LINEAR MAPS PRESERVING INVARIANTS

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ABSTRACT. Let $G \subset \text{GL}(V)$ be a complex reductive group. Let $G'$ denote 
\[ \{ \varphi \in \text{GL}(V) \mid p \circ \varphi = p \text{ for all } p \in \mathbb{C}[V]^G \} \]. We show that, “in general”, 
$G' = G$. In case $G$ is the adjoint group of a simple Lie algebra $\mathfrak{g}$, we show that 
$G'$ is an order 2 extension of $G$. We also calculate $G'$ for all representations of 
$\text{SL}_2$.

1. Introduction

Our base field is $\mathbb{C}$, the field of complex numbers. Let $G \subset \text{GL}(V)$ be a reductive group. Let $G' = \{ \varphi \in \text{GL}(V) \mid p \circ \varphi = p \text{ for all } p \in \mathbb{C}[V]^G \}$. Several authors have studied the problem of determining $G'$. If $G$ is finite, then one easily sees that 
$G' = G$. Solomon [Sol05, Sol06] has classified many triples consisting of reductive groups $H \subset G$ and a $G$-module $V$ such that $\mathbb{C}(V)^H = \mathbb{C}(V)^G$ (rational invariant functions). If $G$ and $H$ are semisimple, then this is the same thing as finding triples where we have equality of the polynomial invariants: 
$\mathbb{C}[V]^H = \mathbb{C}[V]^G$. We show that for “general” faithful $G$-modules $V$ we have that 
$G = G'$. We also compute $G'$ for all representations of $\text{SL}_2$.

First we study the case that $G$ is the adjoint group of a simple Lie algebra $\mathfrak{g}$. Our interest in this case is due to the paper of Raïs [Rai07] where the question of determining $G'$ is raised. The case that $\mathfrak{g} = \mathfrak{sl}_n$ was also settled by him [Rai72], where it is shown that $G'/G$ is generated by the mapping $\mathfrak{sl}_n \ni X \mapsto X^t$ where $X^t$ denotes the transpose of $X$. In §2 we show that, “in general”, $G'/G$ is generated by the element $-\psi$ where $\psi: \mathfrak{g} \to \mathfrak{g}$ is a certain automorphism of $\mathfrak{g}$ of order 2. In the case of $\mathfrak{sl}_n$, $\psi(X) = -X^t$, so that our result reproduces that of Raïs. The computation of $G'$ for $\mathfrak{g}$ semisimple follows easily from the case that $\mathfrak{g}$ is simple. In §3 we prove our result that $G = G'$ for “general” $G$ and “general” $G$-modules $V$.

In §4 we consider representations of $\text{SL}_2$.

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2. The adjoint case

Proposition 2.1. Let $\mathfrak{g}$ be a semisimple Lie algebra, so we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where the $\mathfrak{g}_i$ are simple ideals. Let $\varphi \in G'$. Then $\varphi(\mathfrak{g}_i) = \mathfrak{g}_i$ for all $i$, and $\varphi|_{\mathfrak{g}_i} = \pm \sigma_i$ where $\sigma_i$ is an automorphism of $\mathfrak{g}_i$.

Proof. By a theorem of Dixmier [Dix79] we know that the Lie algebra of $G'$ is $\text{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. Thus $\varphi$ acts on $\mathfrak{ad}(\mathfrak{g}) \simeq \mathfrak{g}$ via an automorphism $\sigma$ where $\varphi \circ \text{ad} X \circ \varphi^{-1} = \text{ad} \sigma(X)$ for $X \in \mathfrak{g}$. Since $\varphi$ induces the identity on $\mathbb{C}[\mathfrak{g}]$, so does $\sigma$, and it follows that $\sigma = \prod_i \sigma_i$ where $\sigma_i \in \text{Aut}(\mathfrak{g}_i)$, $i = 1, \ldots, r$. By Schur’s lemma, $\varphi \circ \sigma^{-1}$ restricted to $\mathfrak{g}_i$ is multiplication by some scalar $\lambda_i \in \mathbb{C}^*$, $i = 1, \ldots, r$. Since $\text{Aut}(\mathfrak{g}_i)$ and $G'$ preserve the invariant of degree 2 corresponding to the Killing form on each $\mathfrak{g}_i$ we must have that $\lambda_i = \pm 1$, $i = 1, \ldots, r$. □

From now on we assume that $\mathfrak{g}$ is simple. Let $\sigma \in \text{Aut}(\mathfrak{g})$. Then we know that, up to multiplication by an element of $G = \text{Aut}(\mathfrak{g})^0$, we can arrange that $\sigma$ preserves a fixed Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Thus we may assume that $\varphi$ preserves $\mathfrak{t}$. Let $T$ denote the corresponding maximal torus of $G$.

Corollary 2.2. We may modify $\varphi$ by an element of $G$ so that $\varphi$ is the identity on $\mathfrak{t}$.

Proof. By Chevalley’s theorem, restriction to $\mathfrak{t}$ gives an isomorphism of $\mathbb{C}[\mathfrak{g}]^G$ with $\mathbb{C}[\mathfrak{t}]^{W}$, where $W$ is the Weyl group of $\mathfrak{g}$. Thus the restriction of $\varphi$ to $\mathfrak{t}$ coincides with an element of $W$, where every element of $W$ is the restriction of an element of $G$ stabilizing $\mathfrak{t}$. Thus we may assume that $\varphi$ is the identity on $\mathfrak{t}$. □

Let $\varphi$ be the set of roots and $\varphi^+$ a choice of positive roots. Let $\Pi$ denote the set of simple roots. Since $\varphi = \pm \sigma$ is the identity on $\mathfrak{t}$, $\sigma(x) = c_\sigma x$ for all $x \in \mathfrak{t}$, where $c_\sigma = \pm 1$. Hence either $\sigma$ sends each $\mathfrak{g}_\alpha$ to itself or it sends each $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$, $\alpha \in \varphi$. Choose nonzero elements $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Pi$, and choose elements $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $(x_\alpha, y_\alpha, [x_\alpha, y_\alpha])$ is an $\mathfrak{sl}_2$-triple. Let $\psi$ denote the unique order 2 automorphism of $\mathfrak{g}$ such that $\psi(x) = -x$, $x \in \mathfrak{t}$ and $\psi(x_\alpha) = -y_\alpha$, $\alpha \in \Pi$ (see [Hum72, 14.3]).

Proposition 2.3. 

(1) If $c_\sigma = 1$, then $\sigma$ is inner.

(2) If $c_\sigma = -1$, then $\sigma$ differs from $\psi$ by an element of $\text{Ad}(T)$.

Proof. If $c_\sigma = 1$, then $\sigma(x_\alpha) = c_\alpha x_\alpha$, $c_\alpha \in \mathbb{C}$, $\alpha \in \Pi$. There is a $t \in T$ such that $\text{Ad}(t)(x_\alpha) = c_\alpha x_\alpha$, $\alpha \in \Pi$. It follows that $\sigma = \text{Ad}(t) \in G$. If $c_\sigma = -1$, we can modify $\sigma$ by an element of $T$ so that it becomes $\psi$. □

Proposition 2.4. Let $\mathfrak{g}$ be simple. Then the following are equivalent.

(1) Every representation of $\mathfrak{g}$ is self-dual.

(2) The automorphism $\psi$ is inner.

(3) The generators of $\mathbb{C}[\mathfrak{g}]^G$ have even degree.

(4) $\mathfrak{g}$ is of the following type:

(a) $B_n$, $n \geq 1$,

(b) $C_n$, $n \geq 3$,

(c) $D_{2n}$, $n \geq 2$,

(d) $E_7$, $E_8$, $F_4$ or $G_2$.

Proof. The equivalence of (1), (3) and (4) is well-known. Now given a highest weight vector $\lambda$ of $\mathfrak{g}$, the highest weight vector of the corresponding dual representation
The group $G'$ has order 2, generated by $-\psi$.

**Proof.** If $\varphi = \sigma \in \text{Aut}(g)$, then Proposition 2.3 shows that $\varphi = \sigma \in G$. If $\varphi = -\sigma$, then by Proposition 2.3 we may assume that $\varphi = -\psi$. Now $-\psi$ induces an automorphism of $\mathbb{C}[g]^G$ and $-\psi$ is the identity on $t$. Hence Chevalley’s theorem shows that $-\psi \in G'$, and we know that $-\psi$ generates $G'/G$. Moreover, $-\psi$ is not in $\text{Aut}(g)$, so that $-\psi \notin G$. □

**Corollary 2.6.** Suppose that $\psi$ is inner. Then $G'/G$ is generated by multiplication by $-1$.

We leave it to the reader to formulate versions of Theorem 2.5 and Corollary 2.6 for the semisimple case.

### 3. The General Case

We have a finite dimensional vector space $V$ and $G$ is a reductive subgroup of $\text{GL}(V)$. Let $G' := \{ \varphi \in \text{GL}(V) \mid p \circ \varphi = p \text{ for all } p \in \mathbb{C}[V]^G \}$. We show that, “in general”, we have $G' = G$.

Let $U$ denote the subset of $V$ consisting of closed $G$-orbits with trivial stabilizer. It follows from Luna’s slice theorem [Hum73] that $U$ is open in $V$.

**Theorem 3.1.** Suppose that $V \setminus U$ is of codimension 2 in $V$. Then $G' = G$.

**Proof.** Let $\varphi \in G'$ and let $x \in U$. Then $\varphi(x) = \psi(x) \cdot x$ where $\psi : U \to G$ is a well-defined morphism. Since $G$ is affine, we may consider $\psi$ as a mapping from $U \to G \subset \mathbb{C}^n$ for some $n$ where $G$ is Zariski closed in $\mathbb{C}^n$. Our condition on the codimension of $V \setminus U$ guarantees that each component of $\psi$ is a regular function on $V$; hence $\psi$ extends to a morphism defined on all of $V$, with image in $G$. Now let $x \in U$. Then

$$\varphi(x) = \lim_{t \to 0} \varphi(tx)/t = \lim_{t \to 0} \psi(tx)tx/t = \psi(0)(x).$$

Thus $\varphi$ is just the action of $\psi(0) \in G$, so $G' = G$. □

### 4. Representations of $\text{SL}_2$

As an illustration, we consider representations of $G = \text{SL}_2$ or $G = \text{SO}_3$. We only consider representations with no nonzero fixed subspace. We let $R_j$ denote the irreducible representation of dimension $j + 1$, $j \geq 0$, and $kR_j$ denotes the direct sum of $k$ copies of $R_j$, $k \geq 1$. When we have a representation only containing copies of $R_j$ for $j$ even, then we are considering representations of $\text{SO}_3$. From [Sch95, 11.9] we know that all representations of $G$ satisfy the hypotheses of Theorem 3.1 except for the following cases, where we compute $G'$.

(1) For $R_1$ we have $G' = \text{GL}_2$, for $2R_1$ we have $G' = \text{O}_4$ and for $3R_1$ we have $G' = G$.  

For $R_2$ we have $G' = O_3$ and for $2R_2$ we have $G' = O_3$. (Here $G = SO_3$.)

(3) For $R_2 \oplus R_1$ we have $G' = \{g' \in GL_2 \mid \det(g') = \pm 1\}$.

(4) For $R_3$ the group $G'$ is the same as in case (3).

(5) For $R_4$ we have $G' = G = SO_3$.

Most of the calculations are easy. We mention some details for some of the non-obvious cases.

Suppose that our representation is $R_4$, which has generating invariants of degrees 2 and 3. The Lie algebra $g'$ acts irreducibly on $R_4$; hence it is the sum of a center and a semisimple Lie algebra (Jac62 Ch. II, Theorem 11). Clearly we cannot have a nontrivial center, so that $g'$ is semisimple. Now a case-by-case check of the possibilities forces $g' = g$. Suppose that $g' \in G' \setminus G$. Then conjugation by $g'$ gives an inner automorphism of $G$; hence we can correct $g'$ by an element of $G$ so that $g'$ commutes with $G$. Thus $g'$ acts on $R_4$ as a scalar. But to preserve the invariants the scalar must be 1. Thus we have $G' = G$. Similar considerations give that $g' = g$ in case (4), so that generators of $G'/G$ act as scalars on $R_2$ and $R_1$. Now generators of the invariants have degrees (2, 0) and (1, 2) so that $G'/G$ is generated by an element which is multiplication by $-1$ on $R_2$ and multiplication by $i$ on $R_1$. Hence $G'$ is as claimed.

In case (3), one sees that $g' = g$, so that generators of $G'/G$ act as scalars on $R_2$ and $R_1$. Now generators of the invariants have degrees (2, 0) and (1, 2) so that $G'/G$ is generated by an element which is multiplication by $-1$ on $R_2$ and multiplication by $i$ on $R_1$. Hence $G'$ is as claimed.

References


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