INCOMPRESSIBILITY OF TORI TRANSVERSE TO AXIOM A FLOWS

C. A. MORALES

Abstract. We prove that a torus transverse to an Axiom A vector field that does not exhibit sinks, sources or null homotopic periodic orbits on a closed irreducible 3-manifold is incompressible. This strengthens the works of Brunella (1993), Fenley (1995), and Mosher (1992).

1. Introduction

It is well known that an Anosov vector field on a closed atoroidal 3-manifold is transitive. This follows from the fact that a torus transverse to an Anosov vector field on a closed 3-manifold is incompressible [2], [3], [15]. In this paper we extend both results to the more general class of vector fields on irreducible 3-manifolds, namely, Axiom A vector fields that do not exhibit sinks, sources or null homotopic periodic orbits. Recall that a sink of a \( C^1 \) vector field is a hyperbolic attracting closed orbit, while a source is a sink for the time-reversed vector field. See [14] where sufficient conditions for the transitivity of Axiom A vector fields on toroidal 3-manifolds are given. Let us present our results in a precise way.

Hereafter \( M, X \) and \( X_t \) will denote a 3-manifold, a \( C^1 \) vector field on \( M \) and the flow generated by \( X \) in \( M \) respectively. We say that \( M \) is closed if it is compact and has empty boundary. We say that \( M \) is irreducible if every embedded 2-sphere in \( M \) is the boundary of a 3-ball in \( M \). A two-side surface \( S \) in \( M \) is incompressible if the homomorphism \( \pi_1(S) \rightarrow \pi_1(M) \) induced by the inclusion is injective. We say that \( X \) is transitive if it has a dense orbit. A compact invariant set \( H \) of \( X \) is hyperbolic if there is a tangent bundle splitting \( T_H M = E^s_H \oplus E^X_H \oplus E^u_H \) over \( H \) such that \( E^s_H \) is contracting, \( E^u_H \) is expanding and \( E^X_H \) is the subbundle generated by \( X \). A nonwandering point of \( X \) is a point \( p \in M \) such that for every neighborhood \( U \) of \( p \) and every \( T > 0 \) there is \( t > T \) such that \( X_t(U) \cap U \neq \emptyset \). We say that \( X \) is Axiom A if its nonwandering set is hyperbolic with dense closed orbits, and Anosov if \( M \) is a hyperbolic set. An Anosov vector field is Axiom A but not conversely. See [8], [6], [10], [17] for details. Our main result is the following.

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Theorem 1.1. Let \( X \) be an Axiom A vector field that does not exhibit sinks, sources or null homotopic periodic orbits on a closed irreducible 3-manifold. Then, every torus transverse to \( X \) is incompressible.

Let us state two short corollaries of Theorem 1.1. Recall that a closed 3-manifold is atoroidal if it has no incompressible tori \([8, 10]\).

Corollary 1.1. Let \( X \) be an Axiom A vector field that does not exhibit sinks, sources or null homotopic periodic orbits on a closed irreducible 3-manifold \( M \). If \( M \) is atoroidal, then \( X \) is a transitive Anosov vector field.

The following corollary was proved early in \([2, 3, 15]\).

Corollary 1.2. Anosov vector fields on closed atoroidal 3-manifolds are transitive.

Theorem 1.1 becomes false if we remove the hypothesis that \( X \) does not exhibit null homotopic periodic orbits \([1]\). On the other hand, if all closed 3-manifolds exhibiting an Axiom A vector field as in the theorem were irreducible, then we could remove the hypothesis that \( M \) is irreducible from the theorem. However, nonirreducible closed 3-manifolds exhibiting such Axiom A vector fields do exist \([13]\). Finally, Corollary 1.1 becomes false if we remove the hypothesis that \( M \) is atoroidal. Indeed, a counterexample can be constructed as in \([4]\).

2. Proofs

A solid torus is a compact 3-manifold diffeomorphic to the product of a 2-disk and a circle. The boundary and the interior of a manifold \( U \) will be denoted by \( \partial U \) and \( \text{Int}(U) \) respectively. The following elementary lemma deals with solid tori inside a solid torus.

Lemma 2.1. Let \( ST^1 \) and \( ST^2 \) be two solid tori contained in a solid torus \( ST \). If \( \partial(ST^1) \subset \text{Int}(ST^2) \), then \( ST^1 \subset \text{Int}(ST^2) \).

Proof. By contradiction assume that \( ST^1 \not\subset \text{Int}(ST^2) \). We claim that \( \partial(ST^2) \subset \text{Int}(ST^1) \). Indeed, \( ST^1 \cap \partial(ST^2) \neq \emptyset \) since \( ST^1 \) is connected and \( \partial(ST^2) \) separates \( ST \). Note that \( ST^1 \cap \partial(ST^2) \) is closed in \( \partial(ST^2) \) since \( ST^1 \) and \( \partial(ST^2) \) are closed in \( \partial(ST^2) \). Moreover, \( ST^1 \cap \partial(ST^2) \) is open in \( \partial(ST^2) \) since \( ST^1 \cap \partial(ST^2) = \text{Int}(ST^1) \cap \partial(ST^2) \). As \( \partial(ST^2) \) is connected we conclude that \( ST^1 \cap \partial(ST^2) = \partial(ST^2) \) and then \( \partial(ST^2) \subset ST^1 \). Actually we have \( \partial(ST^2) \subset \text{Int}(ST^1) \) since \( \partial(ST^1) \subset \text{Int}(ST^2) \), and so \( \partial(ST^2) \cap \partial(ST^1) = \emptyset \). This proves the claim.

As \( \partial(ST^2) \subset \text{Int}(ST^1) \) (by the claim) and \( \partial(ST^1) \subset \text{Int}(ST^2) \) (by hypothesis), we conclude that \( ST^1 \cup ST^2 \) is a closed 3-manifold. Obviously \( ST^1 \cup ST^2 \subset ST \), and so \( ST^1 \cup ST^2 \) is open and closed in \( ST \). Since \( ST \) is connected we conclude that \( ST^1 \cup ST^2 = ST \). But this is a contradiction since \( ST \) has boundary and \( ST^1 \cup ST^2 \) does not. The result follows.

The following lemma deals with compact 3-manifolds with toral boundary inside a solid torus.

Lemma 2.2. Let \( ST \) be a solid torus and let \( U \subset ST \) be a compact connected 3-manifold whose boundary \( \partial U = T_1 \cup \cdots \cup T_n \) is a union of tori. If each \( T_i \) bounds a solid torus \( ST_i \) in \( ST \), then there is \( i_0 \in \{1, \cdots, n\} \) such that \( ST_i \cap U = T_i \), for
all \( i \neq i_0 \) in \( \{1, \cdots, n\} \), and
\[
U \cup \left( \bigcup_{i \in \{1, \cdots, n\}, i \neq i_0} ST_i \right) \subseteq ST_{i_0}.
\]

In particular, if \( n = 1 \), then \( U \) is a solid torus.

**Proof.** Note that each \( T_i \) divides \( ST \) in two connected components, one of which is the solid torus \( ST_i \). Note also that \( \text{Int}(U) \) is contained in one of these components since it is connected and \( T_i \subset \partial U \). If \( \text{Int}(U) \) is not contained in \( ST_i \), for all \( i \), then we would obtain a closed 3-manifold inside \( ST \) by just capping each \( T_i \) with \( ST_i \). But no such manifolds inside \( ST \) exist by the argument in the last part of the proof of Lemma 2.1. So, \( U \subset ST_{i_0} \) for some \( i_0 \); thus \( T_i \subset \text{Int}(ST_{i_0}) \) for each \( i \neq i_0 \) in \( \{1, \cdots, n\} \). Then, \( ST_i \subset \text{Int}(ST_{i_0}) \) for all \( i \neq i_0 \) in \( \{1, \cdots, n\} \) by Lemma 2.1. This proves the first part of the lemma.

Now assume that \( \partial U \) is formed by a single torus \( T_1 \) which bounds a solid torus \( ST_1 \) in \( ST \). By the first part we get \( i_0 = 1 \), hence \( U \subset ST_1 \). Since \( ST_1 \) and \( U \) have common boundary \( T_1 \) we conclude that \( U \) is an open subset of \( ST_1 \) relative to \( ST_1 \). But \( U \) is also a closed subset of \( ST_1 \) relative to \( ST_1 \); therefore \( U = ST_1 \) since \( ST_1 \) is connected. Therefore \( U \) is a solid torus since \( ST_1 \) is also. This proves the lemma.

Next we introduce some basic concepts in hyperbolic dynamics \[6\], \[17\]. Let \( X \) be a \( C^1 \) vector field on a closed 3-manifold \( M \). The **omega-limit set** of \( x \in M \) is defined by
\[
\omega(x) = \left\{ y = \lim_{n \to \infty} X_{t_n}(x) : t_n \text{ is a sequence converging to } \infty \text{ as } n \to \infty \right\}.
\]
A compact invariant set \( \Lambda \) is **transitive** if \( \Lambda = \omega(x) \) for some \( x \in \Lambda \). An **attractor** is a transitive set \( \Lambda \) such that
\[
\Lambda = \bigcap_{t \geq 0} X_t(U)
\]
for some compact neighborhood \( U \). This neighborhood is a **basin of attraction** of \( \Lambda \) if it is a compact 3-manifold with nonempty boundary \( \partial U \) transverse to \( X \). A proper attractor always has a basin of attraction. A **repeller** of \( X \) is an attractor for its time reversed field \( -X \). A **basin of repulsion** of a repeller \( R \) of \( X \) is a basin of attraction of \( R \) viewed as an attractor of \( -X \).

A **hyperbolic attractor** (resp. **hyperbolic repeller**) of \( X \) is a hyperbolic set which is also an attractor (resp. a repeller) of \( X \). In particular, a sink (resp. a source) is a hyperbolic attractor (resp. repeller), but not conversely. It follows from Smale’s Spectral Decomposition Theorem that every Axiom A vector field exhibits a hyperbolic attractor and a hyperbolic repeller which are proper if the vector field is not transitive.

On the other hand, if \( x \in M \) belongs to a hyperbolic set of \( X \), then the so-called Invariant Manifold Theory \[19\] asserts that the strong stable manifold of \( x \) defined by
\[
W^{ss}(x) = \left\{ y \in M : \lim_{t \to -\infty} d(X_t(y), X_t(x)) = 0 \right\}
\]
is a \( C^1 \) immersed submanifold of \( M \). Consequently, the stable manifold
\[
W^s(x) = \bigcup_{t \in \mathbb{R}} W^{ss}(X_t(x))
\]
is an immersed submanifold as well.

Now we state the concept of a half-Reeb component, which will be useful in the next lemma (for basic concepts in foliation theory including the definition of Reeb component; see [5] and the references therein). Let $F$ be a foliation in a solid torus $ST$ transverse to $\partial(ST)$. A half-Reeb component of $F$ is a saturated subset $H \subset ST$, bounded by an annulus leaf $A$ and an annulus $K \subset \partial(ST)$ with $\partial K = \partial A$, such that the double manifold $2H$ is a Reeb component of the double foliation $2F$ (see [2] and Figure 1).

The following lemma was proved in [11]. We outline its proof here for the sake of completeness.

**Lemma 2.3.** A hyperbolic attractor having a solid torus as a basin of attraction is an attracting periodic orbit (hence it is a sink).

**Proof.** It suffices to prove that the expanding subbundle of the attractor is zero dimensional. By contradiction we assume that such a subbundle is not zero dimensional. Then, both the stable and unstable subbundles of the attractor are one dimensional. It follows that the stable manifolds $W^s(x)$ form an invariant foliation $F$ in the solid torus which is transverse to the boundary.

Clearly $F$ has no closed leaves; hence $F$ has no Reeb components. On the other hand, $F$ has no half-Reeb components too, due to an argument based on [2]. It follows that $F$ has neither Reeb nor half-Reeb components, and so the double foliation $2F$ defined in $S^2 \times S^1$ (the double of the solid torus) has no Reeb components.

But $2F$ is supported in $S^2 \times S^1$, which is a manifold with nonzero second homotopy group. So, $2F$ is the trivial product foliation of $S^2 \times S^1$ by spheres (e.g. Theorem 1.10-(iii), p. 92 in [5]), and so $F$ is the trivial product foliation by meridian disks on $ST$. This implies that the leaves of $F$ are invariant disks. Applying the classical Poincare-Bendixon Theorem [16] to one of these disks, we could find a singularity in the solid torus. However, such singularities cannot exist due to the continuity of the hyperbolic splitting. This contradiction proves the result. □

The previous lemmas will be used to prove the following one.

**Lemma 2.4.** Let $X$ be an Axiom A vector field on a closed 3-manifold $M$. If there is a solid torus in $M$ whose boundary is transverse to $X$, then $X$ exhibits either a sink or a source or a null homotopic periodic orbit.
Proof. Let \( ST \) be the solid torus in the statement of the lemma. We can assume that \( X \) points inward to \( ST \) in \( \partial(ST) \), for, otherwise, we replace \( X \) by \(-X\) in the argument below. Then, the Spectral Decomposition Theorem [6] implies that there is a hyperbolic attractor \( \Lambda \) of \( X \) in \( ST \). Of course we can assume that \( \Lambda \) is not a sink, for, otherwise, we are done. Note that \( X \) is not transitive since it points inward to \( ST \) in \( \partial(ST) \). Therefore \( \Lambda \) is proper, and so it exhibits a basin of attraction \( U \) contained in \( ST \). As \( \Lambda \) is not a sink we have that \( \partial U \) is a union of tori transverse to \( X \).

We claim that if \( X \) does not exhibit null homotopic periodic orbits, then every torus in \( \partial U \) bounds a solid torus in \( ST \). By contradiction, assume that there is a torus in \( \partial U \) which does not bound a solid torus in \( ST \). As \( ST \) is irreducible atoroidal we have that such a torus is contained in a 3-ball in \( ST \) (e.g. [8], [10] or (4), p. 11 in [7]). Then such a torus divides \( ST \) in two connected components, one of which contained in the ball (see for instance [13]). Hence we could find a periodic orbit inside the ball, a contradiction because \( X \) does not exhibit null homotopic periodic orbits. This contradiction proves the claim.

Hereafter we assume that \( X \) does not exhibit null homotopic periodic orbits. Then, the previous claim implies that every torus in \( \partial U \) bounds a solid torus in \( ST \). Therefore we can apply Lemma 2.2 to \( U \). Replacing \( ST \) by the solid torus \( ST_0 \) in that lemma, we can assume that \( \partial(ST) \) is one of the boundary components of \( U \). If \( U \) has no more boundary components apart from \( \partial(ST) \), then \( U \) is a solid torus by Lemma 2.2 since \( n = 1 \) in such a case. So, Lemma 2.2 would imply that \( \Lambda \) is a sink, a contradiction since we have assumed that \( \Lambda \) is not a sink. Therefore, there is another torus \( T^1 \) in \( \partial U \). Set \( ST^0 = ST \) and let \( ST^1 \) be the solid torus bounded by \( T^1 \) in \( ST^0 \) (it exists by the claim). Note that \( X \) points outward from \( ST^1 \) at \( T^1 \); therefore there is a repeller \( \Lambda^1 \) in the spectral decomposition of \( X \) inside \( ST^1 \). Of course we can assume that \( \Lambda^1 \) is not a source, for, otherwise, we are done.

By repeating the above argument we get a nested sequence of solid tori

\[
ST^0 \subset ST^1 \subset ST^2 \subset \cdots \subset ST^k \subset \cdots
\]

with \( ST^i \setminus Int(ST^{i+1}) \) containing a hyperbolic attractor of \( X \) (for \( i \) even) or a hyperbolic repeller of \( X \) (for \( i \) odd). As the number of attractors and repellers in the spectral sequence of \( X \) is finite, we have that the sequence must stop. But this occurs precisely when some of the solid tori \( ST^k \) is the basin of attraction (or repulsion) of a hyperbolic attractor (or repeller) of \( X \). Therefore, \( X \) exhibits a sink or a source by Lemma 2.3. This finishes the proof. \( \square \)

Proof of Theorem 1.1. Let \( T \) be a torus transverse to an Axiom A vector field \( X \) in a closed irreducible 3-manifold. In addition, suppose that \( X \) does not exhibit sinks, sources or null homotopic periodic orbits. As the manifold is irreducible we have that \( T \) either bounds a solid torus or is contained in a 3-ball or is incompressible. If \( T \) bounds a solid torus, then \( X \) would exhibit either a sink or a source or a null homotopic periodic orbit by Lemma 2.4. If \( T \) is contained in a 3-ball, then there is a periodic orbit in the ball, and so \( X \) would exhibit a null homotopic periodic orbit. In both cases we get a contradiction; therefore \( T \) is incompressible. This finishes the proof. \( \square \)

Proof of Corollary 1.1. Let \( X \) be an Axiom A vector field that does not exhibit sinks, sources or null homotopic periodic orbits on a closed irreducible atoroidal 3-manifold. Let us prove that \( X \) is transitive. By contradiction assume that it is not
so. Then, $X$ exhibits a proper hyperbolic attractor in its spectral decomposition. Because such an attractor is proper we have that it has a basin of attraction $U$. But the attractor cannot be a sink by hypothesis, so $\partial U$ is formed by finitely many disjoint tori transverse to $X$. Then, there is an incompressible torus by Theorem 1.1 a contradiction since the manifold is atoroidal. This contradiction proves that $X$ is transitive. As every transitive Axiom A vector field is Anosov we get the result. □

Proof of Corollary 1.2. The result follows directly from Corollary 1.1, since every Anosov vector field on a closed atoroidal 3-manifold satisfies the hypotheses of Corollary 1.1. □

References


Instituto de Matemática, Universidade Federal do Rio de Janeiro, P. O. Box 68530, 21945-970 Rio de Janeiro, Brazil.
E-mail address: Morales@impa.br