MINIMAL GENERATORS FOR SYMMETRIC IDEALS

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Abstract. Let \( R = K[X] \) be the polynomial ring in infinitely many indeterminates \( X \) over a field \( K \), and let \( \mathfrak{S}_X \) be the symmetric group of \( X \). The group \( \mathfrak{S}_X \) acts naturally on \( R \), and this in turn gives \( R \) the structure of a module over the group ring \( R[\mathfrak{S}_X] \). A recent theorem of Aschenbrenner and Hillar states that the module \( R \) is Noetherian. We address whether submodules of \( R \) can have any number of minimal generators, answering this question positively.

Let \( R = K[X] \) be the polynomial ring in infinitely many indeterminates \( X \) over a field \( K \). Write \( \mathfrak{S}_X \) (resp. \( \mathfrak{S}_N \)) for the symmetric group of \( X \) (resp. \( \{1, \ldots, N\} \)) and \( R[\mathfrak{S}_X] \) for its (left) group ring, which acts naturally on \( R \). A symmetric ideal \( I \subseteq R \) is an \( R[\mathfrak{S}_X] \)-submodule of \( R \).

Aschenbrenner and Hillar recently proved [1] that all symmetric ideals are finitely generated over \( R[\mathfrak{S}_X] \). They were motivated by finiteness questions in chemistry [3] and algebraic statistics [2]. In proving the Noetherianity of \( R \), it was shown that a symmetric ideal \( I \) has a special, finite set of generators called a minimal Gröbner basis. However, the more basic question of whether \( I \) is always cyclic (already asked by Josef Schicho [4]) was left unanswered in [1]. Our result addresses a generalization of this important issue.

Theorem 1. For every positive integer \( n \), there are symmetric ideals of \( R \) generated by \( n \) polynomials which cannot have fewer than \( n \) \( R[\mathfrak{S}_X] \)-generators.

In what follows, we work with the set \( X = \{x_1, x_2, x_3, \ldots\} \), although as remarked in [1], this is not really a restriction. In this case, \( \mathfrak{S}_X \) is naturally identified with \( \mathfrak{S}_\infty \), the permutations of the positive integers, and \( \sigma x_i = x_{\sigma i} \) for \( \sigma \in \mathfrak{S}_\infty \).

Let \( M \) be a finite multiset of positive integers and let \( i_1, \ldots, i_k \) be the list of its distinct elements, arranged so that \( m(i_1) \geq \cdots \geq m(i_k) \), where \( m(i_j) \) is the multiplicity of \( i_j \) in \( M \). The type of \( M \) is the vector \( \lambda(M) = (m(i_1), m(i_2), \ldots, m(i_k)) \). For instance, the multiset \( M = \{1, 1, 1, 2, 3, 3\} \) has type \( \lambda(M) = (3, 2, 1) \). Multisets are in bijection with monomials of \( R \). Given \( M \), we can construct the monomial:

\[
x_{\lambda(M)}^M = \prod_{j=1}^{k} x_{i_j}^{m(i_j)}.
\]

Conversely, given a monomial, the associated multiset is the set of indices appearing in it, along with multiplicities. The action of \( \mathfrak{S}_\infty \) on monomials coincides with the
Theorem 3. Let \( s_{j} \) be some polynomials \( s_{j} \) for some polynomials \( s_{j} \) with degree larger than \( d \) with corresponding distinct types \( (\lambda) \). Then the submodule \( I = \langle f_{1}, \ldots, f_{n} \rangle \subseteq R[\mathcal{S}_{\infty}] \) generated by the \( n \) polynomials, \( f_{j} = \sum c_{ij} g_{i} \) \( (j = 1, \ldots, n) \), cannot have fewer than \( r \) \( R[\mathcal{S}_{\infty}] \)-generators.

Proof. Suppose that \( p_{1}, \ldots, p_{k} \) are generators for \( I \); we prove that \( k \geq r \). Since each \( p_{l} \in I \), it follows that each is a linear combination, over \( R[\mathcal{S}_{\infty}] \), of monomials in \( G \). Therefore, each monomial occurring in \( p_{l} \) has degree at least \( d \), and, moreover, any degree \( d \) monomial in \( p_{l} \) has the same type as one of the monomials in \( G \).

Write each of the monomials in \( G \) in the form \( g_{i} = x_{i}^{n} \) for multisets \( M_{1}, \ldots, M_{n} \) with corresponding distinct types \( \lambda_{1}, \ldots, \lambda_{n} \), and express each generator \( p_{l} \) as

\[
p_{l} = \sum_{i=1}^{n} \sum_{\lambda(M) = \lambda_{i}} u_{j} x_{M}^{n} + v_{l}, \tag{1}
\]

in which \( u_{j} \in K \) with only finitely many of them nonzero, each monomial in \( v_{l} \) has degree larger than \( d \), and the inner sum is over multisets \( M \) with type \( \lambda_{i} \).

Since each polynomial in \( \langle f_{1}, \ldots, f_{n} \rangle \) is a finite linear combination of the \( p_{l} \), and since only finitely many integers are indices of monomials appearing in \( p_{1}, \ldots, p_{k} \), we may pick \( N \) large enough so that all of these linear combinations can be expressed with coefficients in the subring \( R[\mathcal{S}_{N}] \) (cf. Lemma 2). Therefore, we have

\[
f_{j} = \sum_{l=1}^{k} \sum_{\sigma \in \mathcal{S}_{N}} s_{l} f_{l} \tag{2}
\]

for some polynomials \( s_{l} \in R \). Substituting (1) into (2) gives us that

\[
f_{j} = \sum_{l=1}^{k} \sum_{\sigma \in \mathcal{S}_{N}} \sum_{i=1}^{n} \sum_{\lambda(M) = \lambda_{i}} v_{l} u_{j} x_{M}^{n} + h_{j},
\]

in which each monomial appearing in \( h_{j} \in R \) has degree greater than \( d \) and \( v_{l} f_{l} \) is the constant term of \( s_{l} f_{l} \). Since each \( f_{j} \) has degree \( d \), we have that \( h_{j} = 0 \). Thus,

\[
\sum_{i=1}^{n} c_{ij} x_{M}^{n} = \sum_{l=1}^{k} \sum_{\sigma \in \mathcal{S}_{N}} \sum_{i=1}^{n} \sum_{\lambda(M) = \lambda_{i}} v_{l} u_{j} x_{M}^{n}.
\]

Next, for a fixed \( i \), take the sum on each side in this last equation of the coefficients of monomials with the type \( \lambda_{i} \). This produces the \( n^{2} \) equations

\[
c_{ij} = \sum_{l=1}^{k} \sum_{\sigma \in \mathcal{S}_{N}} \sum_{\lambda(M) = \lambda_{i}} v_{l} u_{j} x_{M}^{n} = \sum_{l=1}^{k} \sum_{\lambda(M) = \lambda_{i}} u_{j} x_{M}^{n} \left( \sum_{\sigma \in \mathcal{S}_{N}} v_{l} \right) = \sum_{l=1}^{k} U_{i} V_{j},
\]

in which \( U_{i} = \sum_{\lambda(M) = \lambda_{i}} u_{j} x_{M}^{n} \) and \( V_{j} = \sum_{\sigma \in \mathcal{S}_{N}} v_{l} \). Set \( U \) to be the \( n \times k \) matrix \((U_{i})\) and similarly let \( V \) denote the \( k \times n \) matrix \((V_{j})\). These \( n^{2} \) equations are
represented by the equation $C = UV$, leading to the following chain of inequalities:

$$r = \text{rank}(C) = \text{rank}(UV) \leq \min\{\text{rank}(U), \text{rank}(V)\} \leq \min\{n, k\} \leq k.$$  

Therefore, we have $k \geq r$, and this completes the proof.  

\[ \square \]

References


