CESÀRO FUNCTION SPACES
FAIL THE FIXED POINT PROPERTY

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Abstract. The Cesàro sequence spaces $ces_p, 1 < p < \infty$, are reflexive but they have the fixed point property. In this paper we prove that in contrast to these sequence spaces the corresponding Cesàro function spaces $Ces_p$ on both $[0, 1]$ and $[0, \infty)$ for $1 < p < \infty$ are not reflexive and they fail to have the fixed point property.

1. Introduction

Let $1 \leq p \leq \infty$. The Cesàro sequence space $ces_p$ is defined as the set of all real sequences $x = \{x_k\}$ such that

$$\|x\|_{c(p)} = \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right) \right)^{1/p} < \infty$$

when $1 \leq p < \infty$,

and

$$\|x\|_{c(\infty)} = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty$$

when $p = \infty$.

The Cesàro function spaces $Ces_p = Ces_p(I)$ are the classes of Lebesgue measurable real functions $f$ on $I = [0, 1]$ or $I = [0, \infty)$ such that the corresponding norms are finite, where

$$\|f\|_{C(p)} = \left( \int_{I} \left( \frac{1}{x} \int_{0}^{x} |f(t)| \, dt \right)^p \, dx \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty,$$

and

$$\|f\|_{C(\infty)} = \sup_{x \in I, \, x > 0} \frac{1}{x} \int_{0}^{x} |f(t)| \, dt < \infty \quad \text{for} \quad p = \infty.$$

The Cesàro sequence spaces $ces_p$ were investigated in the seventies by Shiue, Leibowitz and Jagers. In particular, they proved that $ces_1 = \{0\}, ces_p$ are separable reflexive Banach spaces for $1 < p < \infty$ and the $l^p$ spaces are continuously embedded.

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into $ces_p$ for $1 < p \leq \infty$ with strict embeddings. Also if $1 < p < q \leq \infty$, then $ces_p \subset ces_q$ with continuous strict embeddings. Bennett [3] proved that $ces_p$ for $1 < p < \infty$ are not isomorphic to any $l^q$ space with $1 \leq q \leq \infty$ (see also [15] for another proof). Moreover, Maligranda-Petrot-Suantai [15] proved recently that Cesàro sequence spaces $ces_p$ for $1 < p < \infty$ are not uniformly nonsquare; that is, there are sequences $\{x_n\}$ and $\{y_n\}$ on the unit sphere such that $\lim_{n \to \infty} \min(\|x_n + y_n\|_{c(p)}, \|x_n - y_n\|_{c(p)}) = 2$. They even proved that these spaces are not $B$-convex.

We refer here to [3], [15] and the references given there.

Several geometric properties of the Cesàro sequence spaces $ces_p$ were studied in recent years by many mathematicians, and in 1999-2000 it was also proved by Cui-Hudzik [5], Cui-Hudzik-Li [6] and Cui-Meng-Phuciennik [7] that Cesàro sequence spaces $ces_p$ for $1 < p < \infty$ have the fixed point property (cf. also [4, Part 9]).

Cesàro function spaces $Ces_p[0, \infty)$ for $1 \leq p \leq \infty$ were considered by Shiue [16], Hassard-Hussein [12] and Sy-Zhang-Lee [17]. They proved that $Ces_1[0, \infty) = \{0\}$ and $Ces_p[0, \infty)$ for $1 < p < \infty$ are separable Banach spaces and that $Ces_\infty[0, \infty)$ is a nonseparable Banach space. The space $Ces_\infty[0, 1]$ is known as the Korenblyum-Krein-Levin space already introduced in 1948.

By the Hardy inequality the $L^p(I)$ spaces are continuously embedded into $Ces_p(I)$ for $1 < p \leq \infty$ with strict embedding, where $I = [0, 1]$ or $I = [0, \infty)$ (cf. [11] Theorem 327 and [13] Theorem 2). Also if $1 < p < q \leq \infty$, then $Ces_q[0, 1] \subset Ces_p[0, 1]$ with continuous strict embedding. Moreover, $Ces_1[0, 1]$ is a weighted $L^1_w[0, 1]$ space with the weight $w(t) = \ln \frac{1}{t}$ for $0 < t \leq 1$. In fact,

$$\int_0^1 \left( \frac{1}{x} \int_0^x |f(t)| \, dt \right) \, dx = \int_0^1 \left( \int_t^1 \frac{1}{x} \, dx \right) |f(t)| \, dt = \int_0^1 |f(t)| \ln \frac{1}{t} \, dt.$$ 

We will show that, in contrast to Cesàro sequence spaces, the Cesàro function spaces $Ces_p(I)$ on both $I = [0, 1]$ and $I = [0, \infty)$ for $1 < p < \infty$ are not reflexive and that they do not have the fixed point property.

A Banach space $X$ has the fixed point property (FPP) [resp. weak fixed point property (WFPP)] if every nonexpansive mapping of every closed bounded convex [resp. nonempty weakly compact convex] subset $K$ of $X$ into itself has a fixed point. Recall that $T$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

The spaces $c_0$ and $l^1$ both fail to have the FPP with their classical norms, but they have the WFPP. The space $L^1_w[0, 1]$ does not have the WFPP, as was proved by Alsbach [1].

Our proof that the Cesàro function spaces $Ces_p(I)$ on $I = [0, 1]$ with $1 \leq p \leq \infty$ and on $I = [0, \infty)$ with $1 < p \leq \infty$ fail to have the fixed point property will be carried out by showing that these spaces contain an asymptotically isometric copy of $l^1$.

A Banach space $X$ contains an asymptotically isometric copy of $l^1$ if there exists a null sequence $(\varepsilon_n)$ in $(0, 1)$ and a sequence $(x_n)$ in $X$ such that

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n)|\alpha_n| \leq \|\sum_{n=1}^{\infty} \alpha_n x_n\|_X \leq \sum_{n=1}^{\infty} |\alpha_n|$$

for all $(\alpha_n) \in l^1$ of scalars. This notion was introduced by Dowling and Lennard in [9], where they proved that such spaces fail to have the FPP.
2. Main results

Cesàro sequence spaces $ces_p, 1 < p < \infty$, are reflexive but not $B$-convex and they have the fixed point property. In contrast to these sequence spaces the corresponding Cesàro function spaces $Ces_p(I)$ on both $I = [0, 1]$ and $I = [0, \infty)$ for $1 < p < \infty$ are not reflexive and they do not have the fixed point property. Our main result reads:

**Theorem 1.** Let $1 \leq p \leq \infty$. The Cesàro function space $Ces_p[0, 1]$ contains an asymptotically isometric copy of $l^1$; that is, there exist a sequence $\{\varepsilon_n\} \subset (0, 1), \varepsilon_n \to 0$ as $n \to \infty$ and a sequence of functions $\{f_n\} \subset Ces_p[0, 1]$ such that for arbitrary $\{\alpha_n\} \subset l^1$ we have

$$(2) \quad \sum_{n=1}^{\infty} (1 - \varepsilon_n)|\alpha_n| \leq \| \sum_{n=1}^{\infty} \alpha_n f_n \|_{C(p)} \leq \sum_{n=1}^{\infty} |\alpha_n|.$$ 

Before the proof of this theorem we prove the following auxiliary result.

**Lemma 1.** Let $0 < a < b < 1, f \in Ces_p[0, 1]$ and $supp f := \{t \in [0, 1] : f(t) \neq 0\} \subset [a, b]$. Then

$$(3) \quad (b^{1-p} - 1)^{1/p} \|f\|_1 \leq (p - 1)^{1/p} \|f\|_{C(p)} \leq (a^{1-p} - 1)^{1/p} \|f\|_1,$$

for $1 < p < \infty$ and

$$(4) \quad 1 - \frac{1}{b} \|f\|_1 \leq \|f\|_{C(1)} \leq \frac{1}{a} \|f\|_1, \quad \frac{1}{b} \|f\|_1 \leq \|f\|_{C(\infty)} \leq \frac{1}{a} \|f\|_1,$$

where $\|f\|_1 = \int_0^1 |f(t)| \, dt$.

**Proof.** It is obvious that for any $0 < x \leq 1$ we have

$$\frac{1}{x} \|f\|_1 \chi_{[b, 1]}(x) \leq F_f(x) := \frac{1}{x} \int_0^x |f(t)| \, dt \leq \frac{1}{x} \|f\|_1 \chi_{[a, 1]}(x).$$

Since, for every $c \in (0, 1), \int_c^1 t^{-p} \, dt = \frac{1}{1-p} - \frac{1}{p}$ and $\int_c^1 t^{-1} \, dt = \ln \frac{1}{c}$ we obtain (3) and (4) for $p = 1$. In the case of $p = \infty$ we see that

$$\frac{1}{b} \|f\|_1 \leq \|f\|_{L^\infty[0, 1]} \leq \frac{1}{a} \|f\|_1,$$

and the lemma is proved. \hfill \Box

**Proof of Theorem 1.** For $1 < p < \infty$ we set

$$g_n = \chi_{[a_n, a_{n+1})}, \quad a_n = 2^{1/(1-p)} \left(1 - \frac{1}{2^n}\right), \quad n = 1, 2, \ldots.$$ 

Since $\|g_n\|_1 = a_{n+1} - a_n = 2^{1/(1-p)} \cdot 2^{-n-1}$ and $a_n^{1-p} - 1 = \frac{2}{(1-2^{-n})^{p-1}} - 1$, then, by Lemma 1 (see the second estimate in (3)), this yields that

$$2^{-1/(1-p)} 2^{n+1} (p - 1)^{1/p} \|g_n\|_{C(p)} \leq \left(\frac{2}{(1-2^{-n})^{p-1}} - 1\right)^{1/p}.$$ 

Let $f_n = g_n / \|g_n\|_{C(p)}$ and $\alpha_n \in \mathbb{R}$ for $n = 1, 2, \ldots$. Since $\text{supp} g_n \subset [2^{1/(1-p)} - 1, 2^{1/(1-p)}]$ for every $n \in \mathbb{N}$ it follows from Lemma 1 (see the first estimate in (3))
that
\[ \| \sum_{n=1}^{\infty} \alpha_n f_n \|_{C(p)} \geq \| \sum_{n=1}^{\infty} \alpha_n f_n \|_1 \]
\[ = \sum_{n=1}^{\infty} \frac{|\alpha_n| 2^{1/(1-p)}}{2^{n+1}(p-1)^{1/p} \| g_n \|_{C(p)}} \]
\[ \geq \sum_{n=1}^{\infty} \left( \frac{2}{(1-2^{-n})^{p-1} - 1} \right)^{-1/p} |\alpha_n|. \]

Denote
\[ \varepsilon_n = 1 - \left( \frac{2}{(1-2^{-n})^{p-1} - 1} \right)^{-1/p}. \]

Then \( \{ \varepsilon_n \} \subset (0,1) \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \). This means that the left-hand side of (2) is proved. The right-hand side of (2) is obvious since \( \| f_n \|_{C(p)} = 1 \).

In the case \( p = \infty \) we take \( f_n = g_n / \| g_n \|_{C(\infty)} \), where \( g_n = \chi_{[a_n,a_{n+1}]} \) with \( a_n = 1 - 2^{-n} \), \( n = 1, 2, \ldots \). Then \( \| g_n \|_1 \leq 2^{-n-1} \) and, by Lemma 1 (see the second estimate in (4)) \( \| g_n \|_{C(\infty)} \leq 1/2 - 2^{-n-1} \) or \( 2^{n+1} \| g_n \|_{C(\infty)} \leq 1/2 - 2^{-n} \). Since \( \text{supp} \; g_n \subset [1/2, 1) \), for every \( n \in \mathbb{N} \), it follows from Lemma 1 (see the first estimate in (4)) that:

\[ \| \sum_{n=1}^{\infty} \alpha_n f_n \|_{C(\infty)} \geq \| \sum_{n=1}^{\infty} \alpha_n f_n \|_1 \]
\[ = \sum_{n=1}^{\infty} 2^{n+1} |\alpha_n| \| g_n \|_{C(\infty)} \geq \sum_{n=1}^{\infty} (1 - 2^{-n}) |\alpha_n|, \]

and \( \varepsilon_n = 2^{-n} \) is a required sequence.

In the case \( p = 1 \) we take \( g_n = \chi_{[a_n,a_{n+1})} \), where \( a_n = \frac{1}{3} (1 - 2^{-n}) \), \( n = 1, 2, \ldots \) and argue in a similar way. The proof is complete. \( \square \)

The analogous result holds for Cesàro function spaces on \([0, \infty)\).

**Theorem 2.** Let \( 1 < p \leq \infty \). The Cesàro function space \( \text{Ces}_p[0, \infty) \) contains an asymptotically isometric copy of \( l^1 \).

**Proof.** We consider only the case \( 1 < p < \infty \) (the case \( p = \infty \) can be proved similarly as in Theorem 1). We take \( g_n = \chi_{[a_n,a_{n+1})} \) with \( a_n = 1 - 2^{-n} \), \( n = 1, 2, \ldots \) and continue the proof as in Theorem 1, observing that the estimate corresponding to (3) for \( \text{Ces}_p[0, \infty) \) will be (for \( 0 < a < b < \infty \))

\[ b^{1/p-1} \| f \|_1 \leq (p-1)^{1/p} \| f \|_{C(p)} \leq a^{1/p-1} \| f \|_1, \]

with \( \| f \|_1 = \int_0^\infty |f(t)| \, dt \).
and then for $f_n = g_n / \|g_n\|_{C(p)}$ we have that

$$
\left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_{C(p)} \geq \frac{\| \sum_{n=1}^{\infty} \alpha_n f_n \|_1}{(p-1)^{1/p}}
$$

$$
= \sum_{n=1}^{\infty} \frac{|\alpha_n|}{2^{n+1}(p-1)^{1/p}\|g_n\|_{C(p)}} \geq \sum_{n=1}^{\infty} a_n^{1-1/p}|\alpha_n|
$$

$$
= \sum_{n=1}^{\infty} \left(1 - 2^{-n}\right)^{1-1/p}|\alpha_n| = \sum_{n=1}^{\infty} \left(1 - \varepsilon_n\right)|\alpha_n|,
$$

and $\varepsilon_n = 1 - (1 - 2^{-n})^{1-1/p}$ is a required sequence. The proof is complete. $\square$

**Remark 1.** It is obvious from Theorem 1 and Theorem 2 that the Cesàro function spaces $Ces_p(I)$ for $1 < p \leq \infty$ are not reflexive.

Dowling-Lennard [9] proved that if a Banach space $X$ contains an asymptotically isometric copy of $l^1$, then there exists a nonexpansive mapping defined on a closed bounded convex subset of $X$ without a fixed point, i.e., that $X$ fails to have the fixed point property (see also [10, Theorem 2.3 and Corollary 2.11]). By the Dilworth-Girardi-Hagler result [8, Theorem 2] the dual space $X^*$ does not have the fixed point property since they proved there that $X$ contains an asymptotically isometric copy of $l^1$ if and only if the dual space $X^*$ contains an isometric copy of $L^1[0,1]$. Combining these results with our Theorem 1 and Theorem 2 we obtain immediately our main result on the fixed point property of Cesàro function spaces and their dual spaces.

**Theorem 3.** If either $1 \leq p \leq \infty$ and $I = [0,1]$ or $1 < p \leq \infty$ and $I = [0,\infty)$, then the Cesàro function spaces $Ces_p(I)$ and their dual spaces $Ces_p(I)^*$ fail to have the fixed point property.

Theorem 3 gives information about the fixed point property, and therefore it is natural to ask what one can say about the weak fixed point property.

Note that the space $Ces_1[0,1]$ is isometric to $L^1[0,1]$ by the equality [1], and by the Alspach result [1] $L^1[0,1]$ fails to have WFPP; therefore $Ces_1[0,1]$ also fails to have WFPP.

By combining Theorem 1, Theorem 2 and the Dilworth-Girardi-Hagler result [8, Corollary 13] we have the following result:

**Proposition 1.** The dual spaces to the Cesàro function spaces $Ces_p(I)^*$ do not have the weak fixed point property.

**Proposition 2.** The Cesàro function spaces $Ces_p(I)$ for $1 < p < \infty$ are not isomorphic to any $L^q(I)$ space for $1 \leq q \leq \infty$. In particular, they are not isomorphic to $L^1(I)$.

Of course, $Ces_p(I)$ for $1 < p < \infty$, as nonreflexive and separable spaces, cannot be isomorphic to any $L^q(I)$ with $1 < q < \infty$ or $q = \infty$. The statement for $q = 1$ will be proved in the forthcoming paper [2].
References


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