CESÀRO FUNCTION SPACES
FAIL THE FIXED POINT PROPERTY

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Abstract. The Cesàro sequence spaces \( ces_p \), \( 1 < p < \infty \), are reflexive but they have the fixed point property. In this paper we prove that in contrast to these sequence spaces the corresponding Cesàro function spaces \( Ces_p \) on both \([0, 1]\) and \([0, \infty)\) for \( 1 < p < \infty \) are not reflexive and they fail to have the fixed point property.

1. Introduction
Let \( 1 \leq p \leq \infty \). The Cesàro sequence space \( ces_p \) is defined as the set of all real sequences \( x = \{x_k\} \) such that
\[
\|x\|_{c(p)} = \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right) \right)^{1/p} < \infty \text{ when } 1 \leq p < \infty,
\]
and
\[
\|x\|_{c(\infty)} = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \text{ when } p = \infty.
\]
The Cesàro function spaces \( Ces_p = Ces_p(I) \) are the classes of Lebesgue measurable real functions \( f \) on \( I = [0, 1] \) or \( I = [0, \infty) \) such that the corresponding norms are finite, where
\[
\|f\|_{C(p)} = \left( \int_I \left( \frac{1}{x} \int_0^x |f(t)| \, dt \right)^p \, dx \right)^{1/p} \text{ for } 1 \leq p < \infty,
\]
and
\[
\|f\|_{C(\infty)} = \sup_{x \in I, \, x > 0} \frac{1}{x} \int_0^x |f(t)| \, dt < \infty \text{ for } p = \infty.
\]
The Cesàro sequence spaces \( ces_p \) were investigated in the seventies by Shiue, Leibowitz and Jagers. In particular, they proved that \( ces_1 = \{0\} \), \( ces_p \) are separable reflexive Banach spaces for \( 1 < p < \infty \) and the \( l^p \) spaces are continuously embedded.
into $\text{ces}_p$ for $1 < p \leq \infty$ with strict embeddings. Also if $1 < p < q \leq \infty$, then $\text{ces}_p \subset \text{ces}_q$ with continuous strict embeddings. Bennett \cite{3} proved that $\text{ces}_p$ for $1 < p < \infty$ are not isomorphic to any $l^q$ space with $1 \leq q \leq \infty$ (see also \cite{15} for another proof). Moreover, Maligranda-Petrot-Suantai \cite{15} proved recently that Cesàro sequence spaces $\text{ces}_p$ for $1 < p < \infty$ are not uniformly nonsquare; that is, there are sequences $\{x_n\}$ and $\{y_n\}$ on the unit sphere such that $\lim_{n \to \infty} \min \{\|x_n + y_n\|_{\text{ces}(p)}, \|x_n - y_n\|_{\text{ces}(p)}\} = 2$. They even proved that these spaces are not $B$-convex. We refer here to \cite{3}, \cite{15} and the references given there.

Several geometric properties of the Cesàro sequence spaces $\text{ces}_p$ were studied in recent years by many mathematicians, and in 1999-2000 it was also proved by Cui-Hudzik \cite{5}, Cui-Hudzik-Li \cite{6} and Cui-Meng-Phuciennik \cite{7} that Cesàro sequence spaces $\text{ces}_p$ for $1 < p < \infty$ have the fixed point property (cf. also \cite{4} Part 9).

Cesàro function spaces $\text{Ces}_p[0,\infty)$ for $1 \leq p \leq \infty$ were considered by Shiue \cite{10}, Hassard-Hussein \cite{12} and Sy-Zhang-Lee \cite{17}. They proved that $\text{Ces}_1[0,\infty) = \{0\}$ and $\text{Ces}_p[0,\infty)$ for $1 < p < \infty$ are separable Banach spaces and that $\text{Ces}_\infty[0,\infty)$ is a nonseparable Banach space. The space $\text{Ces}_\infty[0,1]$ is known as the Korenbljum-Krein-Levin space already introduced in 1948.

By the Hardy inequality the $L^p(I)$ spaces are continuously embedded into $\text{Ces}_p(I)$ for $1 < p \leq \infty$ with strict embedding, where $I = [0,1]$ or $I = [0,\infty)$ (cf. \cite{11} Theorem 327 and \cite{13} Theorem 2). Also if $1 < p < q \leq \infty$, then $\text{ces}_q[0,1] \subset \text{ces}_p[0,1]$ with continuous strict embedding. Moreover, $\text{Ces}_1[0,1]$ is a weighted $L^1_w[0,1]$ space with the weight $w(t) = \ln(1/t)$ for $0 < t \leq 1$. In fact,

\begin{equation}
\int_0^1 \left( \frac{1}{x} \int_0^x |f(t)| \, dt \right) \, dx = \int_0^1 \left( \int_t^1 \frac{1}{x} \, dx \right) |f(t)| \, dt = \int_0^1 |f(t)| \ln \frac{1}{t} \, dt.
\end{equation}

We will show that, in contrast to Cesàro sequence spaces, the Cesàro function spaces $\text{Ces}_p(I)$ on both $I = [0,1]$ and $I = [0,\infty)$ for $1 < p < \infty$ are not reflexive and that they do not have the fixed point property.

A Banach space $X$ has the fixed point property (FPP) \textit{[resp. weak fixed point property (WFPP)]} if every nonexpansive mapping of every closed bounded convex \textit{[resp. nonempty weakly compact convex]} subset $K$ of $X$ into itself has a fixed point. Recall that $T$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

The spaces $c_0$ and $l^1$ both fail to have the FPP with their classical norms, but they have the WFPP. The space $L^1[0,1]$ does not have the WFPP, as was proved by Alspach \cite{1}.

Our proof that the Cesàro function spaces $\text{Ces}_p(I)$ on $I = [0,1]$ with $1 \leq p \leq \infty$ and on $I = [0,\infty)$ with $1 < p \leq \infty$ fail to have the fixed point property will be carried out by showing that these spaces contain an asymptotically isometric copy of $l^1$.

A Banach space $X$ contains an asymptotically isometric copy of $l^1$ if there exists a null sequence $(\varepsilon_n)$ in $(0,1)$ and a sequence $(x_n)$ in $X$ such that

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n) |\alpha_n| \leq \| \sum_{n=1}^{\infty} \alpha_n x_n \|_X \leq \sum_{n=1}^{\infty} |\alpha_n|$$

for all $(\alpha_n) \in l^1$ of scalars. This notion was introduced by Dowling and Lennard in \cite{9}, where they proved that such spaces fail to have the FPP.
2. Main results

Cesàro sequence spaces $ces_p, 1 < p < \infty$, are reflexive but not $B$-convex and they have the fixed point property. In contrast to these sequence spaces the corresponding Cesàro function spaces $Ces_p(I)$ on both $I = [0, 1]$ and $I = [0, \infty)$ for $1 < p < \infty$ are not reflexive and they do not have the fixed point property. Our main result reads:

**Theorem 1.** Let $1 \leq p \leq \infty$. The Cesàro function space $Ces_p[0,1]$ contains an asymptotically isometric copy of $l^1$; that is, there exist a sequence $\{\varepsilon_n\} \subset (0,1), \varepsilon_n \to 0$ as $n \to \infty$ and a sequence of functions $\{f_n\} \subset Ces_p[0,1]$ such that for arbitrary $\{\alpha_n\} \in l^1$ we have

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n)|\alpha_n| \leq \left| \sum_{n=1}^{\infty} \alpha_n f_n \right|_{C(p)} \leq \sum_{n=1}^{\infty} |\alpha_n|.$$  

Before the proof of this theorem we prove the following auxiliary result.

**Lemma 1.** Let $0 < a < b < 1, f \in Ces_p[0,1]$ and $supp := \{t \in [0,1] : f(t) \neq 0\} \subset [a,b]$. Then

$$\left( b^{1-p} - 1 \right)^{1/p} \|f\|_1 \leq (p-1)^{1/p} ||f||_{C(p)} \leq (a^{1-p} - 1)^{1/p} \|f\|_1,$$

for $1 < p < \infty$ and

$$\ln \frac{1}{b} \|f\|_1 \leq \|f\|_{C(1)} \leq \ln \frac{1}{a} \|f\|_1, \quad \frac{1}{b} \|f\|_1 \leq \|f\|_{C(\infty)} \leq \frac{1}{a} \|f\|_1,$$

where $\|f\|_1 = \int_0^1 |f(t)| \ dt$.

**Proof.** It is obvious that for any $0 < x \leq 1$ we have

$$\frac{1}{x} \|f\|_1 \chi_{[0,1]}(x) \leq F_f(x) := \frac{1}{x} \int_0^x |f(t)| \ dt \leq \frac{1}{x} \|f\|_1 \chi_{[a,1]}(x).$$

Since, for every $c \in (0,1), \int_c^1 t^{-p} \ dt = \frac{c^{1-p} - 1}{p-1}$ and $\int_c^1 t^{-1} \ dt = \ln \frac{1}{c}$ we obtain (3) and (4) for $p = 1$. In the case of $p = \infty$ we see that

$$\frac{1}{b} \|f\|_1 \leq \|F_f\|_{L^\infty[0,1]} \leq \frac{1}{a} \|f\|_1,$$

and the lemma is proved.

**Proof of Theorem 1.** For $1 < p < \infty$ we set

$$g_n = \chi_{[a_n, a_{n+1})}, \quad a_n = 2^{1/(1-p)} \left( 1 - \frac{1}{2^n} \right), \quad n = 1,2, \ldots.$$ 

Since $\|g_n\|_1 = a_{n+1} - a_n = 2^{1/(1-p)} \cdot 2^{-n-1}$ and $a_n^{1-p} - 1 = \frac{2}{(1 - 2^{-n})^{1-p} - 1}$, then, by Lemma 1 (see the second estimate in (3)), this yields that

$$2^{-1/(1-p)} 2^{n+1} (p-1)^{1/p} \|g_n\|_{C(p)} \leq \left( \frac{2}{(1 - 2^{-n})^{1-p} - 1} \right)^{1/p}.$$ 

Let $f_n = g_n/\|g_n\|_{C(p)}$ and $\alpha_n \in \mathbb{R}$ for $n = 1,2, \ldots$. Since $supp g_n \subset [2^{1/(1-p)} - 1, 2^{1/(1-p)}]$ for every $n \in \mathbb{N}$ it follows from Lemma 1 (see the first estimate in (3))
that
\[ \| \sum_{n=1}^{\infty} \alpha_n f_n \|_{C(p)} \geq \frac{\| \sum_{n=1}^{\infty} \alpha_n f_n \|_1}{(p-1)^{1/p}} = \sum_{n=1}^{\infty} \frac{|\alpha_n| 2^{1/(1-p)}}{2^{n+1}(p-1)^{1/p} \| g_n \|_{C(p)}} \geq \sum_{n=1}^{\infty} \left( \frac{2}{(1 - 2^{-n})^{p-1}} - 1 \right)^{-1/p} |\alpha_n|. \]

Denote
\[ \varepsilon_n = 1 - \left( \frac{2}{(1 - 2^{-n})^{p-1}} - 1 \right)^{-1/p}. \]

Then \( \{ \varepsilon_n \} \subset (0, 1) \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \). This means that the left-hand side of (2) is proved. The right-hand side of (2) is obvious since \( \| f_n \|_{C(p)} = 1 \).

In the case \( p = \infty \) we take \( f_n = g_n / \| g_n \|_{C(\infty)} \), where \( g_n = \chi_{[a_n, a_{n+1}]} \) with \( a_n = 1 - 2^{-n}, n = 1, 2, \ldots \). Then \( \| g_n \|_1 = 2^{-n-1} \) and, by Lemma 1 (see the second estimate in (3)) \( \| g_n \|_{C(\infty)} \leq \frac{1}{1 - 2^{-n-1}} \) or \( 2^{n+1} \| g_n \|_{C(\infty)} \leq \frac{1}{1 - 2^{-n-1}} \). Since \( \text{supp} \ g_n \subset [1/2, 1) \), for every \( n \in \mathbb{N} \), it follows from Lemma 1 (see the first estimate in (3)) that:
\[ \| \sum_{n=1}^{\infty} \alpha_n f_n \|_{C(\infty)} \geq \sum_{n=1}^{\infty} \left( \frac{2}{(1 - 2^{-n})^{p-1}} - 1 \right) |\alpha_n|, \]

and \( \varepsilon_n = 2^{-n} \) is a required sequence.

In the case \( p = 1 \) we take \( g_n = \chi_{[a_n, a_{n+1}]} \), where \( a_n = \frac{1}{2}(1 - 2^{-n}), n = 1, 2, \ldots \) and argue in a similar way. The proof is complete. \( \square \)

The analogous result holds for Cesàro function spaces on \([0, \infty)\).

**Theorem 2.** Let \( 1 < p \leq \infty \). The Cesàro function space \( \text{Ces}_p[0, \infty) \) contains an asymptotically isometric copy of \( l^1 \).

**Proof.** We consider only the case \( 1 < p < \infty \) (the case \( p = \infty \) can be proved similarly as in Theorem 1). We take \( g_n = \chi_{[a_n, a_{n+1}]} \) with \( a_n = 1 - 2^{-n}, n = 1, 2, \ldots \) and continue the proof as in Theorem 1, observing that the estimate corresponding to (3) for \( \text{Ces}_p[0, \infty) \) will be (for \( 0 < a < b < \infty \))
\[ b^{1/p-1} \| f \|_1 \leq (p-1)^{1/p} \| f \|_{C(p)} \leq a^{1/p-1} \| f \|_1, \text{ with } \| f \|_1 = \int_0^\infty |f(t)| \ dt, \]
and then for $f_n = g_n / \|g_n\|_{C(p)}$ we have that

$$
\| \sum_{n=1}^{\infty} \alpha_n f_n \|_{C(p)} \geq \frac{\| \sum_{n=1}^{\infty} \alpha_n f_n \|_1}{(p-1)^{1/p}} = \sum_{n=1}^{\infty} \frac{|\alpha_n|}{2^{n+1}(p-1)^{1/p}\|g_n\|_{C(p)}} \geq \sum_{n=1}^{\infty} a_n^{1-1/p} |\alpha_n| \geq \sum_{n=1}^{\infty} (1 - 2^{-n})^{1-1/p} |\alpha_n|,
$$

and $\varepsilon_n = 1 - (1 - 2^{-n})^{1-1/p}$ is a required sequence. The proof is complete. □

Remark 1. It is obvious from Theorem 1 and Theorem 2 that the Cesàro function spaces $\operatorname{Ces}_p(I)$ for $1 < p \leq \infty$ are not reflexive.

Dowling-Lennard [9] proved that if a Banach space $X$ contains an asymptotically isometric copy of $l^1$, then there exists a nonexpansive mapping defined on a closed bounded convex subset of $X$ without a fixed point, i.e., that $X$ fails to have the fixed point property (see also [10, Theorem 2.3 and Corollary 2.11]). By the Dilworth-Girardi-Hagler result [8, Theorem 2] the dual space $X^*$ does not have the fixed point property since they proved there that $X$ contains an asymptotically isometric copy of $l^1$ if and only if the dual space $X^*$ contains an isometric copy of $L^1[0,1]$. Combining these results with our Theorem 1 and Theorem 2 we obtain immediately our main result on the fixed point property of Cesàro function spaces and their dual spaces.

**Theorem 3.** If either $1 \leq p \leq \infty$ and $I = [0,1]$ or $1 < p \leq \infty$ and $I = [0,\infty)$, then the Cesàro function spaces $\operatorname{Ces}_p(I)$ and their dual spaces $\operatorname{Ces}_p(I)^*$ fail to have the fixed point property.

Theorem 3 gives information about the fixed point property, and therefore it is natural to ask what one can say about the weak fixed point property.

Note that the space $\operatorname{Ces}_1[0,1]$ is isometric to $L^1[0,1]$ by the equality [1], and by the Alsparch result [1] $L^1[0,1]$ fails to have WFPP; therefore $\operatorname{Ces}_1[0,1]$ also fails to have WFPP.

By combining Theorem 1, Theorem 2 and the Dilworth-Girardi-Hagler result [8, Corollary 13] we have the following result:

**Proposition 1.** The dual spaces to the Cesàro function spaces $\operatorname{Ces}_p(I)^*$ do not have the weak fixed point property.

**Proposition 2.** The Cesàro function spaces $\operatorname{Ces}_p(I)$ for $1 < p < \infty$ are not isomorphic to any $L^q(I)$ space for $1 \leq q \leq \infty$. In particular, they are not isomorphic to $L^1(I)$.

Of course, $\operatorname{Ces}_p(I)$ for $1 < p < \infty$, as nonreflexive and separable spaces, cannot be isomorphic to any $L^q(I)$ with $1 < q < \infty$ or $q = \infty$. The statement for $q = 1$ will be proved in the forthcoming paper [2].
References


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