \( l \),
\[
1 = \left| a^{(n_{i_0})}_{i_0} \right| c^{(n_{i_0})}_{i_0} + \sum_{j \neq i_0} a^{(n_{i_0})}_j c^{(n_{i_0})}_j \\
\leq \left| a^{(n_{i_0})}_{i_0} \right| c^{(n_{i_0})}_{i_0} + \sum_{j \neq i_0} |a^{(n_{i_0})}_j| |c^{(n_{i_0})}_j| \\
\leq \sum_{j=1}^{\infty} |c^{(n_{i_0})}_j| = 1.
\]
Thus \( 1 = a^{(n_{i_0})}_{i_0} = |a^{(n_{i_0})}_j| \) for a sufficiently large \( l \), so \( r = 1 \). Therefore \( \lim_{n \to \infty} a^{(n)}_{i_0} = 1 \). We claim that \( \lim_{n \to \infty} a^{(n)}_j = a_j \) for all \( j \in \mathbb{N} \setminus J \). Let \( j_0 \in \mathbb{N} \setminus J \) be fixed. Let \( \{a^{(n_{j_0})}_{j_0}\} \) be any subsequence of \( \{a^{(n)}_{j_0}\} \). As the set \( \{a^{(n_{j_0})}_{j_0}\} \) is a bounded subset of \( \mathbb{C} \), there exist a subsequence \( \{a^{(n_{j_0})}_{j_0}\} \) and a complex number \( \rho \) with \( |\rho| \leq 1 \) such that \( \lim_{n \to \infty} a^{(n_{j_0})}_{j_0} = \rho \). Since
\[
1 = \lim_{n \to \infty} |x^*_n(h(x_n))| \leq \lim_{n \to \infty} |f(x_n)| |x^*_n(z_0)| \leq 1,
\]
we have
\[
1 = \lim_{n \to \infty} |x^*_n(z_0)| = \lim_{n \to \infty} \left| \sum_{j=1}^{\infty} a^{(n)}_j \lambda_j b_j \right|.
\]
It follows that for fixed \( i_0 \in J \),
\[
1 = \lim_{l \to \infty} \left| \sum_{j=1}^{\infty} a^{(n_{i_0})}_j \lambda_j b_j \right| \\
\leq \lim_{l \to \infty} \left| a^{(n_{j_0})}_{j_0} \lambda_j b_{j_0} + a^{(n_{i_0})}_{i_0} b_{i_0} \right| + \sum_{j \in \mathbb{N} \setminus \{i_0, j_0\}}^{\infty} b_j \\
\leq b_{j_0} + b_{i_0} + \sum_{j \in \mathbb{N} \setminus \{i_0, j_0\}} b_j = 1.
\]
Thus
\[ \lim_{l \to \infty} |a^{(n)}_{j_0} \lambda_{j_0} b_{j_0} + a^{(n)}_{i_0} b_{i_0}| = |\rho \lambda_{j_0} b_{j_0} + b_{i_0}| = b_{j_0} + b_{i_0}, \]
showing \( \rho \lambda_{j_0} = 1 \). We have \( \rho = a_{j_0} \). Therefore \( \lim_{n \to \infty} a^{(n)}_{j_0} = a_{j_0} \). Since the \( w^* \)-

By Theorem 2.3 in [3], \( \Pi(l_p) \) is the numerical \( \hat{\text{Si}}lov \) boundary for \( A_\infty(B_{l_p} : l_p) \) for each \( 1 < p < \infty \). The following is an application of Theorem 2.2 to the numerical \( \hat{\text{Si}}lov \) boundary.

**Theorem 2.3.** \( S := \{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\} \) is the numerical \( \hat{\text{Si}}lov \) boundary for \( A_\infty(B_{l_1} : l_1) \).

**Proof.** It is easy to show that \( S \) is \( (\| \| \times w^*) \)-closed. By Theorem 2.2 and the fact that every \( (\| \| \times w^*) \)-closed numerical boundary for \( A_\infty(B_{l_1} : l_1) \) contains all numerical strong-peak points, it suffices to prove that \( S \) is a numerical boundary for \( A_\infty(B_{l_1} : l_1) \). Let \( h \in A_\infty(B_{l_1} : l_1) \) and let \( \epsilon > 0 \). Choose \( (x_0, x^*_0) \in \Pi(l_1) \) such that \( v(h) - \epsilon < |x^*_0(h(x_0))| \). Let \( x_0 := (v_1)_n, x^*_0 := (w_0)_n \) and \( h := (h_n)_n \) for some \( h_n \in A_\infty(B_{l_1}) \). We claim that there exists a complex sequence \( \{\lambda_n\} \) in the unit disk such that if

\[ z^*_n := \sum_{1 \leq j \leq n} \lambda_j e_j + \sum_{j>n} w_j e_j \]

for all \( n \in \mathbb{N} \), then \( (x_0, z^*_n) \in \Pi(l_1) \) and \( |z^*_{n+1}(h(x_0))| \geq |z^*_n(h(x_0))| > v(h) - \epsilon \) for all \( n \in \mathbb{N} \). If \( |w_1| = 1 \), let \( \lambda_1 := w_1 \). Otherwise \( w_1 = 0 \); hence

\[ (x_0, \lambda_1 e_1 + \sum_{n>1} w_n e_n) \in \Pi(l_1) \text{ for all } |\lambda| \leq 1. \]

If \( h_1(x_0) = 0 \), let \( \lambda_1 := 1 \). Assume that \( h_1(x_0) \neq 0 \). We define a nonconstant 1-degree polynomial \( \psi_1 : \mathcal{D}(0,1) \to \mathbb{C} \) by

\[ \psi_1(\lambda) := \lambda h_1(x_0) + \sum_{n>1} w_n h_n(x_0) (|\lambda| \leq 1). \]

Since \( \max_{|\lambda|=1} |\psi_1(\lambda)| \geq |\psi_1(w_1)| = |x^*_0(h(x_0))| \), by the Maximum Modulus Theorem, there exists a complex number \( \lambda_1 \) with \( |\lambda_1| = 1 \) such that

\[ |\psi_1(\lambda_1)| \geq |x^*_0(h(x_0))| > v(h) - \epsilon. \]

Let \( z^*_1 := \lambda_1 e_1 + \sum_{n>1} w_n e_n \). Then \( (x_0, z^*_1) \in \Pi(l_1) \) and \( |z^*_1(h(x_0))| \geq |x^*_0(h(x_0))| > v(h) - \epsilon \). If \( |w_2| = 1 \), let \( \lambda_2 := w_2 \). Otherwise \( w_2 = 0 \); hence

\[ (x_0, \lambda_1 e_1 + \lambda_2 e_2 + \sum_{n>2} w_n e_n) \in \Pi(l_1) \text{ for all } |\lambda| \leq 1. \]

If \( h_2(x_0) = 0 \), let \( \lambda_2 := 1 \). Assume that \( h_2(x_0) \neq 0 \). We define a nonconstant 1-degree polynomial \( \psi_2 : \mathcal{D}(0,1) \to \mathbb{C} \) by

\[ \psi_2(\lambda) := \lambda_1 h_1(x_0) + \lambda h_2(x_0) + \sum_{n>2} w_n h_n(x_0) (|\lambda| \leq 1). \]

Since \( \max_{|\lambda|=1} |\psi_2(\lambda)| \geq |\psi_2(w_2)| = |\psi_1(\lambda_1)| > v(h_0) - \epsilon \), by the Maximum Modulus Theorem, there exists a complex number \( \lambda_2 \) with \( |\lambda_2| = 1 \) such that

\[ |\psi_2(\lambda_2)| \geq |\psi_1(\lambda_1)| > v(h) - \epsilon. \]

Let \( z^*_2 := \sum_{1\leq j \leq 2} \lambda_j e_j + \sum_{n>2} w_n e_n \). Then \( (x_0, z^*_2) \in \Pi(l_1) \) and \( |z^*_2(h(x_0))| \geq |z^*_1(h(x_0))| > v(h) - \epsilon \). Continuing this process, we can get a complex sequence
\{\lambda_n\} in the unit disk satisfying the claim. Let \( z^* := (\lambda_n)_{n=1}^\infty \in l_\infty \). We will show that \((x_0, z^*) \in S\). Indeed, it follows that for each \(n \in \mathbb{N}\),

\[
|z^*(x_0) - 1| = |z^*(x_0) - z^*_n(x_0)| \leq \sum_{j>n} |\lambda_j - w_j| |v_j| \leq 2 \sum_{j>n} |v_j| \to 0,
\]
as \(n \to \infty\). Thus \(z^*(x_0) = 1\). We will show that

\[
\lim_{n \to \infty} |z^*_n(h(x_0))| = |z^*(h(x_0))|.
\]

Let \(h(x_0) := (\beta_n)_{n=1}^\infty \in l_1\). It follows that

\[
|z^*(h(x_0)) - z^*_n(h(x_0))| \leq \sum_{j>n} |\lambda_j - w_j| |\beta_j| \leq 2 \sum_{j>n} |\beta_j| \to 0,
\]
as \(n \to \infty\). Thus we have

\[
v(h) - \epsilon < \limsup_{n \to \infty} |z^*_n(h(x_0))| = |z^*(h(x_0))| \leq \sup_{(x,x^*) \in S} |x^*(h(x))| \leq v(h),
\]
which shows that \(\sup_{(x,x^*) \in S} |x^*(h(x))| = v(h)\). Since \(h \in A_\infty(B_{l_1}: l_1)\) is arbitrary, we complete the proof. \(\square\)

3. Numerical peak points and numerical strong-peak points on \(C(K)\)

**Theorem 3.1.** Let \(K\) be a compact Hausdorff topological space with at least two points. If \((x_0, x^*_0) \in \Pi(C(K))\) is a numerical peak point for \(A_\infty(B_{C(K)}: C(K))\), then:

(a) There exists a unique \(t_0 \in K\) such that

\[
|x_0(t)| = 1, \ \forall t \in K \text{ and } x^*_0 = \text{sign}(x_0(t_0))\delta_{t_0}.
\]

Hence \(x_0\) is an extreme point of \(B_{C(K)}\).

(b) \(x_0\) is a peak point for \(A_\infty(B_{C(K)})\).

**Proof.** Let \(h \in S_{A_\infty(B_{C(K)}: C(K))}\) peak numerically at \((x_0, x^*_0)\). Then \(v(h) = |x^*_0(h(x_0))|\). By Theorem 2.7 in [2], \(1 = v(h) = \|h\|\). Choose an element \(t_0 \in K\) such that \(1 = \|h(x_0)\| = \|\delta_{t_0} \circ h(x_0)\| = \|\delta_{t_0} \circ h\|\). We claim that \(|x_0(t)| = 1, \ \forall t \in K\).

For every complex number \(\lambda\) in the unit disk, the function \(x_0 + \lambda(1 - |x_0|) \in C(K)\), and for every \(t \in K\), it is satisfied that \(|x_0(t) + \lambda(1 - |x_0(t)|)| \leq 1\). Define the continuous function \(\phi : \overline{D}(0,1) \to \mathbb{C}\) by

\[
\phi(\lambda) := \delta_{t_0}(h(x_0 + \lambda(1 - |x_0|))) \ (|\lambda| \leq 1).
\]

Note that \(\phi\) is holomorphic on \(D(0,1)\) and \(|\phi(\lambda)| \leq 1\) for every \(\lambda\) in the unit disk. Also \(|\phi(0)| = |\delta_{t_0} \circ h(x_0)| = 1\). Since \(\phi\) attains its maximum modulus at 0, \(\phi\) is constant. We choose a complex number \(\lambda_0\) satisfying the facts that \(|\lambda_0| = 1\) and \(|x_0(t_0) + \lambda_0(1 - |x_0(t_0)|)| = 1\). So \(\phi(\lambda) = \phi(\lambda_0)\). The element \(z_0 := x_0 + \lambda_0(1 - |x_0|)\) is in the unit ball of \(C(K)\) and \(|z_0(t_0)| = 1\). Then \((z_0, z^*_0) \in \Pi(C(K))\). Since \(|\phi(\lambda_0)| = |\delta_{t_0}(h(x_0)| = v(h) = 1\), we have \(z_0 = x_0\) and \(x^*_0 = x^*_0 \delta_{t_0}\). Thus \(\lambda_0(1 - |x_0(t)|) = 0\) for all \(t \in K\), so \(|x_0(t)| = 1\) for all \(t \in K\). So \(x^*_0 = \text{sign}(x_0(t_0))\delta_{t_0}\).

The uniqueness of \(t_0\) follows from Urysohn’s Lemma. Therefore we have proved assertion (a).
We will show that \( \delta_{t_0} \circ h \) peaks at \( x_0 \). Let \( y \in S_{C(K)} \) such that \( 1 = |\delta_{t_0} \circ h(y)| = \|\delta_{t_0} \circ h\|. \) By the same argument as in the proof of assertion (a), we have \( |y(t_0)| = 1 \). Note that \((y, sign(y(t_0))\delta_{t_0}) \in \Pi(C(K))\). Since
\[
1 = |sign(y(t_0))\delta_{t_0}(h(y))| = \|\delta_{t_0} \circ h\| = \|h\| = v(h),
\]
we have \( y = x_0 \) and \( x_0^* = sign(y(t_0))\delta_{t_0} \), which show assertion (b). \( \square \)

**Theorem 3.2.** (1) Let \( K \) be a compact metrizable space.

Then \( M := \{(x, sign(x(t))\delta_t) : x \in extB_{C(K)}, t \in K \} \) is the set of all numerical peak points for the space of 1-degree polynomials in \( A_{\infty}(B_{C(K)} : C(K)) \).

(2) If \( K \) is any infinite compact topological space, then there are no numerical strong-peak points for \( A_{\infty}(B_{C(K)} : C(K)) \).

**Proof.** (1): By (a) of Theorem 3.1, it suffices to show that if \((x_0, sign(x(t_0))\delta_{t_0}) \in M\), then it is a numerical peak point for the space of 1-degree polynomials in \( A_{\infty}(B_{C(K)} : C(K)) \). Since the subset of peak points in \( S_{C(K)} \) for \( A_{\infty}(B_{C(K)} : C(K)) \) is invariant under surjective linear isometries on \( C(K) \), we can assume that \( x_0(t) = 1 \) for all \( t \in K \). Since \( K \) is a metrizable space, there exists a dense subset \( \{t_n\} \) in \( K \).

Choose \((\alpha_n) \in S_1\) with \( \alpha_n > 0 \) for all \( n \in \mathbb{N} \). We claim that there exists a function \( y_0 \in S_{C(K)} \) such that \( y_0(t_0) = 1 \) and \( 0 \leq y_0(t) < 1 \) for all \( t \in K \setminus \{t_0\} \). Indeed, let \( d \) be a metric in \( K \). Let
\[
A_n := \{t \in K : d(t, t_0) \geq \frac{1}{n} \} \quad (n \in \mathbb{N}).
\]

Clearly \( A_n \) is a closed subset of \( K \) with \( t_0 \notin A_n \) for all \( n \in \mathbb{N} \). By the Tietze Extension Theorem, for each \( n \in \mathbb{N} \), there exists a sequence \( \{z_n\} \) in \( C(K) \) such that \( 0 \leq z_n \leq 1 \), \( z_n(t_0) = 1 \) and \( z_n(A_n) = \{0\} \). We define the function
\[
y_0(t) := \sum_{n=1}^{\infty} \frac{1}{2^n} z_n(t) \quad (t \in K).
\]

We define a 1-degree polynomial \( h \in A_{\infty}(B_{C(K)} : C(K)) \) by
\[
h(x) := \sum_{n=1}^{\infty} \alpha_n(1 + x(t_n))y_0 \quad (x \in C(K)).
\]

We claim that \( h \) peaks numerically at \((x_0, \delta_{t_0})\). Let \((z_0, z_0^*) \in \Pi(C(K))\) be such that \( v(h) = |z_0^*(h(z_0))| \). Since \( 2 \geq \|h\| \geq v(h) \geq \delta_{t_0}(h(x_0)) = 2 \), we have \( 2 = v(h) = \|h\| \).

Since \( 2 = |z_0^*(h(z_0))| = \|h(z_0)\| \), we have \( z_0(t_n) = 1 \) for all \( n \in \mathbb{N} \). Since \( \{t_n\} \) is a dense subset of \( K \), \( z_0(t) = 1 = x_0(t) \) for all \( t \in K \). Thus \( z_0 = x_0 \). By the Riesz Representation Theorem on \( C(K)^* \), there exists a unique regular complex Baire measure \( \mu = v + iw \) on \( K \) (\( v \) and \( w \) are positive measures) satisfying
\[
z_0^*(x) = \int_K x(t) \, d\mu = \int_K x(t) \, dv + i \int_K x(t) \, dw \quad (x \in C(K))
\]
with \( \|z_0^*\| = |\mu| = 1 \). Since
\[
1 = z_0^*(x_0) = \int_K x_0 \, dv + i \int_K x_0 \, dw = v(K) + iw(K),
\]

\( v(K) = 1 \), \( w(K) = 0 \). Thus \( w = 0 \) and
\[
z_0^*(x) = \int_K x(t) \, dv \quad (x \in C(K)).
\]
It follows that
\[ 2 = |z_0^*(h(x_0))| = | \int_K h(x_0) dv | = 2 | \int_K y_0(t) dv | \leq 2 | \int_K |y_0(t)| dv | \leq 2. \]
Thus \( 1 = \int_K |y_0(t)| dv \). We claim that \( v(\{t_0\}) = 1 \). Otherwise, by the regularity of \( v \) and the choice of \( y_0 \), there exists an open subset \( \theta_0 \) of \( K \) containing \( t_0 \) such that \( v(K \setminus \theta_0) > 0 \). Let \( \delta_0 := \max_{t \in K \setminus \theta_0} | y_0(t) | < 1 \). It follows that
\[ 1 = \int_K |y_0(t)| dv \leq v(\theta_0) + \delta_0 \ v(K \setminus \theta_0) < v(K) = 1, \]
which is impossible. Thus \( v(\{t_0\}) = 1 \) and \( v(K \setminus \{t_0\}) = 0 \). Therefore, we have
\[ z_0^*(x) = \int_K x(t) \ dv = x(t_0) \ v(\{t_0\}) = x(t_0) = \delta_0(x) \]
for all \( x \in C(K) \), showing \( z_0^* = \delta_0 \).

(2): By (a) of Theorem 3.1, it suffices to show that if \( (x_0, x_0^*) \in \Pi(C(K)) \) with \( x_0 \in \text{ext}B_{C(K)} \), then \( (x_0, x_0^*) \) is not a numerical strong-peak point. Let \( \{t_n\} \) in \( K \) be the sequence such that there is a sequence \( \{x_n\} \) in \( B_{C(K)} \) such that \( 0 \leq x_n \leq 1, x_n(t_n) = 1, \forall n \) and \( \text{supp}(x_n) \cap \text{supp}(x_m) = \emptyset \) \( (n \neq m) \). Let \( h \in A_\infty(B_{C(K)} : C(K)) \) such that \( 1 = v(h) = |x_0^*(h(x_0))| \). We will show that \( h \) cannot peak strongly at \( (x_0, x_0^*) \). By Theorem 2.7 in \( \text{[2]} \), we have \( v(h) = \| h \| = 1 \). There exists a \( t_0 \in K \) such that \( \text{sign}(x_0(t_0)) \delta_{t_0} \circ h \in A_\infty(B_{C(K)}) \) and
\[ 1 = \| h(x_0) \| = | \delta_{t_0}(x_0) | = | \text{sign}(x_0(t_0)) \delta_{t_0}(h(x_0)) | = \| \delta_{t_0} \circ h \|. \]
Let \( z_n := x_0(1 - x_n) \) for all \( n \in \mathbb{N} \). Since the support of \( x_n \) are pairwise disjoint, there is a positive integer \( N \) such that \( x_n(t_0) = 0 \) for all \( n > N \). Let \( \lambda_n := \text{sign}(z_n(t_0)) \) for all \( n > N \). Thus \( \{z_n, \lambda_n, \delta_{t_0}\} \) in \( \Pi(C(K)) \) for all \( n > N \). Since \( \{x_n\} \) is equivalent to a \( c_0 \)-basis, then it converges weakly to 0. By the Rainwater theorem, the sequence \( \{z_n\} \) is in the unit ball of \( C(K) \) and converges weakly to \( x_0 \). Since \( C(K) \) has the Dunford-Pettis property, then it has also the polynomial Dunford-Pettis property \( \text{[10]} \), and so, if we follow the argument in the proof Proposition 4.1 in \( \text{[4]} \), then
\[ |\lambda_n \delta_{t_0} \circ h(z_n)| \to 1. \]
It follows that
\[ \| z_n - x_0 \| = \| x_0 x_n \| \geq |x_0(t_n)x_n(t_n)| = 1 \]
for all \( n \in \mathbb{N} \). Therefore, we have proved that \( (x_0, x_0^*) \) is not a numerical strong-peak point.

It is known in \( \text{[3]} \), Theorem 5.2, that there is no numerical Šilov boundary for \( A_\infty(B_{C(K)} : C(K)) \) if \( K \) is an infinite compact Hausdorff topological space.

**Corollary 3.3.** Let \( n \in \mathbb{N} \). Then \( M := \{(x, \text{sign}(x(t)) \delta_{t}) \in \Pi(l_\infty^n) : |x(k)| = 1 \text{ for all } k = 1, 2, \ldots, n, \text{ for some } t = 1, 2, \ldots, n \} \) is the numerical Šilov boundary for \( A_\infty(B_{l_\infty^n} : l_\infty^n) \).

**Proof.** (\( \subseteq \)): By Proposition 5.1 in \( \text{[3]} \), it follows.

(\( \supseteq \)): Clearly \( M \) is \( (\| \| \times w^*) \)-closed. By Theorem 3.2 (1), \( M \) is the set of all numerical peak points for \( A_\infty(B_{l_\infty^n} : l_\infty^n) \). Since \( l_\infty^n \) is finite dimensional, \( M \) is the set of all numerical strong-peak points.
References


Department of Mathematics, Kyungpook National University, Daegu 702-701, South Korea
E-mail address: sgk317@knu.ac.kr