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POSITIVE SOLUTIONS OF ANISOTROPIC YAMABE-TYPE EQUATIONS IN $\mathbb{R}^n$

ROBERTO MONTI AND DANIELE MORBIDELLI

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Abstract. We study entire positive solutions to the partial differential equation in $\mathbb{R}^n$,

$$\Delta_x u + (\alpha+1)^2|x|^{2\alpha}\Delta_y u = -|x|^{2\alpha}u^{\frac{n+2}{n-2}},$$

where $x \in \mathbb{R}^2$, $y \in \mathbb{R}^{n-2}$, $n \geq 3$ and $\alpha > 0$. We classify positive solutions with second order derivatives satisfying a suitable growth near the set $x = 0$.

1. Introduction

In this paper, we study positive solutions to the partial differential equation in $\mathbb{R}^n$,

$$\Delta_x u + (\alpha+1)^2|x|^{2\alpha}\Delta_y u = -|x|^{2\alpha}u^{\frac{n+2}{n-2}}, \tag{1.1}$$

where $x \in \mathbb{R}^2$, $y \in \mathbb{R}^{n-2}$, $n \geq 3$, and $\alpha > 0$. This equation is an anisotropic generalization of the Yamabe equation in $\mathbb{R}^n$, the case $\alpha = 0$, and it displays several interesting properties. In particular, the structure of positive entire solutions depends on whether α is an integer or not.

Our technique is based on a Kelvin inversion naturally associated with the equation, which was introduced in [MM], and on an integration-by-parts argument inspired by Obata's classical paper [O]. In this part of the argument, we need to assume a bound on the growth of second order derivatives of solutions near the singular set $x = 0$.

Equation (1.1) has the following geometric interpretation. Let $M = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} : x \neq 0\}$ be endowed with the Riemannian metric

$$g = (\alpha+1)^2|x|^{2\alpha}|dx|^2 + |dy|^2. \tag{1.2}$$

The manifold (M, g) is locally isometric to $\mathbb{R}^n$ with the standard metric, an isometry being $(x, y) \mapsto (x^{\alpha+1}, y)$ with the $x^{\alpha+1}$ power of x as a complex number. If $x \in \mathbb{R}^k$ with $k \geq 3$, this is no longer true. The isometry, however, is not global and (M, g) is not complete. The scalar curvature R of g vanishes identically. The scalar curvature $\widehat{R}$ of the conformal metric $\widehat{g} = u^{\frac{4}{n-2}}g$ is

$$\widehat{R} = -4\frac{n-1}{n-2}u^{-\frac{n+2}{n-2}}\Delta_g u,$$

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where Δ_g is the Laplace-Beltrami operator of g. It can be checked that $\Delta_g = (\alpha+1)^{-2}|x|^{-2\alpha}\Delta_x + \Delta_y$. Then, u solves (1.1) in $\mathbb{R}^n \setminus \{x=0\}$ if and only if $(M, \widehat{g})$ has constant scalar curvature $\widehat{R} = \frac{4(n-1)}{(n-2)(\alpha+1)^2}$.

The partial differential operator

$$\mathcal{L} = \Delta_x + (\alpha+1)^2|x|^{2\alpha}\Delta_y$$

is known as the "Grushin operator" and is an important example of a second order elliptic degenerate operator (see [Gr], [FL], [BGG], [GV1], [GV2]). In [MM], we studied symmetry and uniqueness properties of positive entire solutions of the unweighted equation

$$\mathcal{L}u = -u^{\frac{Q+2}{Q-2}} \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$, $\alpha > 0$ and $Q = k + (\alpha+1)(n-k)$. Equation (1.3) is the critical point equation for a Sobolev inequality whose extremal functions are studied in [Mo] (see also [B]). The number $\frac{Q+2}{Q-2}$, which is the critical exponent for the Grushin operator, is strictly smaller than $\frac{n+2}{n-2}$ for any $\alpha > 0$. The gap between these exponents is balanced by the weight $|x|^{2\alpha}$ in the right-hand side of equation (1.1).

In order to state our assumption in the classification theorem, let us identify $\xi \in \mathbb{R}^2$ with $\xi = \xi_1 + i\xi_2 \in \mathbb{C}$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{\xi \in \mathbb{C} : \operatorname{Re}\xi \le 0 \text{ and } \operatorname{Im}\xi = 0\}$ and take on $\mathbb{C}^*$ a fixed branch $\xi^{1/(\alpha+1)}$ of the $(\alpha+1)$-th root. We say that $u \in C^2(\mathbb{R}^n)$ belongs to the class $\mathcal{A}$, and we write $u \in \mathcal{A}$, if for any $\vartheta \in [0, 2\pi)$ and $r > 0$ there exist $C > 0$ and $\beta < 1$ such that the function

$$v(\xi, y) = u\big(e^{i\vartheta}\xi^{\frac{1}{\alpha+1}}, y\big), \quad \xi \in \mathbb{C}^* \text{ and } y \in \mathbb{R}^{n-2}, \tag{1.4}$$

satisfies

$$|\nabla^2_{\xi\xi} v(\xi, y)| \le C|\xi|^{-\beta}, \qquad \text{for } |y| \le r \text{ and } |\xi| \le 1, \tag{1.5}$$

where $|\nabla^2_{\xi\xi} v| = \sum_{i,j=1}^2 |\partial_{\xi_i}\partial_{\xi_j} v|$.

If u solves (1.1), then the function v in (1.4) solves the Yamabe equation

$$\Delta v = -\frac{1}{(\alpha+1)^2} v^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{C}^* \times \mathbb{R}^{n-2}. \tag{1.6}$$

The positive entire solutions to equation (1.6) in the whole $\mathbb{R}^n = \mathbb{C} \times \mathbb{R}^{n-2}$ are classified. A recent proof for this well-known result and up-to-date references can be found in [LZ]. Our results cannot be obtained from this classification because the solutions to equation (1.6) may be singular on the set where $\operatorname{Re}\xi \le 0$ and $\operatorname{Im}\xi = 0$.

Condition (1.5) has an intrinsic meaning in terms of the metric g in (1.2). This will be clear in Section 3, where we shall look at the manifold M in suitable coordinates.

Our main result is the following theorem.

Theorem 1.1. *Let $u \in C^2(\mathbb{R}^n) \cap \mathcal{A}$ be a positive solution of* (1.1). *Then:*

i) *If $\alpha \in \mathbb{N}$, there are $(x_0, y_0) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ and $a, b > 0$ with $n(n-2)(\alpha+1)^2 ab = 1$ such that*

$$u(x, y) = \big(a\{|x^{\alpha+1} - x_0^{\alpha+1}|^2 + |y - y_0|^2\} + b\big)^{\frac{2-n}{2}}. \tag{1.7}$$

ii) *If* $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, *there are* $y_0 \in \mathbb{R}^{n-2}$ *and* $a, b > 0$ *with* $n(n-2)(\alpha+1)^2 ab = 1$ *such that*

$$u(x,y) = \left(a\{|x|^{2(\alpha+1)} + |y-y_0|^2\} + b\right)^{\frac{2-n}{2}}. \tag{1.8}$$

Both functions in (1.7) and (1.8) are of class $C^2(\mathbb{R}^n)$. Moreover, they satisfy (1.5) with $\beta = 0$. We have not been able to find a counterexample showing the sharpness of condition (1.5). This condition is needed in a crucial step in the argument of Section 3 (see (2.5) and (3.11)–(3.12)).

If $x_0 \neq 0$, the function in (1.7) is not radial in x in spite of the symmetric structure of equation (1.1). A similar phenomenon appears for the Liouville-type equation $\Delta u(x) = -8\pi(\alpha+1)|x|^{2\alpha} e^{u(x)}$ in $\mathbb{R}^2$, which has only radial solutions if $\alpha \notin \mathbb{N}$ and has both radial and nonradial solutions if $\alpha \in \mathbb{N}$ (see the classification result in [PT]).

Notation. In the following, $C > 0$ is a constant which may vary from line to line. We also use the following notation:

$$|\nabla^2_{\xi y} v| = \sum_{k=1}^{2} \sum_{j=1}^{n-2} |\partial_{\xi_k} \partial_{y_j} v|, \quad |\nabla^2_{\xi\xi} v| = \sum_{k,h=1}^{2} |\partial_{\xi_k} \partial_{\xi_h} v|, \quad \text{etc.}$$

2. Asymptotic estimates

Introduce the inversion $\mathcal{I} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ by

$$\mathcal{I}(z) = \left(\frac{x}{\|z\|^2}, \frac{y}{\|z\|^{2(\alpha+1)}} \right), \quad \text{where } \|z\| = \left(|x|^{2(\alpha+1)} + |y|^2\right)^{\frac{1}{2(\alpha+1)}}.$$

The inversion is a conformal map of the metric (1.2) (see [M]). The Kelvin transform of a function u in $\mathbb{R}^n$ is the function u^* in $\mathbb{R}^n \setminus \{0\}$ given by

$$u^*(z) = \|z\|^{(2-n)(\alpha+1)} u\left(\mathcal{I}(z)\right). \tag{2.1}$$

The Kelvin transform (2.1) was introduced in [MM] in connection with the study of equation (1.3) (but see also [LP] for the case $x \in \mathbb{R}$). The Kelvin transform preserves equation (1.1).

Proposition 2.1. *If* $u \in C^2(\mathbb{R}^n)$ *is a positive solution of* (1.1) *in* $\mathbb{R}^n$, *then* $u^* \in C^2(\mathbb{R}^n \setminus \{0\})$ *is a solution of* (1.1) *in* $\mathbb{R}^n \setminus \{0\}$.

Proof. Let $\varphi \in C^1_0(\mathbb{R}^n \setminus \{0\})$ be a test function and denote by φ^* its Kelvin transform. We use identity (2.8) in [MM], which reads

$$\int_{\mathbb{R}^n} \nabla_\alpha u^*(z) \cdot \nabla_\alpha \varphi^*(z)\, dxdy = \int_{\mathbb{R}^n} \nabla_\alpha u(z) \cdot \nabla_\alpha \varphi(z)\, dxdy, \tag{2.2}$$

where $\nabla_\alpha u = (\nabla_x u, (\alpha+1)|x|^\alpha \nabla_y u)$ and $\cdot$ denotes the standard scalar product in $\mathbb{R}^n$. By an integration by parts, we get from (2.2):

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{L}u^*(z)\, \varphi^*(z)\, dxdy &= -\int_{\mathbb{R}^n} \nabla_\alpha u(z) \cdot \nabla_\alpha \varphi(z)\, dxdy \\ &= \int_{\mathbb{R}^n} \mathcal{L}u(z)\, \varphi(z)\, dxdy \\ &= -\int_{\mathbb{R}^n} u(z)^{\frac{n+2}{n-2}}\, \varphi(z)\, |x|^{2\alpha} dxdy. \end{aligned} \tag{2.3}$$

In the last line we used equation (1.1). Denote by $J_{\mathcal{I}}(z')$ the Jacobian determinant of $\mathcal{I}$ at the point $z' = (x', y') \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, $z' \neq 0$. By Lemma 2.2 in [MM], we have $|J_{\mathcal{I}}(z')| = \|z'\|^{-2(n-2)(\alpha+1)-4}$. Performing the change of variable $z = \mathcal{I}(z')$ in the integral on the last line of (2.3), we obtain

$$\int_{\mathbb{R}^n} \mathcal{L}u^*(z)\,\varphi^*(z)\,dxdy = -\int_{\mathbb{R}^n} u^*(z')^{\frac{n+2}{n-2}}\,\varphi^*(z')\,|x'|^{2\alpha}\,dx'dy'.$$

This proves the claim. □

In view of equation (1.6), Proposition 2.1 can also be proved by means of the standard properties of the usual Kelvin transform.

For any function $u : \mathbb{R}^n \to \mathbb{R}$ and $\lambda > 0$, we define the function u_λ in $\mathbb{R}^n \setminus \{0\}$ by

$$u_\lambda(z) = \Big(\frac{\|z\|}{\lambda}\Big)^{(2-n)(\alpha+1)} u\left(\frac{\lambda^2 x}{\|z\|^2}, \frac{\lambda^{2(\alpha+1)} y}{\|z\|^{2(\alpha+1)}}\right).$$

If u solves (1.1) in $\mathbb{R}^n$, then u_λ solves (1.1) in $\mathbb{R}^n \setminus \{0\}$. This follows from Proposition 2.1 and from the fact that equation (1.1) is invariant with respect to the scaling $u \mapsto \delta_\lambda u$, where

$$\delta_\lambda u(z) = \lambda^{(n-2)(\alpha+1)/2} u(\lambda x, \lambda^{\alpha+1} y), \qquad \lambda > 0.$$

Theorem 2.2. *Let $u \in C^2(\mathbb{R}^n)$ be a positive solution of equation* (1.1)*. Then there exists $\lambda > 0$ such that $u = u_\lambda$.*

Proof. The proof of this theorem is identical to the proof of Theorem 2.7 in [MM]. It relies on Proposition 2.1 and uses the moving spheres method. The scheme of the argument is due to Li and Zhang [LZ, Sec. 2].

Here, we give only a very brief sketch of the proof. In a first step, it is shown that for any positive solution $u \in C^2(\mathbb{R}^n)$ of (1.1) there exists $\lambda_0 > 0$ such that

$$u_\lambda(z) \le u(z), \quad \text{for all } \lambda \in (0, \lambda_0) \text{ and } z \in \mathbb{R}^n \text{ with } \|z\| \ge \lambda.$$

Then we can define

$$\bar\lambda = \sup\big\{\lambda_0 > 0 : u_\lambda \le u \ \text{ in } \{z \in \mathbb{R}^n : \|z\| \ge \lambda\}, \text{ for each } \lambda \in (0, \lambda_0]\big\}. \tag{2.4}$$

In a second step, it is shown that if $\bar\lambda < +\infty$, then $u \equiv u_{\bar\lambda}$ on $\mathbb{R}^n \setminus \{0\}$, which ends the proof of the theorem. In order to see this fact, one considers the function $w_\lambda = u - u_\lambda$, which satisfies $w_\lambda(z) = 0$ for $\|z\| = \lambda$, and

$$w_\lambda \ge 0, \quad \mathcal{L}w_\lambda \le 0 \quad \text{in } \{z \in \mathbb{R}^n : \|z\| \ge \lambda\} \text{ for any } \lambda \in (0, \bar\lambda].$$

If, by contradiction, $w_{\bar\lambda}$ does not vanish identically, then one can violate definition (2.4) for $\bar\lambda$.

It remains to check that $\bar\lambda < +\infty$. Using the invariance of equation (1.1) with respect to translations in y, one can prove that $\bar\lambda = +\infty$ implies $u \equiv 0$, which is not the case. □

From now on, we denote by $\zeta = (\xi, y)$ variables in $\mathbb{C}^* \times \mathbb{R}^{n-2}$.

Theorem 2.3. *Let $u \in C^2(\mathbb{R}^n) \cap \mathcal{A}$ be a positive solution to equation* (1.1) *and let v be the function introduced in* (1.4)*, for some $\vartheta \in [0, 2\pi)$. Then, there exists a constant $C > 0$ such that*

$$\frac{1}{C}|\zeta|^{2-n} \le v(\zeta) \le C|\zeta|^{2-n}, \quad |\nabla v(\zeta)| \le C|\zeta|^{1-n}, \quad |\nabla^2 v(\zeta)| \le C|\zeta|^{-n}\Big(\frac{|\zeta|}{|\xi|}\Big)^{\gamma}, \tag{2.5}$$

for all $\zeta \in \mathbb{C}^* \times \mathbb{R}^{n-2}$ *such that* $|\zeta| \geq 1$. *Here, we have* $\gamma = \max\{\beta, \alpha/(\alpha+1)\}$, *where* β *is the constant in* (1.5). ∇v *and* $\nabla^2 v$ *denote the Euclidean gradient and Hessian in the variable* ζ.

Proof. By Theorem 2.2, we have $u = u_\lambda$ for some $\lambda > 0$. Assume without loss of generality that $\lambda = 1$, i.e. $u = u^*$ in $\mathbb{R}^n$. By (1.4), this identity implies

$$v(\zeta) = |\zeta|^{2-n} v\Big(\frac{\zeta}{|\zeta|^2}\Big), \quad \text{for any } \zeta \in \mathbb{C}^* \times \mathbb{R}^{n-2}. \tag{2.6}$$

The first estimate in (2.5) follows directly from (2.6) and from the fact that u is positive and continuous. Upon differentiating both sides of identity (2.6), we deduce that there is a constant $C > 0$ such that for $\zeta \in \mathbb{C}^* \times \mathbb{R}^{n-2}$ we have

$$\begin{aligned} |\nabla v(\zeta)| &\leq C\left\{|\zeta|^{1-n} v\Big(\frac{\zeta}{|\zeta|^2}\Big) + |\zeta|^{-n}\Big|\nabla v\Big(\frac{\zeta}{|\zeta|^2}\Big)\Big|\right\}, \\ |\nabla^2 v(\zeta)| &\leq C\left\{|\zeta|^{-n} v\Big(\frac{\zeta}{|\zeta|^2}\Big) + |\zeta|^{-1-n}\Big|\nabla v\Big(\frac{\zeta}{|\zeta|^2}\Big)\Big| + |\zeta|^{-2-n}\Big|\nabla^2 v\Big(\frac{\zeta}{|\zeta|^2}\Big)\Big|\right\}. \end{aligned} \tag{2.7}$$

We claim that for some $C > 0$ we have

$$|\nabla v(\zeta)| \leq C \quad \text{for } |\zeta| \leq 1. \tag{2.8}$$

Indeed, $|\nabla_y v(\zeta)|$ is uniformly bounded for $|\zeta| \leq 1$ because $u \in C^2(\mathbb{R}^n)$. Moreover, $|\nabla_\xi v(\zeta)|$ is uniformly bounded as soon as $|\zeta| \leq 1$ because, by (1.5), $|\nabla^2_{\xi\xi} v|$ is locally integrable along lines lying in 2-dimensional planes of $\mathbb{R}^n$ of the form $\mathbb{C}^* \times \{y_0\}$ for any $y_0 \in \mathbb{R}^{n-2}$. Now the second estimate in (2.5) follows from (2.8) and from the first line in (2.7).

Next we prove the last estimate. Observe first that there is a constant $C > 0$ such that for $\zeta = (\xi, y) \in \mathbb{C}^* \times \mathbb{R}^{n-2}$ with $|\zeta| \leq 1$ we have

$$|\nabla^2_{\xi y} v(\zeta)| \leq C|\xi|^{-\frac{\alpha}{\alpha+1}} \sup_{\|z\| \leq 1} |\nabla^2_{xy} u(z)|. \tag{2.9}$$

This follows upon taking mixed second order derivatives in (1.4). Now, from the second line in (2.7), by (1.5), (2.8) and (2.9) we deduce that there is a constant $C > 0$ such that, if $|\zeta| \geq 1$, we have

$$|\nabla^2 v(\zeta)| \leq C\Big\{|\zeta|^{-n} + |\zeta|^{-2-n}\left|\frac{\xi}{|\zeta|^2}\right|^{-\gamma}\Big\} \leq C|\zeta|^{-n}\Big(\frac{|\zeta|}{|\xi|}\Big)^{\gamma},$$

with $\gamma = \max\{\beta, \alpha/(\alpha+1)\}$. This is the last estimate in (2.5). □

3. Classification of solutions

In this section, we prove Theorem 1.1. The geometric motivation of the proof can be found in Obata's argument [O] for the classification of all metrics on the sphere conformal to the standard one and with constant scalar curvature. In order to prove that the new metric is Einstein, Obata shows that the total integral of a certain multiple of the squared norm of the traceless Ricci tensor vanishes. Here we follow a similar idea which leads to the system of partial differential equations (3.17). In our case, however, the manifold is noncompact and noncomplete. This requires careful estimates at infinity and near the singular set $x = 0$.

In order to use estimates (2.5), we prefer to work in suitable charts. This also makes the system of equations (3.17) easy to integrate. Let $m = [\alpha] + 2$, where

$[\alpha]$ is the largest integer less than or equal to α, and for $h = 0, 1, ..., m-1$, let $\vartheta_h = 2h\pi/m$. Define the open subsets of $M = \{(x, y) \in \mathbb{C} \times \mathbb{R}^{n-2} : x \neq 0\}$ to be

$$U_h = \Big\{(x, y) \in M \,:\, x = re^{i\vartheta},\, r > 0,\, |\vartheta - \vartheta_h| < \pi/(\alpha+1)\Big\}, \tag{3.1}$$

and let $f_h : U_h \to \mathbb{C}^* \times \mathbb{R}^{n-2}$,

$$f_h(z) = \big((e^{-i\vartheta_h}x)^{\alpha+1}, y), \qquad z = (x, y) \in U_h.$$

Here, the branch of the $(\alpha+1)$-power is such that $f_0(U_0) = \mathbb{C}^* \times \mathbb{R}^{n-2}$. The charts (U_h, f_h), $h = 0, 1, ..., m-1$, form an oriented atlas for M.

Let $u : M \to \mathbb{R}$ be a function. For each $h = 0, 1, ..., m-1$, we define the function $v_h : \mathbb{C}^* \times \mathbb{R}^{n-2} \to \mathbb{R}$ by letting

$$v_h(\zeta) = u\big(f_h^{-1}(\zeta)\big), \quad \zeta \in \mathbb{C}^* \times \mathbb{R}^{n-2}. \tag{3.2}$$

In the intersection of the charts, we have $v_h(f_h(z)) = v_{h+1}(f_{h+1}(z))$, $z \in U_h \cap U_{h+1}$, which is equivalent to

$$v_h(\xi, y) = v_{h+1}\big(e^{-2(\alpha+1)\pi i/m}\xi, y\big), \quad \text{if } \arg\xi \in \Big(\big(2(\alpha+1)/m - 1\big)\pi, \pi\Big). \tag{3.3}$$

Since the pull-back $(f_h^{-1})^*g$ of the metric g in (1.2) is the Euclidean metric on $\mathbb{C}^* \times \mathbb{R}^{n-2}$, it is clear that the left-hand side of (1.5) is part of the length of the Riemannian Hessian of u.

Now we begin the proof of Theorem 1.1.

Step 1. Let $u \in C^2(\mathbb{R}^n) \cap \mathcal{A}$ be a positive solution of equation (1.1). By elliptic regularity, this function is of class C^∞ away from $x = 0$. By the chain rule, we find

$$\Delta v_h(\zeta) = (\alpha+1)^{-2}|\xi|^{-\frac{2\alpha}{\alpha+1}}\Delta_x u(f_h^{-1}(\zeta)) + \Delta_y u(f_h^{-1}(\zeta)), \tag{3.4}$$

where Δ is the standard Laplace operator in $\mathbb{R}^n$. Then, each v_h solves the Yamabe equation

$$\Delta v_h = -\frac{1}{(\alpha+1)^2} v_h^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{C}^* \times \mathbb{R}^{n-2}. \tag{3.5}$$

We let $\varphi^h = v_h^{2/(2-n)}$. In the following, subscript and superscript h always refers to charts. We drop the superscript h for a while and write $\varphi = \varphi^h$. From (3.5), we get

$$2\varphi\Delta\varphi - n|\nabla\varphi|^2 = \frac{4}{(n-2)(\alpha+1)^2} \quad \text{in } \mathbb{C}^* \times \mathbb{R}^{n-2}. \tag{3.6}$$

We denote by $\varphi_j = \partial_j\varphi$ and $\varphi_{jk} = \partial_j\partial_k\varphi$ the first and second order partial derivatives of φ with respect to ζ_j, ζ_k. Let us introduce

$$p_{jk} = \varphi_{jk} - \frac{\Delta\varphi}{n}\delta_{jk}, \tag{3.7}$$

the traceless Euclidean Hessian of φ. Here, δ_{jk} is the Kronecker symbol. Differentiating identity (3.6), we get, for $j = 1, ..., n$,

$$n\varphi_k p_{jk} = \varphi\Delta\varphi_j, \tag{3.8}$$

where, here and in the following, we adopt the convention on summation of repeated indices. Using $\partial_k p_{jk} = \frac{n-1}{n}\Delta\varphi_j$, identity (3.8) becomes $(n-1)\varphi_k p_{jk} = \varphi\partial_k p_{jk}$.

Therefore we get $\partial_k\big(\varphi^{1-n}p_{jk}\big) = (1-n)\varphi^{-n}\varphi_k p_{jk} + \varphi^{1-n}\partial_k p_{jk} = 0$, and consequently

$$\partial_k\big(\varphi^{1-n}p_{jk}\varphi_j\big) = \partial_k\big(\varphi^{1-n}p_{jk}\big)\varphi_j + \varphi^{1-n}p_{jk}\varphi_{jk} = \varphi^{1-n}p_{jk}p_{jk}, \tag{3.9}$$

because $p_{jk}\delta_{jk} = 0$. Note that (3.9) gives a scalar invariant equation on the Riemannian manifold (M, g).

Step 2. Fix positive numbers ε, r, with $0 < \varepsilon < r$, and let

$$A_{\varepsilon r} = \big\{\zeta = (\xi, y) \in \mathbb{C}^* \times \mathbb{R}^{n-2} : |\arg\xi| < (\alpha+1)\pi/m,\ |\xi| > \varepsilon,\ |\zeta| < r\big\}.$$

By the divergence theorem, we get from (3.9),

$$\int_{A_{\varepsilon r}} \varphi^{1-n}p_{jk}p_{jk}\,d\zeta = \int_{\partial A_{\varepsilon r}} \varphi^{1-n}p_{jk}\varphi_j\nu_k\,d\mathcal{H}^{n-1}, \tag{3.10}$$

where $\nu = (\nu_1, ..., \nu_n)$ is the exterior unit normal to $\partial A_{\varepsilon r}$ and $\mathcal{H}^{n-1}$ is the standard hypersurface measure in $\mathbb{R}^n$. The boundary of $A_{\varepsilon r}$ can be split into four parts, $\partial A_{\varepsilon r} = R_{\varepsilon r} \cup S_{\varepsilon r} \cup T^-_{\varepsilon r} \cup T^+_{\varepsilon r}$, where

$$\begin{aligned} R_{\varepsilon r} &= \partial A_{\varepsilon r} \cap \{|\zeta| = r\},\\ S_{\varepsilon r} &= \partial A_{\varepsilon r} \cap \{|\xi| = \varepsilon\},\\ T^{\pm}_{\varepsilon r} &= \partial A_{\varepsilon r} \cap \{\arg\xi = \pm(\alpha+1)\pi/m\}. \end{aligned}$$

Step 3. We claim that, for any $r > 0$, we have

$$\lim_{\varepsilon\to 0^+} \int_{S_{\varepsilon r}} \varphi^{1-n}p_{jk}\varphi_j\nu_k\,d\mathcal{H}^{n-1} = 0. \tag{3.11}$$

Indeed, there exists a constant $C > 0$, depending on r, such that

$$\frac{1}{C} \le \varphi(\zeta) \le C, \quad |\nabla\varphi(\zeta)| \le C, \quad |\nabla^2_{\xi y}\varphi(\zeta)| + |\nabla^2_{\xi\xi}\varphi(\zeta)| \le C|\xi|^{-\gamma}$$

for $|\zeta| \le r$ and $|\xi| \le 1$. The estimate for second order derivatives is a consequence of (1.5) and (2.9), with $\gamma = \max\{\beta, \alpha/(\alpha+1)\}$. It follows that

$$|\varphi^{1-n}p_{jk}\varphi_j\nu_k| = \varphi^{1-n}\left|\left(\varphi_{jk}\varphi_j - \frac{1}{n}\Delta\varphi\varphi_k\right)\nu_k\right| \le C|\xi|^{-\gamma},$$

for all $\zeta \in S_{\varepsilon r}$, where the sum in k ranges over $k = 1, 2$. Thus, we have

$$\int_{S_{\varepsilon r}} \big|\varphi^{1-n}p_{jk}\varphi_j\nu_k\big|\,d\mathcal{H}^{n-1} \le C\varepsilon^{1-\gamma}, \tag{3.12}$$

with $\gamma < 1$, and (3.11) follows.

Step 4. Now we claim that

$$\lim_{r\to+\infty} \int_{|\zeta|=r} \big|\varphi^{1-n}p_{jk}\varphi_j\nu_k\big|\,d\mathcal{H}^{n-1} = 0. \tag{3.13}$$

By Theorem 2.3, the function v_h satisfies estimates (2.5). Then, for some constant $C > 0$ we have, for $|\zeta| \ge 1$,

$$\frac{1}{C}|\zeta|^2 \le \varphi(\zeta) \le C|\zeta|^2, \quad |\nabla\varphi(\zeta)| \le C|\zeta|, \quad |\nabla^2\varphi(\zeta)| \le C\Big(\frac{|\zeta|}{|\xi|}\Big)^{\gamma}, \tag{3.14}$$

where $\gamma < 1$ is the constant given by Theorem 2.3. It follows that, for $r \ge 1$,

$$\int_{|\zeta|=r} \big|\varphi^{1-n}p_{jk}\varphi_j\nu_k\big|\,d\mathcal{H}^{n-1} \le Cr^{2(1-n)+1+\gamma}\int_{|\zeta|=r} |\xi|^{-\gamma}\,d\mathcal{H}^{n-1},$$

where

$$\int_{|\zeta|=r} |\xi|^{-\gamma}\, d\mathcal{H}^{n-1} = \frac{d}{dr}\int_{|\zeta|<r} |\xi|^{-\gamma} d\zeta = (n-\gamma) r^{n-\gamma-1} \int_{|\zeta|<1} |\xi|^{-\gamma} d\zeta.$$

Observe that $|\xi|^{-\gamma}$ is locally integrable in $\mathbb{R}^n$. Now (3.13) follows from

$$\int_{|\zeta|=r} \left|\varphi^{1-n} p_{jk}\varphi_j \nu_k\right| \, d\mathcal{H}^{n-1} \le C r^{2-n},$$

with $n \ge 3$, and $C > 0$ a constant which does not depend on r.

Step 5. Let us recall that p^h_{jk} is the traceless Hessian of $\varphi^h = v_h^{2/(2-n)}$, as in (3.7). Formula (3.10) holds for each φ^h, $h = 0, 1, ..., m-1$. Summing up all these identities, we obtain

$$\sum_{h=0}^{m-1} \int_{A_{\varepsilon r}} (\varphi^h)^{1-n} p^h_{jk} p^h_{jk}\, d\zeta = \sum_{h=0}^{m-1} \int_{S_{\varepsilon r}\cup R_{\varepsilon r}} (\varphi^h)^{1-n} p^h_{jk}\varphi^h_j \nu_k \, d\mathcal{H}^{n-1}. \tag{3.15}$$

The contribution of the piece of boundary $T^{\pm}_{\varepsilon r}$ cancels because

$$\int_{T^+_{\varepsilon r}} (\varphi^h)^{1-n} p^h_{jk}\varphi^h_j \nu_k \, d\mathcal{H}^{n-1} = -\int_{T^-_{\varepsilon r}} (\varphi^{h+1})^{1-n} p^{h+1}_{jk}\varphi^{h+1}_j \nu_k \, d\mathcal{H}^{n-1}, \tag{3.16}$$

for all $h = 0, 1, ..., m-1$. For $h = m-1$, the right-hand side should be read with 0 instead of $h+1$. Indeed, the rotation $\varrho : \mathbb{C}\times\mathbb{R}^{n-2} \to \mathbb{C}\times\mathbb{R}^{n-2}$, $\varrho(\xi, y) = (e^{-2(\alpha+1)\pi i/m}\xi, y)$ transforms $T^+_{\varepsilon r}$ into $T^-_{\varepsilon r}$ and maps the outward unit normal to $T^+_{\varepsilon r}$, the vector ν, to the inward unit normal to $T^-_{\varepsilon r}$, which is $-\nu$. Now (3.16) follows from the relation $p^h_{jk}\varphi^h_j = \varrho_{ik}(p^{h+1}_{\ell i}\varphi^{h+1}_\ell)\circ\varrho$, which can be obtained from (3.3).

This cancellation is nothing else but the fact that we are integrating the scalar invariant equation (3.9) on the set $\bigcup_{h=0}^{m-1} f_h^{-1}(\bar{A}_{\varepsilon r}) \subset M$ with the Riemannian divergence theorem.

Step 6. By (3.11) and (3.13), the right-hand side of (3.15) converges to 0 when we let, first, $\varepsilon \to 0^+$ and afterwards $r \to +\infty$. The integrand in the left-hand side of (3.15) is nonnegative, and we deduce that for all $h = 0, 1, ..., m-1$ and $j, k = 1, ..., n$, it is $p^h_{jk} = 0$, that is,

$$\varphi^h_{jk} - \frac{1}{n}\Delta\varphi^h\, \delta_{jk} = 0. \tag{3.17}$$

This system of equations can be integrated: there exist $a_h, c_h \in \mathbb{R}$ and $b_h = (b'_h, b''_h) \in \mathbb{R}^2\times\mathbb{R}^{n-2} = \mathbb{R}^n$ such that

$$\varphi^h(\zeta) = a_h|\zeta|^2 + b_h\cdot\zeta + c_h = a_h\left\{|\xi-\xi_{0,h}|^2 + |y-y_{0,h}|^2\right\} + c_h - \frac{|b_h|^2}{4a_h},$$

where $\xi_{0,h} = -\frac{b'_h}{2a_h}$ and $y_{0,h} = -\frac{b''_h}{2a_h}$.

Now let $z = (x, y) \in U_h$ with $x = re^{i\vartheta}$, $|\vartheta - \vartheta_h| < \pi/(\alpha+1)$. Relation (3.2) gives:

$$\begin{aligned} u(z) = v_h(f_h(z)) &= \Big(a_h\Big\{\left|(e^{-i\vartheta_h}x)^{\alpha+1} - \xi_{0,h}\right|^2 + |y-y_{0,h}|^2\Big\} + c_h - \frac{|b_h|^2}{4a_h}\Big)^{(2-n)/2} \\ &= \Big(a_h\Big\{\left|r^{\alpha+1}e^{i(\alpha+1)\vartheta} - e^{i(\alpha+1)\vartheta_h}\xi_{0,h}\right|^2 + |y-y_{0,h}|^2\Big\} + c_h - \frac{|b_h|^2}{4a_h}\Big)^{(2-n)/2}. \end{aligned}$$

If $z \in U_h \cap U_{h+1}$, the last equation holds both for h and $h+1$. Therefore we conclude that $a_h, y_{0,h}$, $c_h - \frac{|b_h|^2}{4a_h}$ and $e^{i(\alpha+1)\vartheta_h}\xi_{0,h}$ must be independent from h. In other words, after some relabeling, we have

$$u(re^{i\vartheta}, y) = \Big(a\Big\{\Big|r^{\alpha+1}e^{i(\alpha+1)\vartheta} - \xi_0\Big|^2 + |y-y_0|^2\Big\} + b\Big)^{(2-n)/2}$$

for all $\vartheta \in \big(-\pi/(\alpha+1), \vartheta_{m-1} + \pi/(\alpha+1)\big)$. In order to have a well-defined (one-valued) function on $\mathbb{R}^2 \times \mathbb{R}^{n-2}$, we must have either $\xi_0 = 0$ or, in the case $\xi_0 \neq 0$, $\alpha \in \mathbb{N}$.

Ultimately, u must have the form (1.7) if $\alpha \in \mathbb{N}$ and (1.8) if $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$.

Step 7. In order to determine the parameters $a > 0$ and $b > 0$, we briefly show that the function u in (1.7) satisfies

$$\Delta_x u + (\alpha+1)^2|x|^{2\alpha}\Delta_y u = -n(n-2)(\alpha+1)^2 ab|x|^{2\alpha}u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n. \tag{3.18}$$

In order to check (3.18), let $\varphi = u^{1/\mu}$ with $\mu = (2-n)/2$. Denoting by $\partial_x, \partial_{\bar{x}}$ complex derivatives, we have

$$\partial_x u = \mu(\alpha+1)ax^\alpha(\bar{x}^{\alpha+1} - \bar{x}_0^{\alpha+1})\varphi^{\mu-1},$$

and then a short computation gives

$$\Delta_x u = 4\partial_{\bar{x}}\partial_x u = 4\mu(\alpha+1)^2 a|x|^{2\alpha}\Big\{\mu a|x^{\alpha+1} - x_0^{\alpha+1}|^2 + a|y-y_0|^2 + b\Big\}\varphi^{\mu-2}.$$

Analogously, we have

$$\Delta_y u = 2\mu a\Big\{-2a|y-y_0|^2 + (n-2)a|x^{\alpha+1} - x_0^{\alpha+1}|^2 + (n-2)b\Big)\Big\}\varphi^{\mu-2}.$$

Summing up we get (3.18).

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Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Trieste, 63, 35121 Padova, Italy

E-mail address: monti@math.unipd.it

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato, 5, 40127 Bologna, Italy

E-mail address: morbidel@dm.unibo.it