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ON THE COHEN-MACAULAYNESS OF FIBER CONES

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Abstract. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and I an ideal of A. Denote by $F_A(I) = \bigoplus_{n\geq 0} (I^n/\mathfrak{m}I^n)$ the fiber cone of I. This paper characterizes the multiplicity and the Cohen-Macaulayness of fiber cones in terms of minimal reductions of ideals.

1. Introduction

Throughout this paper, let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue field $k = A/\mathfrak{m}$, and Krull dimension $\dim A = d > 0$. Define $F_A(I) = \bigoplus_{n\geq 0} (I^n/\mathfrak{m}I^n)$ to be the *fiber cone* of an ideal I in A. Denote by $\ell(I) = \dim F_A(I)$ the *analytic spread* of I, by $\mu(I)$ the minimum number of generators of I, by $r = r_J(I)$ the reduction number of I with respect to a minimal reduction J, and by $v(A) = \mu(\mathfrak{m})$ the embedding dimension of A. In recent years, the Cohen-Macaulayness and other properties of special fiber rings have attracted much attention (see for instance [C-G-P-U], [C-P-V2], [Co-Z], [D-R-V], [H-H], [J-V-V1], [J-P-V]).

Using different approaches, some authors gave criteria for the Cohen-Macaulayness of fiber cones. For instance, Huneke and Sally [H-S] proved that if A is Cohen-Macaulay and I is $\mathfrak{m}$-primary with $r(I) \leqslant 1$, then $F_A(I)$ is Cohen-Macaulay. More generally, Shah [Sh1, Sh2] showed the sufficient condition that $F_A(I)$ is Cohen-Macaulay if I is an equimultiple ideal with grade $I = \operatorname{ht} I$ and $r(I) \leqslant 1$. Under the assumption that a minimal reduction J of I is generated by a regular sequence and $J \cap I^n = JI^{n-1}$ for all $n \leqslant r_J(I)$, Cortadellas and Zarzuela [C-Z] proved that the Cohen-Macaulayness of $F_A(I)$ is equivalent to $J \cap \mathfrak{m}I^n = J\mathfrak{m}I^{n-1}$ for all $n \leqslant r_J(I)$.

This paper characterizes the multiplicity (Theorem 2.3, Section 2) and the Cohen-Macaulayness (Theorem 3.1, Section 3) of fiber cones in terms of minimal reductions of ideals. As interesting consequences of main results, we get e.g. a generalization of results of Huneke-Sally and Shah [op. cit.] (Proposition 3.2, Section 3). In the case that minimal reductions of I are generated by a regular sequence, we get a result that seems to account well for the essence of the theorem of Cortadellas-Zarzuela [op. cit.] (see Corollary 3.5, Section 3).

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A ring A is said to have minimal multiplicity if $e(A) = v(A) - d + 1$. It is also known by Sally [Sa] that A is Cohen-Macaulay with minimal multiplicity if and only if A is Cohen-Macaulay and $r(\mathfrak{m}) \leqslant 1$. By using the main results, we answer the question as to when the fiber cone is Cohen-Macaulay with minimal multiplicity (Theorem 3.6, Section 3).

This paper is divided into three sections. Section 2 investigates the multiplicity of fiber cones. Section 3 is devoted to the discussion of the Cohen-Macaulayness of fiber cones.

2. The multiplicity of fiber cones

Using the concept of weak-(FC)-sequences in [Vi1], this section presents some results on the multiplicity of the fiber cone.

We first review some results and notions. Set

$$\mathfrak{a} : \mathfrak{b}^\infty = \{x \in A \mid \text{there is a positive integer } n \text{ such that } x\mathfrak{b}^n \subseteq \mathfrak{a}\}.$$

Definition ([Vi1])**.** Let I be a non-nilpotent ideal of A. Set $A^* = A/0 : I^\infty$, $\mathfrak{m}^* = \mathfrak{m}A^*$, and $I^* = IA^*$. Recall that an element $x \in I$ (x^* is the image of x in A^*) is called a weak-(FC)-element with respect to $(\mathfrak{m}, I)$ if there exists an integer n_1 such that

(FC$_1$): $x \in I \setminus \mathfrak{m}I$, and for all $n \geq n_1$ and for all non-negative integers m,

$$\mathfrak{m}^{*m} I^{*n} \cap (x^*) = \mathfrak{m}^{*m} I^{*n-1} x^*.$$

(FC$_2$): x is a filter-regular element with respect to I, i.e., $0 : x \subseteq 0 : I^\infty$.

Let $x_1, x_2, \ldots, x_t$ be a sequence in I. For each $i = 0, 1, \ldots, t-1$, set $A_i = A/(x_1, x_2, \ldots, x_i)$; $\bar{x}_{i+1}$ is the image of x_{i+1} in A_i, $I_i = IA_i$, and $\mathfrak{m}_i = \mathfrak{m}A_i$. The sequence $x_1, \ldots, x_t$ is called a weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$ if $\bar{x}_{i+1}$ is a weak-(FC)-element with respect to $(\mathfrak{m}_i, I_i)$ for each $i = 0, 1, \ldots, t-1$.

An ideal J is called a *reduction* of ideal I if $J \subseteq I$ and $I^{n+1} = JI^n$ for some n. A reduction J is called a *minimal reduction* of I if it does not properly contain any other reduction of I [N-R]. The least integer n such that $I^{n+1} = JI^n$ is called the *reduction number* of I with respect to J, and we denote it by $r_J(I)$. The reduction number of I is defined by $r(I) = \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}$.

By using the Rees lemma in [Re], the author [Vi2]–[Vi3] showed that if I is non-nilpotent, the length of any maximal weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$ is equal to $\ell(I)$, and if $x_1, x_2, \ldots, x_\ell$ is a weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$, then $(x_1, x_2, \ldots, x_\ell)$ is a minimal reduction of I. Moreover, as in the proof of Theorem 3.3 of [Vi4], then any minimal reduction of I is generated by a maximal weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$. So we have:

Note 1. An ideal $J \subseteq I$ is a minimal reduction of I if and only if J is generated by a weak-(FC)-sequence of length $\ell(I)$ in I with respect to $(\mathfrak{m}, I)$.

The following proposition plays an important role in the proofs of this paper.

Proposition 2.1. *Let I be an ideal with analytic spread $\ell(I) = \ell > 0$. Let $x_1, x_2, \ldots, x_\ell$ be a weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$. Set $J = (x_1, x_2, \ldots, x_\ell)$. For any $i < \ell$, set*

$$A_i = A/(x_1, \ldots, x_i), I_i = IA_i, P_i = (x_1, x_2, \ldots, x_i) : I^\infty, A'_i = A/P_i, I'_i = IA'_i.$$

Then

(i) $e(F_A(I)) = e(F_{A_i}(I_i)) = e(F_{A'_i}(I'_i))$.

(ii) $l[\frac{F_{A'_i}(I'_i)}{JF_{A'_i}(I'_i)}] \leqslant l[\frac{F_{A_i}(I_i)}{JF_{A_i}(I_i)}] \leqslant l[\frac{F_A(I)}{JF_A(I)}]$.

(iii) $l[\frac{F_{A'_i}(I'_i)}{JF_{A'_i}(I'_i)}] = l[\frac{F_A(I)}{JF_A(I)}]$ *if and only if* $I^n \cap P_i \subseteq JI^{n-1} (\mathrm{mod}\ \mathfrak{m}I^n)$ *for all* $1 \leqslant n \leqslant r_J(I)$.

Proof. Set $N = 0 : I^\infty$, $A^* = A/N$, $\mathfrak{m}^* = \mathfrak{m}A^*$, $I^* = IA^*$. By the Artin-Rees lemma, there exists an integer u such that $N \cap I^n = I^{n-u}(N \cap I^u) \subseteq I^{n-u}N$ for all $n \geq u$. Since $I^{n-u}N = 0$ for all large enough n, $N \cap I^n = 0$ for all large enough n. From this it follows that

$$l_{A^*}(\frac{I^{*n}}{\mathfrak{m}^*I^{*n}}) = l_A(\frac{I^n + N}{\mathfrak{m}I^n + N}) = l_A(\frac{I^n}{\mathfrak{m}I^n + N \cap I^n}) = l_A(\frac{I^n}{\mathfrak{m}I^n})$$

for all large enough n. Hence

$$e(F_A(I)) = e(F_{A^*}(I^*)). \tag{1}$$

Let $x \in I$ be a weak-(FC)-element with respect to $(\mathfrak{m}, I)$. If x^* is the image of x in A^*, then x^* is not contained in any prime ideal belonging to Ass A^* (see condition (FC_2)). Hence x^* is a non-zero divisor in A^*. Set $B = A^*/x^*A^*, \mathfrak{b} = \mathfrak{m}^*B, \mathfrak{c} = I^*B$. For all large enough n, we have

$$\begin{aligned}
l_B(\frac{\mathfrak{c}^n}{\mathfrak{b}\mathfrak{c}^n}) &= l_{A^*}(\frac{I^{*n} + (x^*)}{\mathfrak{m}^*I^{*n} + (x^*)}) = l_{A^*}(\frac{I^{*n}}{\mathfrak{m}^*I^{*n} + (x^*) \cap I^{*n}}) \\
&= l_{A^*}(\frac{I^{*n}}{\mathfrak{m}^*I^{*n} + x^*I^{*n-1}}) \text{ (see condition } (\mathrm{FC}_1)) \\
&= l_{A^*}(\frac{I^{*n}}{\mathfrak{m}^*I^{*n}}) - l_{A^*}(\frac{\mathfrak{m}^*I^{*n} + x^*I^{*n-1}}{\mathfrak{m}^*I^{*n}}) \\
&= l_{A^*}(\frac{I^{*n}}{\mathfrak{m}^*I^{*n}}) - l_{A^*}(\frac{x^*I^{*n-1}}{\mathfrak{m}^*I^{*n} \cap (x^*) \cap I^{*n}}) \\
&= l_{A^*}(\frac{I^{*n}}{\mathfrak{m}^*I^{*n}}) - l_{A^*}(\frac{x^*I^{*n-1}}{x^*\mathfrak{m}^*I^{*n-1}}).
\end{aligned}$$

Since x^* is a non-zero divisor in A^*, it follows that $l_{A^*}(\frac{I^{*n-1}}{\mathfrak{m}^*I^{*n-1}}) = l_{A^*}(\frac{x^*I^{*n-1}}{x^*\mathfrak{m}^*I^{*n-1}})$, so that

$$l_B(\frac{\mathfrak{c}^n}{\mathfrak{b}\mathfrak{c}^n}) = l_{A^*}(\frac{I^{*n}}{\mathfrak{m}^*I^{*n}}) - l_{A^*}(\frac{I^{*n-1}}{\mathfrak{m}^*I^{*n-1}})$$

for all large enough n. Hence if $\ell > 1$, then

$$e(F_{A^*}(I^*)) = e(F_B(\mathfrak{c})). \tag{2}$$

The case of $\ell(I) > 1$, $\mathfrak{c}$ is non-nilpotent in B. Set $B' = B/0 : \mathfrak{c}^\infty$ and $\mathfrak{c}' = \mathfrak{c}B'$. Then by (1) we get that

$$e(F_B(\mathfrak{c})) = e(F_{B'}(\mathfrak{c}')). \tag{3}$$

Set $A_1 = A/(x)$, $I_1 = IA_1$, $A'_1 = A/(x) : I^\infty$, and $I'_1 = IA'_1$. Note that $B' \cong A/(x) : I^\infty = A'_1$. Then by (1), we have

$$e(F_{A_1}(I_1)) = e(F_{A'_1}(I'_1)) = e(F_{B'}(\mathfrak{c}')). \tag{4}$$

Combining (4) with (3), (2) and (1), we get

$$e(F_A(I)) = e(F_{A_1}(I_1)) = e(F_{A'_1}(I'_1)). \tag{5}$$

Assume that the analytic spread $\ell = \ell(I) > 1$ and $x_1, x_2, \ldots, x_i$ $(i < \ell)$ is a weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$. Set $A_i = A/(x_1, \ldots, x_i), I_i = IA_i, A'_i = A/(x_1, \ldots, x_i) : I^\infty$, and $I'_i = IA'_i$. Then using (5) and by induction on $i < \ell = \ell(I)$, we easily show that $e(F_A(I)) = e(F_{A_i}(I_i)) = e(F_{A'_i}(I'_i))$. Note that this equation is true too in the case of $\ell = 1$ by (1). This establishes (i).

Assume that $r_J(I) = r$. Set $Q_i = (x_1, \ldots, x_i)$, $P_i = Q_i : I^\infty$. It can be verified that

$$\begin{aligned}
l[F_A(I)/JF_A(I)] &= 1 + \sum_{1 \leqslant n \leqslant r} l[I^n/(JI^{n-1} + \mathfrak{m}I^n)],\\
l[F_{A_i}(I_i)/JF_{A_i}(I_i)] &= 1 + \sum_{1 \leqslant n \leqslant r} l[Q_i + I^n/(Q_i + JI^{n-1} + \mathfrak{m}I^n)]\\
&= 1 + \sum_{1 \leqslant n \leqslant r} l[I^n/(Q_i \cap I^n + JI^{n-1} + \mathfrak{m}I^n)],\\
l[F_{A'_i}(I'_i)/JF_{A'_i}(I'_i)] &= 1 + \sum_{1 \leqslant n \leqslant r} l[P_i + I^n/(P_i + JI^{n-1} + \mathfrak{m}I^n)]\\
&= 1 + \sum_{1 \leqslant n \leqslant r} l[I^n/(P_i \cap I^n + JI^{n-1} + \mathfrak{m}I^n)].
\end{aligned}$$

It is clear that

$$(JI^{n-1} + \mathfrak{m}I^n) \subseteq (Q_i \cap I^n + JI^{n-1} + \mathfrak{m}I^n) \subseteq (P_i \cap I^n + JI^{n-1} + \mathfrak{m}I^n)$$

for all $1 \leqslant n \leqslant r_J(I)$. Hence we immediately get (ii) and (iii). □

Recall that $\mu(I)$ is the minimum number of generators of I and

$$\mu(I) = \dim_k(I/\mathfrak{m}I) = l(I/\mathfrak{m}I).$$

Note 2. It can be easily seen that if a is a non-zero divisor in A, then $\mu(aI) = \mu(I)$.

At this point we remark that: On the one hand, by Proposition 2.1 we have $e(F_A(I)) = e(F_{A_{\ell-1}}(I_{\ell-1}))$. On the other hand, since $J = (x_1, x_2, \ldots, x_\ell)$ is a minimal reduction of I, the analytic spread of $I_{\ell-1}$ is 1.

Thus the problem is reduced to the case of the fiber cone of an ideal with analytic spread $\ell = 1$. This is the reason why we need the following lemma.

Lemma 2.2. *Let I be an ideal with analytic spread $\ell(I) = 1$ and x a weak-(FC)-element in I with respect to $(\mathfrak{m}, I)$. Set $r_{xA}(I) = r$ and $I^0 = A$. Then*

(i) $e(F_A(I)) = \mu(I^n)$ *for all* $n \geq r$ *if* grade $I = 1$.

(ii) $e(F_A(I)) = \mu(\frac{I^n + 0 : I^\infty}{0 : I^\infty})$ *for all* $n \geq r$.

(iii) $e(F_A(I)) \leqslant \mu(I^r)$ *with equality if and only if* $(0 : I^\infty) \cap I^r = (0 : I^\infty) \cap \mathfrak{m}I^r$.

Proof. Since $\ell(I) = 1, l(I^n/\mathfrak{m}I^n) = \mu(I^n)$ takes a constant value for all large enough n. From this it follows that $e(F_A(I)) = \mu(I^n)$ for all large enough n. Remember that $r_{(x)}(I) = r$. Hence $I^n = I^r x^{n-r}$ for all $n > r$. Now, if grade $I > 0$, then x^{n-r} is a non-zero divisor in A. By Note 2 we have $\mu(I^n) = \mu(I^r x^{n-r}) = \mu(I^r)$ for all $n \geq r$. We get (i).

Set $A^* = A/0 : I^\infty$ and $I^* = IA^*$. We recall that by Proposition 2.1 we have $e(F_A(I)) = e(F_{A^*}(I^*))$. Since $(0 : I^\infty) : I = 0 : I^\infty$, it follows that $\operatorname{grade}_{A^*} I^* > 0$. Hence by (i) we get

$$e(F_A(I)) = e(F_{A^*}(I^*)) = \mu(I^{*r}) = \mu(\frac{I^r + 0 : I^\infty}{0 : I^\infty}).$$

This completes the proof of (ii).

It is easily seen that

$$\begin{aligned}\mu(\frac{I^r+0:I^\infty}{0:I^\infty}) &= \mu(\frac{I^r}{I^r\cap(0:I^\infty)})\\ &= l_A(\frac{I^r}{\mathfrak{m}I^r}) - l_A(\frac{I^r\cap(0:I^\infty)+\mathfrak{m}I^r}{\mathfrak{m}I^r})\\ &= \mu(I^r) - l_A(\frac{I^r\cap(0:I^\infty)}{\mathfrak{m}I^r\cap I^r\cap(0:I^\infty)})\\ &= \mu(I^r) - l_A(\frac{I^r\cap(0:I^\infty)}{\mathfrak{m}I^r\cap(0:I^\infty)}).\end{aligned}$$

Hence by (ii), $e(F_A(I)) \leqslant \mu(I^r)$ with equality if and only if $(0:I^\infty)\cap I^r = (0:I^\infty)\cap \mathfrak{m}I^r$. Thus, we get (iii). □

By combining Proposition 2.1 with Lemma 2.2, we obtain the following theorem.

Theorem 2.3. *Let $(A,\mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $k = A/\mathfrak{m}$. Let I be an ideal with analytic spread $\ell(I) = \ell > 0$ and J a minimal reduction of I. Suppose that J is generated by $x_1, x_2, \ldots, x_\ell$, a weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$. Set $J = (x_1, x_2, \ldots, x_\ell)$, $r_J(I) = r$, and $Q = (x_1, x_2, \ldots, x_{\ell-1})$. Then*

(i) $e(F_A(I)) = \mu(\frac{I^n+Q:I^\infty}{Q:I^\infty})$ *for all* $n \geq r$.

(ii) $e(F_A(I)) \leqslant \mu[(I^r+Q)/Q]$ *with equality if and only if*

$$(Q:I^\infty)\cap I^r = (Q:I^\infty)\cap \mathfrak{m}I^r (\text{mod } Q).$$

(iii) $e(F_A(I)) = \mu[(I^r+Q)/Q]$ *if* $(Q:I^\infty)\cap I^r \subseteq Q$.

(iv) $e(F_A(I)) \leqslant \mu(I^r) - \ell(I) + 1$.

Proof. Set $A_{\ell-1} = A/Q, I_{\ell-1} = IA_{\ell-1}, A'_{\ell-1} = A/Q : I^\infty$, and $I'_{\ell-1} = IA'_{\ell-1}$. Then by Proposition 2.1 we have $e(F_A(I)) = e(F_{A_{\ell-1}}(I_{\ell-1}))$, since $(x_1, x_2, \ldots, x_\ell)$ is a minimal reduction of I. Hence $\ell(I_{\ell-1}) = 1$. Lemma 2.2(ii) immediately gives $e(F_{A_{\ell-1}}(I_{\ell-1})) = \mu(I'^n_{\ell-1})$ for all $n \geq r$. Thus, we get (i).

By Lemma 2.2(iii), $e(F_{A_{\ell-1}}(I_{\ell-1})) \leqslant \mu(I^r_{\ell-1})$, since $e(F_A(I)) = e(F_{A_{\ell-1}}(I_{\ell-1}))$ and

$$\mu(I^r_{\ell-1}) = \mu[(I^r+Q)/Q],$$

it follows that $e(F_A(I)) \leqslant \mu[(I^r+Q)/Q]$, since $e(F_A(I)) = \mu[(I^r+Q)/Q]$ if and only if

$$e(F_{A_{\ell-1}}(I_{\ell-1})) = \mu(I^r_{\ell-1}).$$

Hence by Lemma 2.2(iii) we get (ii).

Since $(Q:I^\infty)\cap I^r \subseteq Q$, it follows that $Q + (Q:I^\infty)\cap I^r = Q = Q + (Q:I^\infty)\cap \mathfrak{m}I^r$. Hence by (ii) we have (iii).

Set $\mu(I^r) = s$. Since $(x_1, x_2, \ldots, x_\ell)$ is a minimal reduction of I, $(x_1^r, x_2^r, \ldots, x_\ell^r)$ is a minimal reduction of I^r. Hence there exist elements $y_1, y_2, \ldots, y_{s-\ell+1}$ in I^r such that $(x_1^r, \ldots, x_{\ell-1}^r, y_1, \ldots, y_{s-\ell+1}) = I^r$. Set $P = (y_1, y_2, \ldots, y_{s-\ell+1})$,

$$e(F_A(I)) \leqslant \mu[(I^r+Q)/Q] = \mu[(P+Q)/Q] \leqslant \mu(P) = s - \ell(I) + 1.$$

We get (iv). Theorem 2.3 has been proved. □

If ht I is the height of I, then ht $(I) \leqslant \ell(I)$. In the case of ht $(I) = \ell(I)$, I is called *equimultiple*.

Remark 2.4. Let I be an equimultiple ideal with $h = \operatorname{ht} I = \ell(I) > 0$. If grade $I = h$ or A is Cohen-Macaulay, then $x_1, x_2, \ldots, x_h$ is a regular sequence, so that

$$(Q : I^\infty) \cap I^r = Q \cap I^r \subseteq Q.$$

Hence by Theorem 2.3 we get $e(F_A(I)) = \mu[(I^n + Q)/Q]$ for all $n \geq r = r(I)$. If in addition $r(I) \leqslant 1$, then $e(F_A(I)) = \mu[(I + Q)/Q] = \mu(I) - \ell(I) + 1$.

Note that $F_A(\mathfrak{m}) = G(\mathfrak{m})$ is the associated graded ring of $\mathfrak{m}$. It is well-known that $e(G(\mathfrak{m})) = e(A)$ and $\ell(\mathfrak{m}) = d = \dim A$. Recall that $v(A) = \mu(\mathfrak{m})$ is the embedding dimension of A. As an immediate consequence of Theorem 2.3 we have the following result for the multiplicity of a Noetherian local ring.

Corollary 2.5. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ with maximal ideal $\mathfrak{m}$ and infinite residue field $k = A/\mathfrak{m}$. Suppose that $x_1, x_2, \ldots, x_d$ is a weak-(FC)-sequence in $\mathfrak{m}$ with respect to $\mathfrak{m}$ and $r_J(\mathfrak{m}) = r$; here $J = (x_1, x_2, \ldots, x_d)$. Set $Q = (x_1, x_2, \ldots, x_{d-1})$. Then*

(i) $e(A) = \mu(\frac{\mathfrak{m}^n + Q:\mathfrak{m}^\infty}{Q:\mathfrak{m}^\infty})$ *for all* $n \geq r$.

(ii) $e(A) \leqslant \mu[(Q + \mathfrak{m}^r)/Q]$ *with equality if and only if*

$$(Q : \mathfrak{m}^\infty) \cap \mathfrak{m}^r = (Q : \mathfrak{m}^\infty) \cap \mathfrak{m}^{r+1} (\operatorname{mod} Q).$$

In addition to $r \leqslant 1$, $e(A) \leqslant v(A) - d + 1$.

(iii) $e(A) = \mu[(Q + \mathfrak{m}^r)/Q]$ *if A is Cohen-Macaulay.*

Example 2.6. Let $S = k[[x^3, x^2y, xy^2, y^3, z]]$, where x, y, z are indeterminates over the infinite field k. Then S is a Cohen-Macaulay ring with maximal ideal $\mathfrak{M} = (x^3, x^2y, xy^2, y^3, z)$ and $\dim S = 3$. Set $J = (x^3, y^3, z)$. A direct computation shows that $\mathfrak{M}^2 = J\mathfrak{M}$. Therefore, $r_J(\mathfrak{M}) \leqslant 1$. It can be verified that $v(S) = 5$. Hence by Remark 2.4, we get $e(S) = 3$.

Remark 2.7. In a Cohen-Macaulay ring A we have $e(A) \geq v(A) - \dim A - 1$. On the other hand if $r(\mathfrak{m}) \leqslant 1$, then $e(A) \leqslant v(A) - \dim A - 1$ by Corollary 2.5(ii). So we immediately get that A is Cohen-Macaulay with minimal multiplicity; i.e., $e(A) = v(A) - \dim A - 1$ if and only if A is Cohen-Macaulay and $r(\mathfrak{m}) \leqslant 1$. This result is proved by Sally in [Sa].

3. The Cohen-Macaulayness of fiber cones

In this section we will discuss the Cohen-Macaulayness of fiber cones.

As remarked in Note 1, a reduction J of an ideal I is a minimal reduction if and only if J is generated by a weak-(FC)-sequence of the length $\ell(I)$ in I with respect to $(\mathfrak{m}, I)$.

We begin by establishing the following lemma.

Lemma 3.0. *Let I be an ideal with analytic spread $\ell(I) = 1$ and grade $I = 1$. Let x be a regular element in I such that xA is a minimal reduction of I. Set $r_{xA}(I) = r$. Then $F_A(I)$ is Cohen-Macaulay if and only if $xI^{n-1} \cap \mathfrak{m}I^n = x\mathfrak{m}I^{n-1}$for all $1 \leqslant n \leqslant r$.*

Proof. Since x is a regular element, by Note 2 we have $\mu(xI^n) = \mu(I^n)$. By Lemma 2.2, $\mu(I^r) = e(F_A(I))$. From this it follows that

$$\begin{aligned} l[F_A(I)/xF_A(I)] &= 1 + \sum_{1\leqslant n\leqslant r} l[I^n/xI^{n-1} + \mathfrak{m}I^n] \\ &= 1 + \sum_{1\leqslant n\leqslant r} (l[I^n/\mathfrak{m}I^n] - l[(xI^{n-1} + \mathfrak{m}I^n)/\mathfrak{m}I^n]) \\ &= 1 + \sum_{1\leqslant n\leqslant r} (\mu(I^n) - l[xI^{n-1}/xI^{n-1} \cap \mathfrak{m}I^n]) \\ &\geq 1 + \sum_{1\leqslant n\leqslant r} (\mu(I^n) - l[xI^{n-1}/\mathfrak{m}xI^{n-1}]) \\ &= 1 + \sum_{1\leqslant n\leqslant r} (\mu(I^n) - \mu(xI^{n-1})) = \mu(I^r) = e(F_A(I)). \end{aligned}$$

Therefore, $l[F_A(I)/xF_A(I)] = e(F_A(I))$ and hence $F_A(I)$ is Cohen-Macaulay if and only if

$$xI^{n-1} \cap \mathfrak{m}I^n = x\mathfrak{m}I^{n-1} \text{ for all } 1 \leqslant n \leqslant r. \qquad \square$$

The main result of this paper is established in the following theorem.

Theorem 3.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $k = A/\mathfrak{m}$. Let I be an ideal with analytic spread $\ell(I) = \ell > 0$ and J a minimal reduction of I. Suppose that J is generated by $x_1, x_2, \ldots, x_\ell$, a weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$. Set $Q = (x_1, \ldots, x_{\ell-1})$. Then $F_A(I)$ is Cohen-Macaulay if and only if the following conditions are satisfied:*

(i) $(Q : I^\infty) \cap I^n \subseteq JI^{n-1} (\text{mod } \mathfrak{m}I^n)$ *for all* $1 \leqslant n \leqslant r_J(I)$.

(ii) $(JI^{n-1} + Q : I^\infty) \cap \mathfrak{m}I^n = \mathfrak{m}JI^{n-1} (\text{mod } Q : I^\infty)$ *for all* $1 \leqslant n \leqslant r_J(I)$.

Proof. Set $A_{\ell-1} = A/Q, I_{\ell-1} = IA_{\ell-1}, A'_{\ell-1} = A/Q : I^\infty$, and $I'_{\ell-1} = IA'_{\ell-1}$. Note that $(x_1, x_2, \ldots, x_\ell)$ is a minimal reduction of I, $\ell(I'_{\ell-1}) = 1$. By Proposition 2.1 we have

$$e(F_A(I)) = e(F_{A_{\ell-1}}(I_{\ell-1})) = e(F_{A'_{\ell-1}}(I'_{\ell-1}))$$

and

$$\begin{aligned} l[F_{A'_{\ell-1}}(I'_{\ell-1})/JF_{A'_{\ell-1}}(I'_{\ell-1})] &\leqslant l[F_{A_{\ell-1}}(I_{\ell-1})/JF_{A_{\ell-1}}(I_{\ell-1})] \\ &\leqslant l[F_A(I)/JF_A(I)]. \end{aligned}$$

Note that $F_A(I)$ is Cohen-Macaulay if and only if $e(F_A(I)) = l[F_A(I)/JF_A(I)]$. This is equivalent to the fact that $F_{A'_{\ell-1}}(I'_{\ell-1})$ is Cohen-Macaulay and

$$\begin{aligned} l[F_{A'_{\ell-1}}(I'_{\ell-1})/JF_{A'_{\ell-1}}(I'_{\ell-1})] &= l[F_{A_{\ell-1}}(I_{\ell-1})/JF_{A_{\ell-1}}(I_{\ell-1})] \\ &= l[F_A(I)/JF_A(I)]. \end{aligned} \tag{6}$$

By Proposition 2.1(iii), (6) is equivalent to

$$(Q : I^\infty) \cap I^n \subseteq JI^{n-1} (\text{mod } \mathfrak{m}I^n) \text{ for all } 1 \leqslant n \leqslant r_J(I).$$

It is clear that grade $I'_{\ell-1} = 1 = \ell(I'_{\ell-1})$. Hence by Lemma 3.0, $F_{A'_{\ell-1}}(I'_{\ell-1})$ is Cohen-Macaulay if and only if

$$(JI^{n-1} + Q : I^\infty) \cap (\mathfrak{m}I^n + Q : I^\infty) = \mathfrak{m}JI^{n-1} (\text{mod } Q : I^\infty)$$

for all $1 \leqslant n \leqslant r_J(I)$. It can be verified that this condition is equivalent to

$$(JI^{n-1} + Q : I^\infty) \cap \mathfrak{m}I^n = \mathfrak{m}JI^{n-1} (\text{mod } Q : I^\infty) \text{ for all } 1 \leqslant n \leqslant r_J(I).$$

Therefore, $F_A(I)$ is Cohen-Macaulay if and only if the following conditions are satisfied:

$$(Q : I^\infty) \cap I^n \subseteq JI^{n-1} (\text{mod } \mathfrak{m}I^n)$$
$$\text{and } (JI^{n-1} + Q : I^\infty) \cap \mathfrak{m}I^n = \mathfrak{m}JI^{n-1} (\text{mod } Q : I^\infty)$$

for all $1 \leqslant n \leqslant r_J(I)$. This completes the proof of Theorem 3.1. □

We now examine how particular cases of Theorem 3.1 can be treated. In the case of $r(I) \leqslant 1$, by different approaches, Huneke and Sally [H-S] proved that if A is Cohen-Macalay and I is $\mathfrak{m}$-primary, then $F_A(I)$ is Cohen-Macaulay. Shah [Sh1] extended this result and showed that $F_A(I)$ is Cohen-Macaulay if I is an equimultiple ideal with grade $I = \text{ht } I$. These results are particular cases of the following proposition:

Proposition 3.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $k = A/\mathfrak{m}$. Let I be an ideal with analytic spread $\ell(I) = \ell > 0$ and J a minimal reduction of I. Suppose that J is generated by $x_1, x_2, \ldots, x_\ell$, a weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$ and $r_J(I) \leqslant 1$. Set $Q = (x_1, \ldots, x_{\ell-1})$. Then $F_A(I)$ is Cohen-Macaulay if $Q : I^\infty \subseteq J$.*

Proof. Since $r_J(I) \leqslant 1$, by Theorem 3.1 it follows that $F_A(I)$ is Cohen-Macaulay if and only if

$$(Q : I^\infty) \cap I \subseteq J (\text{mod } \mathfrak{m}I)$$

and $(J + Q : I^\infty) \cap \mathfrak{m}I = \mathfrak{m}J (\text{mod } Q : I^\infty)$. But $Q : I^\infty \subseteq J$, the first equation is obvious, and the second equation is equivalent to $J \cap \mathfrak{m}I = \mathfrak{m}J (\text{mod } Q : I^\infty)$. To obtain this equation, it is enough to show $J \cap \mathfrak{m}I = \mathfrak{m}J$. Since J is a minimal reduction of I, there exist elements $y_1, \ldots, y_m$ in I such that $x_1, x_2, \ldots, x_\ell, y_1, \ldots, y_m$ is a minimal base of I. Now assume that $x = a_1x_1 + \cdots + a_\ell x_\ell \in J \cap \mathfrak{m}I$,

$$x = b_1x_1 + \cdots + b_\ell x_\ell + c_1y_1 + \cdots + c_my_m,$$

and $b_i \in \mathfrak{m}$ for $1 \leqslant i \leqslant \ell$ and $c_j \in \mathfrak{m}$ for $1 \leqslant j \leqslant m$. Therefore,

$$(a_1 - b_1)x_1 + \cdots + (a_\ell - b_\ell)x_\ell + c_1y_1 + \cdots + c_my_m = 0.$$

Since $x_1, x_2, \ldots, x_\ell, y_1, \ldots, y_m$ is a minimal base of I, $a_i - b_i \in \mathfrak{m}$ for $1 \leqslant i \leqslant \ell$. Hence $a_i \in \mathfrak{m}$ for $1 \leqslant i \leqslant \ell$. Thus, $J \cap \mathfrak{m}I = \mathfrak{m}J$. We get the proof of Proposition 3.2. □

Remark 3.3. Note that if $r(I) \leqslant 1$ and I is an equimultiple ideal with grade $I = \text{ht } I = h > 0$ and $x_1, x_2, \ldots, x_h$ is a weak-(FC)-sequence in I, then $x_1, x_2, \ldots, x_h$ is a regular sequence. Hence

$$(x_1, \ldots, x_{h-1}) : I^\infty = (x_1, \ldots, x_{h-1}).$$

By Proposition 3.2, we immediately get the results of Huneke-Sally and Shah as in [op. cit.].

Remark 3.4. Corso-Polini-Vasconcelos [C-P-V1], Corso-Polini [C-P], and Corso-Huneke-Vasconcelos [C-H-V] proved that if A is Cohen-Macaulay with $e(A) > 1$ and J is a parameter ideal of A, then $I^2 = IJ$, where $I = J : \mathfrak{m}$. Therefore, $r(I) \leqslant 1$. Hence we have $F_A(I)$ Cohen-Macaulay by Remark 3.3 and $e(F_A(I)) = \mu(I) - d + 1$ by Remark 2.7.

In the case in which a minimal reduction J of I is generated by a regular sequence and $J \cap I^n = JI^{n-1}$ for all $n \leqslant r_J(I)$, Cortadellas and Zarzuela [C-Z] proved that $F_A(I)$ is Cohen-Macaulay if and only if $J \cap \mathfrak{m}I^n = J\mathfrak{m}I^{n-1}$ for all $n \leqslant r_J(I)$.

As an immediate consequence of Theorem 3.1 we get a result that seems to account well for the theorem of Cortadellas-Zarzuela [op. cit.].

Corollary 3.5. *Let I be an equimultiple ideal of height* $\operatorname{ht} I = h > 0$ *and* $\operatorname{grade} I = h$; *$J$ is a minimal reduction of I. Suppose that J is generated by $x_1, x_2, \ldots, x_h$, a weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$. Set $Q = (x_1, \ldots, x_{h-1})$. Then $F_A(I)$ is Cohen-Macaulay if and only if the following conditions are satisfied:*

(i) $Q \cap I^n \subseteq JI^{n-1} (\operatorname{mod} \mathfrak{m}I^n)$ *for all* $1 \leqslant n \leqslant r_J(I)$.

(ii) $(JI^{n-1} + Q) \cap \mathfrak{m}I^n = \mathfrak{m}JI^{n-1} (\operatorname{mod} Q)$ *for all* $1 \leqslant n \leqslant r_J(I)$.

If $J \cap I^n = JI^{n-1}$ for all $n \leqslant r_J(I)$, then condition (i) is always true and

$$\begin{aligned}(JI^{n-1} + Q) \cap \mathfrak{m}I^n &= (JI^{n-1} + Q) \cap I^n \cap \mathfrak{m}I^n \\ &= (JI^{n-1} + Q \cap I^n) \cap \mathfrak{m}I^n \\ &= JI^{n-1} \cap \mathfrak{m}I^n \\ &= J \cap I^n \cap \mathfrak{m}I^n = J \cap \mathfrak{m}I^n.\end{aligned}$$

Thus, if $J \cap \mathfrak{m}I^n = J\mathfrak{m}I^{n-1}$ for all $n \leqslant r_J(I)$, then condition (ii) is satisfied. This recovers the theorem of Cortadellas-Zarzuela [op. cit.].

By Corollary 2.5(ii), $e(A) \leqslant v(A) - d + 1$ if $r(I) \leqslant 1$. In the case that A is Cohen-Macaulay it is well-known that $e(A) \geq v(A) - d + 1$. If $e(A) = v(A) - d + 1$, it is said that A has minimal multiplicity. We know by Sally [Sa] that A is Cohen-Macaulay with minimal multiplicity if and only if A is Cohen-Macaulay and $r(\mathfrak{m}) \leqslant 1$.

In answer to the question as to when the fiber cone is Cohen-Macaulay with minimal multiplicity, we have the following theorem.

Theorem 3.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $k = A/\mathfrak{m}$. Let I be an ideal with analytic spread $\ell(I) = \ell > 0$ and J a minimal reduction of I. Suppose that J is generated by $x_1, x_2, \ldots, x_\ell$, a weak-(FC)-sequence in I with respect to $(\mathfrak{m}, I)$. Set $Q = (x_1, \ldots, x_{\ell-1})$. Then $F_A(I)$ is Cohen-Macaulay with minimal multiplicity if and only if $r_J(I) \leqslant 1$ and the following conditions are satisfied:*

(i) $(Q : I^\infty) \cap I \subseteq J (\operatorname{mod} \mathfrak{m}I)$.

(ii) $(J + Q : I^\infty) \cap \mathfrak{m}I = \mathfrak{m}J (\operatorname{mod} Q : I^\infty)$.

Proof. Note that $v(F_A(I)) = \mu(I)$. Therefore, $F_A(I)$ has minimal multiplicity if $e(F_A(I)) = \mu(I) - \ell(I) + 1$. Hence $F_A(I)$ is Cohen-Macaulay with minimal multiplicity if and only if $F_A(I)$ is Cohen-Macaulay and

$$l[F_A(I)/JF_A(I)] = \mu(I) - \ell(I) + 1 + \sum_{2 \leqslant n \leqslant r} l[I^n/(JI^{n-1} + \mathfrak{m}I^n)] = \mu(I) - \ell(I) + 1.$$

This is equivalent to $r_J(I) \leqslant 1$ and $F_A(I)$ is Cohen-Macaulay. Hence the proof is complete by Theorem 3.1. □

We recall that $F_A(\mathfrak{m}) = G(\mathfrak{m})$ and $e(A) = e(G(\mathfrak{m}))$; $v(A) = v(G(\mathfrak{m}))$. Hence $G(\mathfrak{m})$ has minimal multiplicity if and only if A has minimal multiplicity. If $I = \mathfrak{m}$ and $r_J(I) \leqslant 1$, then $\mathfrak{m}I = \mathfrak{m}^2 \subseteq J$, and hence the condition $(Q : I^\infty) \cap I \subseteq J (\operatorname{mod} \mathfrak{m}I)$ is equivalent to $Q : \mathfrak{m}^\infty \subseteq J$. Then as an immediate consequence of Proposition 3.2 and Theorem 3.6, we have:

Corollary 3.7. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $k = A/\mathfrak{m}$. Suppose that J is generated by $x_1, x_2, \ldots, x_d$, a weak-(FC)-sequence in $\mathfrak{m}$ with respect to $\mathfrak{m}$. Set $Q = (x_1, \ldots, x_{d-1})$. Then $G(\mathfrak{m})$ is Cohen-Macaulay with minimal multiplicity if and only if $r_J(\mathfrak{m}) \leqslant 1$ and $Q : \mathfrak{m}^\infty \subseteq J$.*

Remark 3.8. Denote by $R(\mathfrak{m})$ the Rees algebra of $\mathfrak{m}$. By Goto and Shimoda in [G-S], if A is Cohen-Macaulay, $R(\mathfrak{m})$ is Cohen-Macaulay if and only if $G(\mathfrak{m})$ is Cohen-Macaulay and $r(\mathfrak{m}) \leqslant \dim A - 1$. Now if A is Cohen-Macaulay of $\dim A > 1$ and $r(\mathfrak{m}) \leqslant 1$, then by Corollary 3.7 $G(\mathfrak{m})$ is Cohen-Macaulay. Hence $R(\mathfrak{m})$ is also Cohen-Macaulay.

Example 3.9. Let $S = k[[x^3, x^2y, xy^2, y^3, z]]$, where x, y, z are indeterminates over the infinite field k. Then S is Cohen-Macaulay with maximal ideal $\mathfrak{M} = (x^3, x^2y, xy^2, y^3, z)$ and $\dim S = 3$. Set $J = (x^3, y^3, z)$, $\mathfrak{M}^2 = J\mathfrak{M}$. So $r(\mathfrak{M}) = 1$. Hence $R(\mathfrak{M})$ is Cohen-Macaulay.

From Theorem 3.6 and Proposition 3.2, we also immediately get the following result.

Corollary 3.10. *Let I be an equimultiple ideal of* $\operatorname{ht} I = h > 0$ *and* $\operatorname{grade} I = h$. *Then $F_A(I)$ is Cohen-Macaulay with minimal multiplicity if and only if $r(I) \leqslant 1$.*

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