ON THE COHEN-MACULAYNESS OF FIBER CONES

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Abstract. Let \( A \) be a Noetherian local ring with the maximal ideal \( m \) and \( I \) an ideal of \( A \). Denote by \( F_A(I) = \bigoplus_{n \geq 0} (I^n/mI^n) \) to be the fiber cone of \( I \). This paper characterizes the multiplicity and the Cohen-Macaulayness of fiber cones in terms of minimal reductions of ideals.

1. Introduction

Throughout this paper, let \((A, m)\) be a Noetherian local ring with maximal ideal \( m \), infinite residue field \( k = A/m \), and Krull dimension \( \dim A = d > 0 \). Define \( F_A(I) = \bigoplus_{n \geq 0} (I^n/mI^n) \) to be the fiber cone of an ideal \( I \) in \( A \). Denote by \( \ell(I) = \dim F_A(I) \) the analytic spread of \( I \), by \( \mu(I) \) the minimum number of generators of \( I \), by \( r(I) = r_J(I) \) the reduction number of \( I \) with respect to a minimal reduction \( J \), and by \( v(A) = \mu(m) \) the embedding dimension of \( A \). In recent years, the Cohen-Macaulayness and other properties of special fiber rings have attracted much attention (see for instance [C-G-P-U], [C-P-V2], [Co-Z], [D-R-V], [H-H], [J-V-V1], [J-P-V]).

Using different approaches, some authors gave criteria for the Cohen-Macaulayness of fiber cones. For instance, Huneke and Sally [H-S] proved that if \( A \) is Cohen-Macaulay and \( I \) is \( m \)-primary with \( r(I) \leq 1 \), then \( F_A(I) \) is Cohen-Macaulay. More generally, Shah [Sh1, Sh2] showed the sufficient condition that \( F_A(I) \) is Cohen-Macaulay if \( I \) is an equimultiple ideal with grade \( \text{ht } I \) and \( r(I) \leq 1 \). Under the assumption that a minimal reduction \( J \) of \( I \) is generated by a regular sequence and \( J \cap I^n = JmI^n - 1 \) for all \( n \leq r_J(I) \), Cortadellas and Zarzuela [C-Z] proved that the Cohen-Macaulayness of \( F_A(I) \) is equivalent to \( J \cap mI^n = JmI^n - 1 \) for all \( n \leq r_J(I) \).

This paper characterizes the multiplicity (Theorem 2.3, Section 2) and the Cohen-Macaulayness (Theorem 3.1, Section 3) of fiber cones in terms of minimal reductions of ideals. As interesting consequences of main results, we get e.g. a generalization of results of Huneke-Sally and Shah [op. cit.] (Proposition 3.2, Section 3). In the case that minimal reductions of \( I \) are generated by a regular sequence, we get a result that seems to account well for the essence of the theorem of Cortadellas-Zarzuela [op. cit.] (see Corollary 3.5, Section 3).

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A ring $A$ is said to have minimal multiplicity if $e(A) = v(A) - d + 1$. It is also known by Sally [Sa] that $A$ is Cohen-Macaulay with minimal multiplicity if and only if $A$ is Cohen-Macaulay and $r(m) \leq 1$. By using the main results, we answer the question as to when the fiber cone is Cohen-Macaulay with minimal multiplicity (Theorem 3.6, Section 3).

This paper is divided into three sections. Section 2 investigates the multiplicity of fiber cones. Section 3 is devoted to the discussion of the Cohen-Macaulayness of fiber cones.

2. The Multiplicity of Fiber Cones

Using the concept of weak-(FC)-sequences in [Vi1], this section presents some results on the multiplicity of the fiber cone.

We first review some results and notions. Set $a : b^n = \{ x \in A \mid \text{there is a positive integer } n \text{ such that } xb^n \subseteq a \}$.

**Definition** ([Vi1]). Let $I$ be a non-nilpotent ideal of $A$. Set $A^* = A/0 : I^\infty$, $m^* = mA^*$, and $I^* = IA^*$. Recall that an element $x \in I$ ($x^*$ is the image of $x$ in $A^*$) is called a weak-(FC)-element with respect to $(m, I)$ if there exists an integer $n_1$ such that

$$(FC_1): x \in I \setminus mI, \text{ and for all } n \geq n_1 \text{ and for all non-negative integers } m,$$

$$m^*mI^n \cap (x^*) = m^*mI^{n-1}x^*.$$ 

$$(FC_2): x \text{ is a filter-regular element with respect to } I, \text{ i.e., } 0 : x \subseteq 0 : I^\infty.$$ 

Let $x_1, x_2, \ldots, x_t$ be a sequence in $I$. For each $i = 0, 1, \ldots, t - 1$, set $A_i = A/(x_1, x_2, \ldots, x_i)$; $\overline{x}_{i+1}$ is the image of $x_{i+1}$ in $A_i$, $I_i = IA_i$, and $m_i = mA_i$.

The sequence $x_1, \ldots, x_t$ is called a weak-(FC)-sequence in $I$ with respect to $(m, I)$ if $\overline{x}_{i+1}$ is a weak-(FC)-element with respect to $(m_i, I_i)$ for each $i = 0, 1, \ldots, t - 1$.

An ideal $J$ is called a reduction of ideal $I$ if $J \subseteq I$ and $I^{n+1} = JI^n$ for some $n$. A reduction $J$ is called a minimal reduction of $I$ if it does not properly contain any other reduction of $I$ [NR]. The least integer $n$ such that $I^{n+1} = JI^n$ is called the reduction number of $I$ with respect to $J$, and we denote it by $r_J(I)$. The reduction number of $I$ is defined by $r(I) = \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}$.

By using the Rees lemma in [Re], the author [Vi2, Vi3] showed that if $I$ is non-nilpotent, the length of any maximal weak-(FC)-sequence in $I$ with respect to $(m, I)$ is equal to $\ell(I)$, and if $x_1, x_2, \ldots, x_t$ is a weak-(FC)-sequence in $I$ with respect to $(m, I)$, then $(x_1, x_2, \ldots, x_t)$ is a minimal reduction of $I$. Moreover, as in the proof of Theorem 3.3 of [Vi1], then any minimal reduction of $I$ is generated by a maximal weak-(FC)-sequence in $I$ with respect to $(m, I)$. So we have:

**Note 1.** An ideal $J \subseteq I$ is a minimal reduction of $I$ if and only if $J$ is generated by a weak-(FC)-sequence of length $\ell(I)$ in $I$ with respect to $(m, I)$.

The following proposition plays an important role in the proofs of this paper.

**Proposition 2.1.** Let $I$ be an ideal with analytic spread $\ell(I) = \ell > 0$. Let $x_1, x_2, \ldots, x_\ell$ be a weak-(FC)-sequence in $I$ with respect to $(m, I)$. Set $J = (x_1, x_2, \ldots, x_\ell)$. For any $i < \ell$, set

$A_i = A/(x_1, \ldots, x_i), I_i = IA_i, P_i = (x_1, x_2, \ldots, x_i) : I^\infty, A'_i = A/P_i, I'_i = IA'_i.$
Then
(i) \( e(F_A(I)) = e(F_A(I)) = e(F_A(I)) \).
(ii) \( l[F_A(I)] = l[F_A(I)] \leq l[F_A(I)] \).
(iii) \( l[F_A(I)] = l[F_A(I)] \) if and only if \( I^n \cap P_i \subseteq J I^{n-1}(mod \ mI^n) \) for all \( 1 \leq n \leq r_J(I) \).

Proof. Set \( N = 0 : I^\infty, A^* = A/N, m^* = mA^*, I^* = IA^* \). By the Artin-Rees lemma, there exists an integer \( u \) such that \( N \cap I^n = I^{n-u}(N \cap I^u) \subseteq I^{n-u}N \) for all \( n \geq u \). Since \( I^{n-u}N = 0 \) for all large enough \( n \), \( N \cap I^n = 0 \) for all large enough \( n \). From this it follows that

\[
l_A^*(\frac{I^n}{m^n I^n}) = l_A^*(\frac{I^n + N}{m^n I^n + N}) = l_A^*(\frac{I^n}{m^n I^n + N}) = l_A^*(\frac{I^n}{m^n I^n})
\]

for all large enough \( n \). Hence

\[
e(F_A(I)) = e(F_A(I^*)).
\]

Let \( x \in I \) be a weak-(FC)-element with respect to \((m, I)\). If \( x^* \) is the image of \( x \) in \( A^* \), then \( x^* \) is not contained in any prime ideal belonging to \( Ass A^* \) (see condition \((FC_2)\)). Hence \( x^* \) is a non-zero divisor in \( A^* \). Set \( B = A^*/x^*A^*, b = m^*B, e = I^*B \). For all large enough \( n \), we have

\[
l_B^*(\frac{c^n}{b^c^n}) = l_A^*(\frac{x^n + (x^*)}{m^nI^n + (x^*)}) = l_A^*(\frac{x^n}{m^nI^n + (x^*)})
\]

(see condition \((FC_1)\))

\[
= l_A^*(\frac{x^n}{m^nI^n + x^nI^{n-1}})
\]

(\(I^* = 1\))

\[
= l_A^*(\frac{x^n}{m^nI^n}) - l_A^*(\frac{x^n}{m^nI^n + x^nI^{n-1}})
\]

(\(I^* = 1\))

\[
= l_A^*(\frac{x^n}{m^nI^n}) - l_A^*(\frac{x^n}{m^nI^n + x^nI^{n-1}})
\]

(\(I^* = 1\))

\[
= l_A^*(\frac{x^n}{m^nI^n}) - l_A^*(\frac{x^n}{m^nI^n + x^nI^{n-1}}).
\]

Since \( x^* \) is a non-zero divisor in \( A^* \), it follows that \( l_A^*(\frac{x^n}{m^nI^n + x^nI^{n-1}}) = l_A^*(\frac{x^n}{m^nI^n + x^nI^{n-1}}) \), so that

\[
l_B^*(\frac{c^n}{b^c^n}) = l_A^*(\frac{x^n}{m^nI^n}) - l_A^*(\frac{x^n}{m^nI^n + x^nI^{n-1}})
\]

for all large enough \( n \). Hence if \( \ell > 1 \), then

\[
e(F_A(I)) = e(F_B(e))
\]

The case of \( \ell(I) > 1 \), \( e \) is non-nilpotent in \( B \). Set \( B' = B/0 : c^\infty \) and \( e' = eB' \). Then by (1) we get that

\[
e(F_B(e)) = e(F_B(e'))
\]

Set \( A_1 = A/(x), I_1 = IA_1, A'_1 = A/(x) : I^\infty, \) and \( I'_1 = IA'_1 \). Note that \( B' \cong A/(x) : I^\infty = A'_1 \). Then by (1), we have

\[
e(F_A(I)) = e(F_A(I)) = e(F_A'(I'_1))
\]

Combining (4) with (3), (2) and (1), we get

\[
e(F_A(I)) = e(F_A(I)) = e(F_A'(I'_1)).
\]
Assume that the analytic spread $\ell = \ell(I) > 1$ and $x_1, x_2, \ldots, x_i$ ($i < \ell$) is a weak-\((FC)\)-sequence in $I$ with respect to $(m, I)$. Set $A_i = A / (x_1, \ldots, x_i), I_i = I A_i, A_i' = A / (x_1, \ldots, x_i) : I_i^\infty$, and $I_r = I A_r'$. Then using (5) and by induction on $i < \ell = \ell(I)$, we easily show that $e(F_A(I)) = e(F_{A_i}(I_i)) = e(F_{A_i'}(I_i'))$. Note that this equation is true too in the case of $\ell = 1$ by (1). This establishes (i).

Assume that $r_j(I) = r$. Set $Q_i = (x_1, \ldots, x_i)$, $P_i = Q_i : I_i^\infty$. It can be verified that

\[
\begin{align*}
l[F_A(I)/JF_A(I)] &= 1 + \sum_{1 \leq n \leq r} l[I^n/(JI^{n-1} + mI^n)], \\
l[F_{A_i}(I_i)/JF_{A_i}(I_i)] &= 1 + \sum_{1 \leq n \leq r} l[Q_i + I^n/(Q_i + JI^{n-1} + mI^n)] \\
&= 1 + \sum_{1 \leq n \leq r} l[I^n/(Q_i \cap I^n + JI^{n-1} + mI^n)], \\
l[F_{A_i'}(I_i')/JF_{A_i'}(I_i') &= 1 + \sum_{1 \leq n \leq r} l[P_i + I^n/(P_i + JI^{n-1} + mI^n)] \\
&= 1 + \sum_{1 \leq n \leq r} l[I^n/(P_i \cap I^n + JI^{n-1} + mI^n)].
\end{align*}
\]

It is clear that

\[
(JI^{n-1} + mI^n) \subseteq (Q_i \cap I^n + JI^{n-1} + mI^n) \subseteq (P_i \cap I^n + JI^{n-1} + mI^n)
\]

for all $1 \leq n \leq r_j(I)$. Hence we immediately get (ii) and (iii).

Recall that $\mu(I)$ is the minimum number of generators of $I$ and $\mu(I) = \dim_k(I/mI) = l(I/mI).

Note 2. It can be easily seen that if $a$ is a non-zero divisor in $A$, then $\mu(aI) = \mu(I)$.

At this point we remark that: On the one hand, by Proposition 2.1 we have $e(F_A(I)) = e(F_{A_{i-1}}(I_{i-1}))$. On the other hand, since $J = (x_1, x_2, \ldots, x_\ell)$ is a minimal reduction of $I$, the analytic spread of $I_{i-1}$ is 1.

Thus the problem is reduced to the case of the fiber cone of an ideal with analytic spread $\ell = 1$. This is the reason why we need the following lemma.

**Lemma 2.2.** Let $I$ be an ideal with analytic spread $\ell(I) = 1$ and $x$ a weak-\((FC)\)$-element in $I$ with respect to $(m, I)$. Set $r_{xA}(I) = r$ and $I^0 = A$. Then

1. $e(F_A(I)) = \mu(I^n)$ for all $n \geq r$ if grade $I = 1$.
2. $e(F_A(I)) = \mu(I^{n+0:I})$ for all $n \geq r$.
3. $e(F_A(I)) \leq \mu(I^r) \mbox{ with equality if and only if } (0 : I^\infty) \cap (0 : I^\infty) \cap mI^r$.

**Proof.** Since $\ell(I) = 1$, $l(I^n/mI^n) = \mu(I^n)$ takes a constant value for all large enough $n$. From this it follows that $e(F_A(I)) = \mu(I^n)$ for all large enough $n$. Remember that $r_{(x)}(I) = r$. Hence $I^n = I^n x^{n-r}$ for all $n > r$. Now, if grade $I > 0$, then $x^{n-r}$ is a non-zero divisor in $A$. By Note 2 we have $\mu(I^n) = \mu(I^{n+0:I}) = \mu(I^r)$ for all $n \geq r$. We get (i).

Set $A^* = A/0 : I^\infty$ and $I^* = IA^*$. We recall that by Proposition 2.1 we have $e(F_A(I)) = e(F_A(I^*))$. Since $(0 : I^\infty) : I = 0 : I^\infty$, it follows that grade$_{A^*} I^* > 0$. Hence by (i) we get

\[
e(F_A(I)) = e(F_A(I^*)) = \mu(I^{n+r}) = \mu(I^{r+0 : I^\infty}).
\]
This completes the proof of (ii).

It is easily seen that

\[
\mu\left(\frac{I^r + 0 : I^\infty}{0 : I^\infty}\right) = \mu\left(\frac{I^r}{I^r \cap (0 : I^\infty)}\right)
= l_A\left(\frac{I^r}{mI^r}\right) - l_A\left(\frac{I^r \cap (0 : I^\infty) + mI^r}{mI^r}\right)
= \mu(I^r) - l_A\left(\frac{I^r \cap (0 : I^\infty)}{mI^r \cap (0 : I^\infty)}\right)
= \mu(I^r) - l_A\left(\frac{I^r \cap (0 : I^\infty)}{mI^r \cap (0 : I^\infty)}\right).
\]

Hence by (ii), \(e(F_A(I)) \leq \mu(I^r)\) with equality if and only if \((0 : I^\infty) \cap I^r = (0 : I^\infty) \cap mI^r\). Thus, we get (iii). \(\square\)

By combining Proposition 2.1 with Lemma 2.2, we obtain the following theorem.

**Theorem 2.3.** Let \((A, m)\) be a Noetherian local ring with maximal ideal \(m\) and infinite residue field \(k = A/m\). Let \(I\) be an ideal with analytic spread \(\ell(I) = \ell > 0\) and \(J\) a minimal reduction of \(I\). Suppose that \(J\) is generated by \(x_1, x_2, \ldots, x_r\), a weak-(FC)-sequence in \(I\) with respect to \((m, I)\). Set \(J = (x_1, x_2, \ldots, x_r), r_J(I) = r,\) and \(Q = (x_1, x_2, \ldots, \ell_{r-1})\). Then

(i) \(e(F_A(I)) = \mu(\frac{I^r + Q}{Q})\) for all \(n \geq r\).

(ii) \(e(F_A(I)) \leq \mu(\frac{I^r + Q}{Q})\) with equality if and only if \((Q : I^\infty) \cap I^r = (Q : I^\infty) \cap mI^r (\mod Q)\).

(iii) \(e(F_A(I)) \leq \mu(I^r)\) if \((Q : I^\infty) \cap I^r \subseteq Q\).

(iv) \(e(F_A(I)) \leq \mu(I^r) - \ell(I) + 1\).

**Proof.** Set \(A_{r-1} = A/Q, I_{r-1} = IA_{r-1}, A'_{r-1} = A/Q : I^\infty,\) and \(I'_{r-1} = IA'_{r-1}\). Then by Proposition 2.1 we have \(e(F_A(I)) = e(F_{A_{r-1}}(I_{r-1}))\), since \((x_1, x_2, \ldots, x_r)\) is a minimal reduction of \(I\). Hence \(\ell(I_{r-1}) = 1\). Lemma 2.2(ii) immediately gives \(e(F_{A_{r-1}}(I_{r-1})) = \mu(I_{r-1}^n)\) for all \(n \geq r\). Thus, we get (i).

By Lemma 2.2(iii), \(e(F_{A_{r-1}}(I_{r-1})) \leq \mu(I_{r-1}^n)\), since \(e(F_A(I)) = e(F_{A_{r-1}}(I_{r-1}))\) and

\[
\mu(I_{r-1}^n) = \mu(\frac{I^r + Q}{Q}),
\]

it follows that \(e(F_A(I)) \leq \mu(\frac{I^r + Q}{Q})\), since \(e(F_A(I)) = \mu(\frac{I^r + Q}{Q})\) if and only if

\[
e(F_{A_{r-1}}(I_{r-1})) = \mu(I_{r-1}^n).
\]

Hence by Lemma 2.2(iii) we get (ii).

Since \((Q : I^\infty) \cap I^r \subseteq Q\), it follows that \(Q + (Q : I^\infty) \cap I^r = Q + (Q : I^\infty) \cap mI^r\). Hence by (ii) we have (iii).

Set \(\mu(I^r) = s\). Since \((x_1, x_2, \ldots, x_r)\) is a minimal reduction of \(I\), \((x_1', x_2', \ldots, x_r')\) is a minimal reduction of \(I^r\). Hence there exist elements \(y_1, y_2, \ldots, y_{s-\ell+1} \in I^r\) such that \((x_1', x_2', \ldots, x_{s-\ell+1}) = I^r\). Set \(P = (y_1, y_2, \ldots, y_{s-\ell+1})\),

\[
e(F_A(I)) \leq \mu(\frac{I^r + Q}{Q}) = \mu(P + Q) / Q \leq \mu(P) = s - \ell(I) + 1.
\]

We get (iv). Theorem 2.3 has been proved. \(\square\)

If \(\text{ht} I\) is the height of \(I\), then \(\text{ht} (I) = \ell(I)\). In the case of \(\text{ht} (I) = \ell(I)\), \(I\) is called *equimultiple*. 

Remark 2.4. Let $I$ be an equimultiple ideal with $h = \text{ht } I = \ell(I) > 0$. If grade $I = h$ or $A$ is Cohen-Macaulay, then $x_1, x_2, \ldots, x_h$ is a regular sequence, so that

$$(Q : I^\infty) \cap I^r = Q \cap I^r \subseteq Q.$$ 

Hence by Theorem 2.3 we get $e(F_A(I)) = \mu((I^n + Q)/Q)$ for all $n \geq r = r(I)$. If in addition $r(I) \leq 1$, then $e(F_A(I)) = \mu((I + Q)/Q) = \mu(I) - \ell(I) + 1$.

Note that $F_A(m) = G(m)$ is the associated graded ring of $m$. It is well-known that $e(G(m)) = e(A)$ and $\ell(m) = d = \dim A$. Recall that $v(A) = \mu(m)$ is the embedding dimension of $A$. As an immediate consequence of Theorem 2.3 we have the following result for the multiplicity of a Noetherian local ring.

**Corollary 2.5.** Let $(A, m)$ be a Noetherian local ring of dimension $d > 0$ with maximal ideal $m$ and infinite residue field $k = A/m$. Suppose that $x_1, x_2, \ldots, x_d$ is a weak-(FC)-sequence in $m$ with respect to $m$ and $r_J(m) = r$; here $J = (x_1, x_2, \ldots, x_d)$. Set $Q = (x_1, x_2, \ldots, x_d-1)$. Then

(i) $e(A) = \mu((m^n + Q : m^{n+\infty})$ for all $n \geq r$.

(ii) $e(A) \leq \mu((Q + m^r)/Q)$ with equality if and only if

$$(Q : m^\infty) \cap m^r = (Q : m^\infty) \cap m^{r+1} (\text{mod } Q).$$

In addition to $r \leq 1$, $e(A) \leq v(A) - d + 1$.

(iii) $e(A) = \mu((Q + m^r)/Q)$ if $A$ is Cohen-Macaulay.

**Example 2.6.** Let $S = k[[x^3, x^2y, xy^2, y^3, z]]$, where $x, y, z$ are indeterminates over the infinite field $k$. Then $S$ is a Cohen-Macaulay ring with maximal ideal $\mathfrak{m} = (x^3, x^2y, xy^2, y^3, z)$ and $\dim S = 3$. Set $J = (x^3, y^3, z)$. A direct computation shows that $\mathfrak{m}^2 = J\mathfrak{m}$. Therefore, $r_J(\mathfrak{m}) = 1$. It can be verified that $v(S) = 5$. Hence by Remark 2.4, we get $e(S) = 3$.

Remark 2.7. In a Cohen-Macaulay ring $A$ we have $e(A) \geq v(A) - \dim A - 1$. On the other hand if $r(m) \leq 1$, then $e(A) \leq v(A) - \dim A - 1$ by Corollary 2.5(ii). So we immediately get that $A$ is Cohen-Macaulay with minimal multiplicity; i.e., $e(A) = v(A) - \dim A - 1$ if and only if $A$ is Cohen-Macaulay and $r(m) \leq 1$. This result is proved by Sally in [Sa].

### 3. The Cohen-Macaulayness of Fiber Cones

In this section we will discuss the Cohen-Macaulayness of fiber cones.

As remarked in Note 1, a reduction $J$ of an ideal $I$ is a minimal reduction if and only if $J$ is generated by a weak-(FC)-sequence of the length $\ell(I)$ in $I$ with respect to $(m, I)$.

We begin by establishing the following lemma.

**Lemma 3.0.** Let $I$ be an ideal with analytic spread $\ell(I) = 1$ and grade $I = 1$. Let $x$ be a regular element in $I$ such that $xA$ is a minimal reduction of $I$. Set $r_xA(I) = r$. Then $F_A(I)$ is Cohen-Macaulay if and only if $xI^{n-1} \cap mI^n = xmI^{n-1}$ for all $1 \leq n \leq r$.
Proof. Since \( x \) is a regular element, by Note 2 we have \( \mu(xI^n) = \mu(I^n) \). By Lemma 2.2, \( \mu(I^n) = e(F_A(I^n)) \). From this it follows that

\[
l[F_A(I)/xF_A(I)] = 1 + \sum_{1 \leq n \leq r} l[I^n/xI^n-1 + mI^n]
\]

\[
= 1 + \sum_{1 \leq n \leq r} (l[I^n/mI^n] - l((xI^n-1 + mI^n)/mI^n))
\]

\[
= 1 + \sum_{1 \leq n \leq r} (\mu(I^n) - l[xI^n-1/xI^n-1 \cap mI^n])
\]

\[
\geq 1 + \sum_{1 \leq n \leq r} (\mu(I^n) - l[xI^n-1/mxI^n-1])
\]

\[
= 1 + \sum_{1 \leq n \leq r} (\mu(I^n) - \mu(xI^n-1)) = \mu(I^n) = e(F_A(I^n)).
\]

Therefore, \( l[F_A(I)/xF_A(I)] = e(F_A(I^n)) \) and hence \( F_A(I^n) \) is Cohen-Macaulay if and only if

\[
xI^n-1 \cap mI^n = xmI^n-1 \text{ for all } 1 \leq n \leq r.
\]

The main result of this paper is established in the following theorem.

Theorem 3.1. Let \( (A, \mathfrak{m}) \) be a Noetherian local ring with maximal ideal \( \mathfrak{m} \) and infinite residue field \( k = A/\mathfrak{m} \). Let \( I \) be an ideal with analytic spread \( \ell(I) = \ell > 0 \) and \( J \) a minimal reduction of \( I \). Suppose that \( J \) is generated by \( x_1, x_2, \ldots, x_\ell \), a weak-(FC)-sequence in \( I \) with respect to \( (\mathfrak{m}, I) \). Set \( Q = (x_1, \ldots, x_{\ell-1}) \). Then \( F_A(I) \) is Cohen-Macaulay if and only if the following conditions are satisfied:

(i) \( (Q : I^\infty) \cap I^n \subseteq JI^n-1 mod mI^n \) for all \( 1 \leq n \leq r_J(I) \).

(ii) \( (JI^n-1 + Q : I^\infty) \cap mI^n = mJI^n-1 mod Q : I^\infty \) for all \( 1 \leq n \leq r_J(I) \).

Proof. Set \( A_{\ell-1} = A/Q, I_{\ell-1} = IA_{\ell-1}, A_{\ell-1}' = A/Q : I^\infty, \) and \( I_{\ell-1}' = IA_{\ell-1}' \). Note that \( (x_1, x_2, \ldots, x_{\ell-1}) \) is a minimal reduction of \( I \), \( \ell(I_{\ell-1}) = 1 \). By Proposition 2.1 we have

\[
e(F_A(I^n)) = e(F_{A_{\ell-1}}(I_{\ell-1})) = e(F_{A_{\ell-1}'}(I_{\ell-1}'))
\]

and

\[
l[F_{A_{\ell-1}'}(I_{\ell-1}')/JF_{A_{\ell-1}'}(I_{\ell-1}')] \leq l[F_{A_{\ell-1}}(I_{\ell-1})/JF_{A_{\ell-1}}(I_{\ell-1})]
\]

\[
\leq l[F_A(I)/JF_A(I)].
\]

Note that \( F_A(I) \) is Cohen-Macaulay if and only if \( e(F_A(I^n)) = l[F_A(I^n)/JF_A(I^n)] \).

This is equivalent to the fact that \( F_{A_{\ell-1}'}(I_{\ell-1}') \) is Cohen-Macaulay and

\[
l[F_{A_{\ell-1}'}(I_{\ell-1}')/JF_{A_{\ell-1}'}(I_{\ell-1}')] = l[F_{A_{\ell-1}}(I_{\ell-1})/JF_{A_{\ell-1}}(I_{\ell-1})]
\]

\[
= l[F_A(I)/JF_A(I)].
\]

By Proposition 2.1(iii), (6) is equivalent to

\[
(Q : I^\infty) \cap I^n \subseteq JI^n-1 mod mI^n \text{ for all } 1 \leq n \leq r_J(I).
\]

It is clear that \( I_{\ell-1}' = 1 = \ell(I_{\ell-1}) \). Hence by Lemma 3.0, \( F_{A_{\ell-1}'}(I_{\ell-1}') \) is Cohen-Macaulay if and only if

\[
(JI^n-1 + Q : I^\infty) \cap (mI^n + Q : I^\infty) = mJI^n-1 mod Q : I^\infty
\]
for all $1 \leq n \leq r_J(I)$. It can be verified that this condition is equivalent to

$$(J^{n-1}I + Q : I^n) \cap mI^n = mJ^{n-1}(\text{mod } Q : I^n)$$

for all $1 \leq n \leq r_J(I)$. Therefore, $F_A(I)$ is Cohen-Macaulay if and only if the following conditions are satisfied:

$$(Q : I^n) \cap I^n \subseteq JI^{n-1}(\text{mod } mI^n)$$

and

$$(J^{n-1}I + Q : I^n) \cap mI^n = mJ^{n-1}(\text{mod } Q : I^n)$$

for all $1 \leq n \leq r_J(I)$. This completes the proof of Theorem 3.1. □

We now examine how particular cases of Theorem 3.1 can be treated. In the case of $r(I) \leq 1$, by different approaches, Huneke and Sally [H-S] proved that if $A$ is Cohen-Macaulay and $I$ is $m$-primary, then $F_A(I)$ is Cohen-Macaulay. Shah [Sh] extended this result and showed that $F_A(I)$ is Cohen-Macaulay if $I$ is an equimultiple ideal with grade $I = \text{ht } I$. These results are particular cases of the following proposition:

**Proposition 3.2.** Let $(A, m)$ be a Noetherian local ring with maximal ideal $m$ and infinite residue field $k = A/m$. Let $I$ be an ideal with analytic spread $\ell(I) = \ell > 0$ and $J$ a minimal reduction of $I$. Suppose that $J$ is generated by $x_1, x_2, \ldots, x_\ell$, a weak-(FC)-sequence in $I$ with respect to $(m, I)$ and $r_J(I) \leq 1$. Set $Q = (x_1, \ldots, x_{\ell-1})$. Then $F_A(I)$ is Cohen-Macaulay if $Q : I^\infty \subseteq J$.

**Proof.** Since $r_J(I) \leq 1$, by Theorem 3.1 it follows that $F_A(I)$ is Cohen-Macaulay if and only if

$$(Q : I^n) \cap I \subseteq J(\text{mod } mI)$$

and

$$(J + Q : I^n) \cap mI = mJ(\text{mod } Q : I^n).$$

But $Q : I^\infty \subseteq J$, the first equation is obvious, and the second equation is equivalent to $J \cap mI = mJ(\text{mod } Q : I^\infty)$. To obtain this equation, it is enough to show $J \cap mI = mJ$. Since $J$ is a minimal reduction of $I$, there exist elements $y_1, \ldots, y_m$ in $I$ such that $x_1, x_2, \ldots, x_\ell, y_1, \ldots, y_m$ is a minimal base of $I$. Now assume that $x = a_1x_1 + \cdots + a_\ell x_\ell \in J \cap mI$,

$$x = b_1x_1 + \cdots + b_\ell x_\ell + c_1 y_1 + \cdots + c_m y_m,$$

and $b_i \in m$ for $1 \leq i \leq \ell$ and $c_j \in m$ for $1 \leq j \leq m$. Therefore,

$$(a_1 - b_1)x_1 + \cdots + (a_\ell - b_\ell) x_\ell + c_1 y_1 + \cdots + c_m y_m = 0.$$

Since $x_1, x_2, \ldots, x_\ell, y_1, \ldots, y_m$ is a minimal base of $I$, $a_i - b_i \in m$ for $1 \leq i \leq \ell$. Hence $a_i \in m$ for $1 \leq i \leq \ell$. Thus, $J \cap mI = mJ$. We get the proof of Proposition 3.2. □

**Remark 3.3.** Note that if $r(I) \leq 1$ and $I$ is an equimultiple ideal with grade $I = \text{ht } I = h > 0$ and $x_1, x_2, \ldots, x_h$ is a weak-(FC)-sequence in $I$, then $x_1, x_2, \ldots, x_h$ is a regular sequence. Hence

$$(x_1, \ldots, x_{h-1}) : I^\infty = (x_1, \ldots, x_{h-1}).$$

By Proposition 3.2, we immediately get the results of Huneke-Sally and Shah as in [op. cit.].

**Remark 3.4.** Corso-Polini-Vasconcelos [CP-V], Corso-Polini [CP], and Corso-Huneke-Vasconcelos [CH-V] proved that if $A$ is Cohen-Macaulay with $e(A) > 1$ and $J$ is a parameter ideal of $A$, then $I^2 = JJ$, where $I = J : m$. Therefore, $r(I) \leq 1$. Hence we have $F_A(I)$ Cohen-Macaulay by Remark 3.3 and $e(F_A(I)) = \mu(I) - d + 1$ by Remark 2.7.
In the case in which a minimal reduction $J$ of $I$ is generated by a regular sequence and $J \cap I^n = JI^{n-1}$ for all $n \leq r_J(I)$, Cortadellas and Zarzuela [C-Z] proved that $F_A(I)$ is Cohen-Macaulay if and only if $J \cap mI^n = JmI^{n-1}$ for all $n \leq r_J(I)$.

As an immediate consequence of Theorem 3.1 we get a result that seems to account well for the theorem of Cortadellas-Zarzuela [op. cit.].

**Corollary 3.5.** Let $I$ be an equimultiple ideal of height $h = \text{ht} I = h$; $J$ is a minimal reduction of $I$. Suppose that $J$ is generated by $x_1, x_2, \ldots, x_\ell$, a weak-(FC)-sequence in $I$ with respect to $(m, I)$. Set $Q = (x_1, \ldots, x_{\ell-1})$. Then $F_A(I)$ is Cohen-Macaulay if and only if the following conditions are satisfied:

(i) $Q \cap I^n \subseteq JI^{n-1}(\text{mod } mI^n)$ for all $1 \leq n \leq r_J(I)$.

(ii) $(JI^{n-1} + Q) \cap mI^n = mJI^{n-1}(\text{mod } Q)$ for all $1 \leq n \leq r_J(I)$.

If $J \cap I^n = JI^{n-1}$ for all $n \leq r_J(I)$, then condition (i) is always true and

$(JI^{n-1} + Q) \cap mI^n = (JI^{n-1} + Q) \cap I^n \cap mI^n = (JI^{n-1} + Q) \cap I^n \cap mI^n = J \cap I^n \cap mI^n = J \cap I^n.$

Thus, if $J \cap mI^n = JmI^{n-1}$ for all $n \leq r_J(I)$, then condition (ii) is satisfied. This recovers the theorem of Cortadellas-Zarzuela [op. cit.].

By Corollary 2.5(ii), $e(A) \leq v(A) - d + 1$ if $r(I) \leq 1$. In the case that $A$ is Cohen-Macaulay it is well-known that $e(A) \geq v(A) - d + 1$. If $e(A) = v(A) - d + 1$, it is said that $A$ has minimal multiplicity. We know by Sally [Sa] that $A$ is Cohen-Macaulay with minimal multiplicity if and only if $A$ is Cohen-Macaulay and $r(m) \leq 1$.

In answer to the question as to when the fiber cone is Cohen-Macaulay with minimal multiplicity, we have the following theorem.

**Theorem 3.6.** Let $(A, m)$ be a Noetherian local ring with maximal ideal $m$ and infinite residue field $k = A/m$. Let $I$ be an ideal with analytic spread $\ell(I) = \ell > 0$ and $J$ a minimal reduction of $I$. Suppose that $J$ is generated by $x_1, x_2, \ldots, x_\ell$, a weak-(FC)-sequence in $I$ with respect to $(m, I)$. Set $Q = (x_1, \ldots, x_{\ell-1})$. Then $F_A(I)$ is Cohen-Macaulay with minimal multiplicity if and only if $r_J(I) \leq 1$ and the following conditions are satisfied:

(i) $(Q : I^\infty) \cap I \subseteq J(\text{mod } mI)$.

(ii) $(J + Q : I^\infty) \cap mI = mJ(\text{mod } Q : I^\infty)$.

**Proof.** Note that $e(F_A(I)) = \mu(I)$. Therefore, $F_A(I)$ has minimal multiplicity if $e(F_A(I)) = \mu(I) - \ell(I) + 1$. Hence $F_A(I)$ is Cohen-Macaulay with minimal multiplicity if and only if $F_A(I)$ is Cohen-Macaulay and

$l[F_A(I)/JF_A(I)] = \mu(I) - \ell(I) + 1 + \sum_{2 \leq n \leq r} l[I^n / (JI^{n-1} + mI^n)] = \mu(I) - \ell(I) + 1.$

This is equivalent to $r_J(I) \leq 1$ and $F_A(I)$ is Cohen-Macaulay. Hence the proof is complete by Theorem 3.1. □

We recall that $F_A(m) = G(m)$ and $e(A) = e(G(m))$; $v(A) = v(G(m))$. Hence $G(m)$ has minimal multiplicity if and only if $A$ has minimal multiplicity. If $I = m$ and $r_J(I) \leq 1$, then $mI = m^2 \subseteq J$, and hence the condition $(Q : I^\infty) \cap I \subseteq J(\text{mod } mI)$ is equivalent to $Q : m^\infty \subseteq J$. Then as an immediate consequence of Proposition 3.2 and Theorem 3.6, we have:
Corollary 3.7. Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $k = A/\mathfrak{m}$. Suppose that $J$ is generated by $x_1, x_2, \ldots, x_d$, a weak-(FC)-sequence in $\mathfrak{m}$ with respect to $\mathfrak{m}$. Set $Q = (x_1, \ldots, x_{d-1})$. Then $G(\mathfrak{m})$ is Cohen-Macaulay with minimal multiplicity if and only if $r_J(\mathfrak{m}) \leq 1$ and $Q : \mathfrak{m}^\infty \subseteq J$.

Remark 3.8. Denote by $R(\mathfrak{m})$ the Rees algebra of $\mathfrak{m}$. By Goto and Shimoda in [G-S], if $A$ is Cohen-Macaulay, $R(\mathfrak{m})$ is Cohen-Macaulay if and only if $G(\mathfrak{m})$ is Cohen-Macaulay and $r(\mathfrak{m}) \leq \dim A - 1$. Now if $A$ is Cohen-Macaulay of dim $A > 1$ and $r(\mathfrak{m}) \leq 1$, then by Corollary 3.7 $G(\mathfrak{m})$ is Cohen-Macaulay. Hence $R(\mathfrak{m})$ is also Cohen-Macaulay.

Example 3.9. Let $S = k[[x^3, x^2y, xy^2, y^3, z]]$, where $x, y, z$ are indeterminates over the infinite field $k$. Then $S$ is Cohen-Macaulay with maximal ideal $\mathfrak{m} = (x^3, x^2y, xy^2, y^3, z)$ and $\dim S = 3$. Set $J = (x^3, y^3, z)$, $\mathfrak{m}^2 = J\mathfrak{m}$. So $r(\mathfrak{m}) = 1$. Hence $R(\mathfrak{m})$ is Cohen-Macaulay.

From Theorem 3.6 and Proposition 3.2, we also immediately get the following result.

Corollary 3.10. Let $I$ be an equimultiple ideal of $ht I = h > 0$ and grade $I = h$. Then $F_A(I)$ is Cohen-Macaulay with minimal multiplicity if and only if $r(I) \leq 1$.

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