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THE KADISON-SINGER PROBLEM AND THE UNCERTAINTY PRINCIPLE

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ABSTRACT. We compare and contrast the Kadison-Singer problem to the Uncertainty Principle via exponential frames. Our results suggest that the Kadison-Singer problem, if true, is in a sense a stronger version of the Uncertainty Principle.

In 1959, Kadison and Singer answered in the negative [13] the well-known question of unique pure state extensions: can a pure state on a C^*-subalgebra of $B(H)$ be extended uniquely to a pure state on all of $B(H)$? They showed that in general the extension is not unique. However, they were unable to solve a special case of this question: can a pure state on the algebra of diagonal operators in $B(\ell^2(\mathbb{Z}))$ be extended uniquely to a pure state on all of $B(\ell^2(\mathbb{Z}))$? The answer to this special case is still unknown, and the question is now called the Kadison-Singer problem (KSP).

In 1979, Anderson stated what is now called the Paving Conjecture (PC): if $T \in B(\ell^2(\mathbb{Z}))$ has zeroes on the diagonal, given $\epsilon > 0$, does there exist a finite partition $\{A_j\}_{j=1}^N$ of $\mathbb{Z}$ such that

$$\|Q_{A_j} T Q_{A_j}\| < \epsilon?$$

Here Q_{A_j} is the canonical projection onto the subspace $\ell^2(A_j) \subset \ell^2(\mathbb{Z})$. Anderson proved that the PC is equivalent to the KSP [1, 2].

There is a finite dimensional version of the PC, which is equivalent to the infinite dimensional version stated above, as follows:

Anderson Paving Conjecture. For $\epsilon > 0$, there is a natural number N so that for every natural number n and every linear operator T on l_2^n whose matrix has zero diagonal, we can find a partition (i.e. a *paving*) $\{A_j\}_{j=1}^N$ of $\{1, \dots, n\}$, such that

$$\|Q_{A_j} T Q_{A_j}\| \le \epsilon \|T\| \quad \text{for all } j = 1, 2, \dots, N.$$

It is now known that the PC is equivalent to just paving any one of the following special classes of operators [9] (when we talk of paving a class of operators without zero diagonal, we mean paving the operators with the diagonal removed): unitary operators, orthogonal projections, positive operators, selfadjoint operators, invertible operators, Gram matrices. Recently [8], it was shown that the PC is

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equivalent to paving operators U with zero diagonal and satisfying $U^2 = I$. It is also shown in [8] that the PC is equivalent to paving orthogonal projections with the constant $1/2$ on the diagonal (and that this class cannot be paved with $N = 2$ paving sets). We believe that if it can be shown that this class is not $N = 3$ pavable, then a counterexample to the whole problem will follow. Also, it was recently shown that the PC is equivalent to paving triangular matrices [15]. For an up-to-date analysis of the status of the KSP and the PC we recommend the website: http://www.aimath.org/pastworkshops/kadisonsinger.html.

There have been numerous (seemingly weaker) related problems and special cases considered: paving Laurent operators [12]; invertibility of large submatrices [4, 5]; the Feichtinger conjecture [6]. All of these related problems remain unsolved in their respective full generality. This is unsurprising, since it is now known that the KSP is equivalent to the Bourgain-Tzafriri conjecture and the Feichtinger conjecture, along with a host of other problems [7, 9, 10, 14].

Our purpose in this paper is to connect the KSP to the Uncertainty Principle, indeed to demonstrate that the KSP is, if true, a much stronger version of the Uncertainty Principle. We will show the extent to which the Uncertainty Principle falls short of the KSP. We shall consider specifically two versions of the Uncertainty Principle: the discrete version on the group $\mathbb{Z}_N$, and the semi-discrete version on the dual groups $\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$.

By "Suppose that the KSP is true" we mean that we suppose that every pure state extension is unique. We point out here that Kadison and Singer themselves "incline to the view" [13] that the KSP is false, that is, that the extensions are not in general unique. Thus, the Uncertainty Principle provides one direction for locating the strongest results which are true if (and when) it is shown that the KSP is false in general.

1. Uncertainty and discrete exponential frames

Our first comparison between the KSP and the Uncertainty Principle is accomplished via discrete exponential frames. These arise from considering the Fourier basis in the finite dimensional Hilbert space $\ell^2(\mathbb{Z}_N)$. For $v \in \ell^2(\mathbb{Z}_N)$, let $v[k]$ denote the k-th coordinate; or equivalently, we consider v to be a function on $\mathbb{Z}_N$, and $v[k]$ is the value at $k \in \mathbb{Z}_N$.

Let f_j denote the j-th Fourier basis element in $\ell^2(\mathbb{Z}_N)$, given by:

$$f_j[k] = \frac{1}{\sqrt{N}} e^{2\pi i jk/N}.$$

For $E \subset \mathbb{Z}_N$, $v \in \ell^2(\mathbb{Z}_N)$, we let $v\chi_E$ denote the element in $\ell^2(\mathbb{Z}_N)$ such that $(v\chi_E)[k] = v[k]$ if $k \in E$ and 0 otherwise. This is equivalent to projecting v onto the subspace $\ell^2(E) \subset \ell^2(\mathbb{Z}_N)$. Our discrete exponential frames are of the form $\{f_j\chi_E : j \in F \subset \mathbb{Z}_N; E \subset \mathbb{Z}_N\}$.

We recall the relevant definitions from frame theory. Let $\mathbb{J}$ be a finite or countable index set and let H be a Hilbert space. We say $\mathbb{X} := \{x_j\}_{j\in\mathbb{J}} \subset H$ is *Bessel* if the following (formally densely defined) operator is well-defined and bounded:

$$\Theta_{\mathbb{X}}^* : \ell^2(\mathbb{J}) \to H : (c_j)_j \mapsto \sum_{j\in\mathbb{J}} c_j x_j.$$

This is called the synthesis operator. If $\mathbb{X}$ is Bessel, then the adjoint $\Theta_{\mathbb{X}}$ of $\Theta_{\mathbb{X}}^*$ is

$$\Theta_{\mathbb{X}} : H \to \ell^2(\mathbb{J}) : v \mapsto (\langle v, x_j \rangle)_j.$$

This is called the analysis operator.

The set $\mathbb{X} = \{x_j\}_{j\in\mathbb{J}}$ is a *Riesz basic sequence* if

$$\Theta_{\mathbb{X}}\Theta_{\mathbb{X}}^* : \ell^2(\mathbb{J}) \to \ell^2(\mathbb{J})$$

is invertible. If so, there are constants $B_1, B_2 > 0$, called the Riesz basis bounds, such that

$$B_1\|(c_j)\|_{\ell^2(\mathbb{J})} \leq \|\sum_{j\in J} c_j x_j\|_H \leq B_2\|(c_j)\|_{\ell^2(\mathbb{J})} \quad \forall (c_j) \in \ell^2(\mathbb{J}).$$

The set $\mathbb{X}$ is a *frame* if

$$\Theta_{\mathbb{X}}^*\Theta_{\mathbb{X}} : H \to H$$

is invertible. If so, there are constants $C_1, C_2 > 0$, called the frame bounds, such that

$$C_1\|v\|_H^2 \leq \sum_j |\langle v, x_j\rangle|^2 \leq C_2\|v\|_H^2 \quad \forall v \in H.$$

The operator $\Theta_{\mathbb{X}}\Theta_{\mathbb{X}}^*$ is called the Grammian, or Gram matrix, for $\mathbb{X}$; the entries are given by $\langle x_j, x_k\rangle$.

As mentioned in the introduction, the following conjecture is related to the KSP:

Conjecture (Feichtinger)**.** If $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence for which there is a constant $M > 0$ such that $\|x_j\| \geq M$ for all $j \in \mathbb{J}$, is there a finite partition $\{A_k\}_{k=1}^N$ of $\mathbb{J}$ such that the subsets $\{x_j\}_{j\in A_k}$ are Riesz basic sequences?

It turns out that the Feichtinger conjecture is equivalent to the KSP: it was shown in [6] that the KSP implies the Feichtinger conjecture, while the converse was shown in [10]. Via the Feichtinger conjecture, we obtain a consequence of the KSP for exponential frames.

Theorem 1. *Suppose that the KSP is true. For the pair of sequences $\{E_N\}$, $\{F_N\}$, with $E_N, F_N \subset \{0, \ldots, N-1\}$ and $|E_N| = O(N)$, then there exist constants K, L independent of N such that F_N can be partitioned into at most K subsets $\{A_N^j\}$ where $\{f_l\chi_{E_N} | l \in A_N^j\}$ is a Riesz basic sequence with lower basis bound greater than L.*

Proof. We consider the Hilbert space $H = \bigoplus_{N=1}^\infty \ell^2(\mathbb{Z}_N)$, and imbed the set $\{f_l\chi_{E_N} | l \in F_N\}$ in H by

$$f_l\chi_{E_N} \mapsto \underbrace{0 \oplus \cdots \oplus 0}_{N-1} \oplus f_l\chi_{E_N} \oplus 0 \oplus \cdots .$$

Thus, we identify the set $\bigcup_N\{f_l\chi_{E_N} | l \in F_N\}$ with its image under the above imbedding, and consider the resulting sequence as indexed by $\mathbb{J} = \{(N, l) : N, l \in \mathbb{N};\ l \in F_N\}$. We claim that this sequence is Bessel; indeed, for each $N \in \mathbb{N}$, the set $\{0 \oplus \cdots \oplus 0 \oplus f_l\chi_{E_N} \oplus 0 \oplus \cdots\}$ enjoys the property that for any $v \in H$, writing

$v = \bigoplus_{N=1}^{\infty} v_N$, we have

$$\begin{aligned}\sum_{l \in F_N} |\langle v, 0 \oplus \cdots \oplus 0 \oplus f_l \chi_{E_N} \oplus 0 \oplus \cdots \rangle|^2 &= \sum_{l \in F_N} |\langle v_N, f_l \chi_{E_N} \rangle|^2 \\ &\leq \sum_{l=1}^{N} |\langle \chi_{E_N} v_N, f_l \rangle|^2 \\ &\leq \|v_N\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{N=1}^{\infty} \sum_{l \in F_N} |\langle v, 0 \oplus \cdots \oplus 0 \oplus f_l \chi_{E_N} \oplus 0 \oplus \cdots \rangle|^2 &= \sum_{N=1}^{\infty} \sum_{l \in F_N} |\langle v_N, f_l \chi_{E_N} \rangle|^2 \\ &\leq \sum_{N=1}^{\infty} \|v_N\|^2 \\ &= \|v\|^2.\end{aligned}$$

Moreover, this sequence is bounded: that is, there exists an $M > 0$ such that $M \leq \|0 \oplus \cdots \oplus 0 \oplus f_l \chi_{E_N} \oplus 0 \oplus \cdots \|^2$. Indeed, we have

$$\begin{aligned}\|0 \oplus \cdots \oplus 0 \oplus f_l \chi_{E_N} \oplus 0 \oplus \cdots \|^2 &= \|f_l \chi_{E_N}\|^2 \\ &= \frac{|E_N|}{N}.\end{aligned}$$

By assumption, $|E_N| = O(N)$, and thus there exists an $M > 0$ such that $M \leq \dfrac{|E_N|}{N}$ for all N.

Thus, we apply the Feichtinger conjecture formulation of the KSP to the bounded Bessel sequence, which gives us that we can partition the index set $\mathbb{J}$ into $A_1, \ldots, A_K$ with the property that each subset

$$\{0 \oplus \cdots \oplus 0 \oplus f_l \chi_{E_N} \oplus 0 \oplus \cdots | (N, l) \in A_k\}$$

is a Riesz basic sequence with lower basis bound L_k. Let $L = min\{L_1, \ldots, L_K\}$. For each $N \in \mathbb{N}$, the set

$$\{0 \oplus \cdots 0 \oplus f_l \chi_{E_N} \oplus 0 \oplus \cdots | (N, l) \in A_k\}$$

is a Riesz basic sequence with lower basis bound at least L, since it is a subset of a Riesz basic sequence with lower basis bound at least L. Since the embedding $f_l \chi_{E_N} \mapsto 0 \oplus \cdots 0 \oplus f_l \chi_{E_N} \oplus 0 \oplus \cdots$ is isometric, we have that for N fixed, the set $\{f_l \chi_{E_N} : (N, l) \in A_k\}$ is again a Riesz basic sequence, with lower basis bound at least L. This partitions $\{f_l \chi_{E_N} | l \in F_N\}$ into K subsets, each with lower basis bound at least L. $\square$

Turning to the Uncertainty Principle for discrete exponential frames, we have the following statement.

Theorem (Donoho-Stark [11]). *If $v \in \ell^2(\mathbb{Z}_N)$ (non-zero), then*

$$|\{k = 0, \ldots, N-1 : v(k) \neq 0\}| \cdot |\{k = 0, \ldots, N-1 : \hat{v}(k) \neq 0\}| \geq N.$$

Corollary 1. *If $E, F \subset \{0, \ldots, N-1\}$ and $|F| < \dfrac{N}{N - |E|}$, then $\{f_j \chi_E : j \in F\}$ is a Riesz basic sequence.*

Proof. Suppose that $\sum_{j\in F} c_j f_j \chi_E = 0$. Let $v = \sum_{j\in F} c_j f_j$. By definition, $|\{k = 1,\dots,N : \hat{v}(k) \neq 0\}| \leq |F|$. Moreover, since $v\chi_E = 0$, we have that $|\{k = 1,\dots,N : v(k) \neq 0\}| \leq N - |E|$. By assumption, we have

$$|\{k = 1,\dots,N : v(k) \neq 0\}| \cdot |\{k = 1,\dots,N : \hat{v}(k) \neq 0\}| \leq |F|(N - |E|) < N;$$

therefore the Uncertainty Principle yields that $v = 0$, and hence $c_j = 0$ for every j. Thus, we have that $\{f_j \chi_E : j \in F\}$ is a finite linearly independent set and hence is a Riesz basic sequence. □

Theorem 2. *Let $\{M_N\}$ be a sequence of positive integers such that $\alpha := \sup_N N - M_N < \infty$. For the pair of sequences $\{E_N\}$, $\{F_N\}$, with $E_N, F_N \subset \{0,\dots,N-1\}$ and $|E_N| \geq M_N$, then there exists a constant $K := K(\alpha)$ independent of N such that there is a partition of $\{0,\dots,N-1\}$ into at most K subsets $\{A_N^j\}$ where $\{f_l \chi_{E_N} : l \in A_N^j \cap F_N\}$ is a Riesz basic sequence.*

Proof. The proof is a consequence of Corollary 1. If $E_N \subset \{0,\dots,N-1\}$ with $|E_N| \geq M_N$, then $N - |E_N| \geq \alpha$. If $K = \lceil \alpha + 1 \rceil$, we have that $\dfrac{N}{K} < \dfrac{N}{N - |E_N|}$ for all N, and thus we can arbitrarily partition F_N into K subsets $\{A_N^k : k = 1,\dots,K\}$, each of which has cardinality at most $\dfrac{N}{K}$. For each subset A_N^k, we have $|A_N^k| < \dfrac{N}{N - |E_N|}$, and thus by Corollary 1 the set $\{f_l \chi_{E_N} : l \in A_N^k\}$ is a Riesz basic sequence. □

Note 1. Of course, we can choose $F_N = \{0,\dots,N-1\}$. So, if we restrict the Fourier basis to a subset of its domain, we can partition that frame into finitely many linearly independent sets, and the number depends on the size of the subset of the domain.

We mention here that the bound in Corollary 1, and hence also Theorem 2, is the best possible. This is because the inequality in the Uncertainty Principle is sharp; i.e., there exist extremal vectors $v \in \ell^2(\mathbb{Z}_N)$ such that the bound is attained whenever N is composite [11]. In fact, these extremal vectors have the form $v = \chi_E$. Therefore, the proof of Corollary 1 breaks down if $|F| \geq \dfrac{N}{N - |E|}$ and E corresponds to one of these extremal cases. We refer to [11] for the details.

In view of Theorems 1 and 2, we see that both the KSP and the Uncertainty Principle decompose discrete exponential frames into Riesz basic sequences. Note how the Uncertainty Principle falls short: the primary weakness of the Uncertainty Principle is that it requires a much stronger growth rate on the sets E_N than does the KSP. This is necessitated by the sharpness of the bound just mentioned in Corollary 1. The secondary weakness is that the Uncertainty Principle does not imply a common lower basis bound for the Riesz basic sequences, whereas the KSP does.

Note that this suggests a possibility of more quantitative versions of the discrete Uncertainty Principle. These quantitative versions would conceivably do the following: 1) account for the extremal cases; 2) give information regarding a lower basis bound, and 3) close the current gap between the Uncertainty Principle and the KSP.

2. Uncertainty and Laurent operators

We now turn our attention to a similar comparison between the Uncertainty Principle and the KSP applied to Laurent Operators via the PC. This comparison is less illustrative than that of the discrete version in the previous section, but here the direct application of the PC is more transparent.

A Laurent operator is a bounded operator T on $\ell^2(\mathbb{Z})$ which commutes with the bilateral shift. The matrix form of T satisfies the condition that $T[m+1, n+1] = T[m, n]$. Thus, the $(m+1)$th row of T is the mth row shifted to the right by 1. If $x \in \ell^2(\mathbb{Z})$, taking the Fourier transform yields:

$$\widehat{Tx}(\xi) = \phi(\xi)\hat{x}(\xi),$$

where $\phi(\xi)$ is the Fourier transform of the 0 row of T; ϕ is essentially bounded and is called the symbol of T. We normalize ϕ in the $L^1([0,1])$ norm, so that the diagonal of T is 1.

We shall say that T is pavable if $T - I$ is pavable in the sense of the definition given in the introduction. Note here that the diagonal of $T - I$ is 0.

It is not known if every Laurent operator is pavable, but many of them are [12]:

Theorem (Halpern-Kaftal-Weiss)**.** *If the symbol ϕ of the Laurent operator T_ϕ is bounded and Riemann integrable, then T_ϕ is pavable (in fact uniformly pavable by arithmetic progressions).*

It is a long-standing open problem whether paving the Laurent operators is equivalent to KSP.

If we let $\{x_n : n \in \mathbb{Z}\}$ denote the rows of T_ϕ, then the Fourier transform of the rows yields the set $\{e^{2\pi i n\xi}\phi(\xi) : n \in \mathbb{Z}\}$. This sequence of exponentials is a Bessel sequence in $L^2([0,1])$, since $\phi \in L^\infty([0,1])$ [3]. Conversely, if $\{e^{2\pi i n\xi}\psi(\xi) : n \in \mathbb{Z}\}$ is a Bessel sequence in $L^2([0,1])$, then there is a corresponding Laurent operator T_ψ on $\ell^2(\mathbb{Z})$.

Paving a Laurent operator decomposes an exponential Bessel sequence into finitely many Riesz basic sequences, just as the KSP decomposes discrete exponential frames into Riesz basic sequences.

Theorem 3. *Suppose that T_ϕ is a Laurent operator with normalized symbol ϕ. If T_ϕ is pavable, then the Bessel sequence $\{e^{2\pi i n\xi}\phi(\xi) : n \in \mathbb{Z}\}$ satisfies the Feichtinger conjecture. That is to say, there is a finite partition $\{A_k\}_{k=1}^K$ such that each subset $\{e^{2\pi i n\xi}\phi(\xi) : n \in A_k\}$ is a Riesz basic sequence. Moreover, given $1 > \epsilon > 0$ we may choose the partition so that each subset has lower Riesz basis bound greater than $1 - \epsilon$.*

Proof. Let $1 > \epsilon > 0$ be given, and choose a $\delta > 0$ such that $\delta^2 + 2\delta < \epsilon$. Pave T_ϕ with the threshhold of δ; i.e., find a finite partition $\{A_k\}_{k=1}^K$ such that $\|Q_{A_k}(T_\phi - I)Q_{A_k}\| < \delta$. We claim that for each $k = 1, \dots, K$, $\{e^{2\pi i n\xi}\phi(\xi) : n \in A_k\}$ is a Riesz basic sequence with lower basis bound greater than $1 - \epsilon$.

Let M_k denote the submatrix of T_ϕ given by $Q_{A_k}(T_\phi)Q_{A_k}$. Regarding $M_k : \ell^2(A_k) \to \ell^2(A_k)$, we can write $M_k = I + B$, where $\|B\| < \delta$, owing to the fact that M_k has 1's on the diagonal, whence $\|M_k - I\| = \|Q_{A_k}(T_\phi - I)Q_{A_k}\| < \delta$. Furthermore, we regard M_k as the analysis operator of the vectors given by its rows (adjusting suitably for the necessary conjugate); since $\|B\| < 1$, M_k is invertible, and therefore the rows form a Riesz basis for $\ell^2(A_k)$. Now, the basis bounds for the rows of M_k can be estimated by the spectrum of $M_kM_k^*$: if

$$AI \leq M_kM_k^* \leq BI,$$

then $\sqrt{A}$ and $\sqrt{B}$ are basis bounds for the rows of M_k. Note that $M_kM_k^* = I + B + B^* + BB^*$, so by virtue of $\|B + B^* + BB^*\| < 2\delta + \delta^2$ and the spectral mapping theorem, we have that

$$(1 - \epsilon)I \leq M_kM_k^* \leq (1 + \epsilon)I.$$

Therefore, the rows of M_k form Riesz basic sequences with lower basis bound greater than $\sqrt{1 - \epsilon} > 1 - \epsilon$.

Now, the rows of T_ϕ corresponding to A_k enjoy the same property. Indeed, let $\{x_n : n \in A_k\}$ denote the rows of T_ϕ; we can write $x_n = Q_{A_k}x_n + y_n$, where $\{y_n \in \ell^2(\tilde{A}_k) : n \in A_k\}$ is a Bessel sequence (the rows of T_ϕ form a Bessel sequence, and thus $\{y_n : n \in A_k\}$ is the projection of a Bessel sequence). Thus, we have $x_n = z_n \oplus y_n \in \ell^2(A_k) \oplus \ell^2(\tilde{A}_k)$, with $\{z_n : n \in A_k\}$ the rows from the matrix M_k. Appealing to Lemma 1 below yields the claim that $\{x_n : n \in A_k\}$ is a Riesz basic sequence with lower basis bound greater than $1 - \epsilon$.

Note that the nth row of T_ϕ, x_n, satisfies the condition $S^n x_0 = x_n$, where $S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the bilateral shift. If we take the Fourier transform of the set $\{x_n : n \in A_k\} = \{S^n x_0 : n \in A_k\}$, we obtain the set $\{e^{2\pi in\xi}\phi(\xi) : n \in A_k\}$, since the Fourier transform maps x_0 to ϕ. Since the Fourier transform is a unitary operator, we have that $\{e^{2\pi in\xi}\phi(\xi) : n \in A_k\}$ is a Riesz basic sequence with lower basis bound greater than $1 - \epsilon$. □

It is not known at this time if the KSP for Laurent operators is equivalent to the Feichtinger Conjecture for Bessel sequences of exponential functions.

Lemma 1. *Suppose $\{v_n\} \subset H$ is a Riesz basic sequence with lower bound A, and $\{w_n\} \subset \tilde{H}$ is a Bessel sequence. Then $\{v_n \oplus w_n\} \in H \oplus \tilde{H}$ is a Riesz basic sequence with lower basis bound A.*

Proof. Let c_n be such that $\sum_n |c_n|^2 < \infty$. Since $\{v_n\}$ is a Riesz basic sequence with lower bound A, we have:

$$\begin{aligned}\|\sum_n c_n(v_n \oplus w_n)\|^2 &= \|\sum_n c_n v_n\|^2 + \|\sum_n c_n w_n\|^2 \\ &\geq \|\sum_n c_n v_n\|^2 \\ &\geq A^2 \sum_n |c_n|^2.\end{aligned}$$

□

We have seen that the KSP, via the PC, wants to decompose a Bessel sequence of exponential functions into Riesz basic sequences. Now consider the results obtained from the Uncertainty Principle.

Theorem (Smith [16])**.** *If $f \in L^2[0,1]$ is nonzero, supported on $V \subset [0,1]$, and $\hat{f}$ is supported on $W \subset \mathbb{Z}$, then $|V||W| \geq 1$, where $|V|$ is the Lebesgue measure of V and $|W|$ is the cardinality of W.*

Corollary 2. *If $E \subset [0,1]$ (measurable) and $W \subset \mathbb{Z}$ such that $(1 - |E|)|W| < 1$, then*

$$\{e^{2\pi i n \xi}\chi_E(\xi) : n \in W\}$$

is a Riesz basic sequence.

Proof. Suppose that $\sum_{n\in W} c_n e^{2\pi i n\xi}\chi_E(\xi) = 0$. Then the function $f(\xi) = \sum_{n\in W} c_n e^{2\pi i n \xi}$ satisfies the conditions $\mathrm{supp}(\hat{f}) \subset W$ and $\mathrm{supp}(f) \subset [0,1] \setminus E$, so if f is not zero, we would have $|V||W| < 1$, where $V = [0,1] \setminus E$, a violation of the Uncertainty Principle. Since W is necessarily finite, we have $\{e^{2\pi i n\xi}\chi_E(\xi) : n \in W\}$ is a Riesz basic sequence. $\square$

Therefore, the Uncertainty Principle allows us to decompose $\{e^{2\pi i n\xi}\chi_E(\xi) : n \in \mathbb{Z}\}$ into linearly independent subsets, each of which is finite, and thus each subset is a Riesz basic sequence. Notice that each partition element is finite, so the size of the partition is infinite. Furthermore, the Uncertainty Principle is actually weaker here than other considerations, since we know that for any measurable non-null E and any finite W, the set $\{e^{2\pi i n\xi}\chi_E(\xi) : n \in W\}$ is linearly independent, regardless of whether $(1 - |E|)|W| < 1$. Also, from [6] it is known that every bounded Bessel sequence is a finite union of (finitely) linearly independent sets.

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