THE RESTRICTED ISOMORPHISM PROBLEM
FOR METACYCLIC RESTRICTED LIE ALGEBRAS

HAMID USEFI

(Communicated by Gail R. Letzter)

Abstract. Let \( L \) be a restricted Lie algebra with the restricted enveloping algebra \( u(L) \) over a perfect field of positive characteristic \( p \). The restricted isomorphism problem asks what invariants of \( L \) are determined by \( u(L) \). This problem is the analogue of the modular isomorphism problem for finite \( p \)-groups. Bagiński and Sandling have given a positive answer to the modular isomorphism problem for metacyclic \( p \)-groups. In this paper, we provide a positive answer to the restricted isomorphism problem in case \( L \) is metacyclic and \( p \)-nilpotent.

1. Introduction

Let \( L \) be a Lie algebra over a field \( F \) of positive characteristic \( p \) and denote by \( \text{ad} : L \rightarrow L \) the adjoint representation of \( L \) given by \( (\text{ad} x)(y) = [y, x] \), where \( x, y \in L \). Recall that \( L \) is called restricted if \( L \) additionally affords a \( p \)-map \( [p] : L \rightarrow L \), satisfying

1. \( (\text{ad} x)^p = \text{ad}(x[p]) \), for every \( x \in L \);
2. \( (\alpha x)^p = \alpha^p x[p] \), for every \( x \in L \) and \( \alpha \in F \); and
3. \( (x + y)^p = x[p] + y[p] + \sum_{i=1}^{p-1} s_i(x, y) \), for all \( x, y \in L \), where \( s_i(x, y) \) is the coefficient of \( \lambda^{i-1} \) in \( \text{ad}(\lambda x + y)^{p-1}(x) \).

Let \( L \) be a restricted Lie algebra and denote by \( u(L) \) the restricted enveloping algebra of \( L \). The restricted isomorphism problem asks what invariants of \( L \) are determined by \( u(L) \); i.e. given another restricted Lie algebra \( H \) with the property that \( u(L) \cong u(H) \), as algebras, can we deduce that \( L \) and \( H \) have the same invariants? Of course, the strongest invariant of \( L \) is its isomorphism type. We have considered the restricted isomorphism problem for abelian restricted Lie algebras in [10]. One main result of [10] states that if \( L \) is an abelian restricted Lie algebra in the class \( \mathcal{F}_p \), then the isomorphism type of \( L \) is determined. Recall that \( L \) is said to be in the class \( \mathcal{F}_p \) if \( L \) is finite-dimensional and there exists an integer \( k \) such that \( L[p]^k = 0 \). It is also proved in [10] that if \( F \) is algebraically closed, then every finite-dimensional abelian restricted Lie algebra over \( F \) is determined by its enveloping algebra.
The restricted isomorphism problem is the analogue of the modular isomorphism problem for group algebras of finite $p$-groups [8]. This sort of isomorphism problem also makes sense for ordinary Lie algebras and has been considered by David Riley and the present author in [6]. Bagiński [2] and Sandling [7] proved that every metacyclic $p$-group $G$ is determined by its modular group algebra $\mathbb{F}_p G$, where $\mathbb{F}_p$ denotes the field of $p$ elements. Motivated by this result of Bagiński and Sandling, in this paper we consider the isomorphism problem for metacyclic restricted Lie algebras in the class $\mathcal{F}_p$. Recall that a restricted Lie algebra $L$ is called metacyclic if $L$ has a cyclic restricted ideal $I$ such that $L/I$ is cyclic. Our main result is as follows:

**Theorem.** Let $L \in \mathcal{F}_p$ be a metacyclic restricted Lie algebra over a perfect field of positive characteristic. Then $L$ is determined by its enveloping algebra $u(L)$.

Before closing this section, I would like to thank Luzius Grunenfelder for many useful discussions.

2. Preliminaries

Let $L$ be a restricted Lie algebra with the restricted enveloping algebra $u(L)$ over a field $\mathbb{F}$ of characteristic $p$. Let $X$ be an ordered basis of $L$ over $\mathbb{F}$. The analogue of the Poincaré–Birkhoff–Witt (PBW) Theorem for restricted Lie algebras (see [3]) allows us to view $L$ as a restricted Lie subalgebra of $u(L)$ in such a way that $u(L)$ has a basis consisting of PBW monomials, that is, monomials of the form

$$x_1^{a_1} \cdots x_t^{a_t},$$

where $x_1 < \cdots < x_t$ in $X$, $0 \leq a_i \leq p-1$, and $t$ is a non-negative integer. Henceforth, we identify the $[p]$-map of $L$ by the exponentiation by $p$ in $u(L)$.

The augmentation ideal, $\omega(L)$, of $u(L)$ is the associative ideal of $u(L)$ generated by $L$. Let $\gamma_1(L) := L$. We denote by $\gamma_n(L) := [\gamma_{n-1}(L), L]$ the $n^{th}$ term of the lower central series of $L$. Hence, $L' = \gamma_2(L)$. Recall that $L$ is said to be nilpotent if $\gamma_n(L) = 0$ for some $n$; the nilpotence class of $L$, denoted by $cl(L)$, is the minimal integer $c$ such that $\gamma_c(L) = 0$. We shall denote by $L'_p$ the restricted Lie subalgebra of $L$ generated by $L'$. For a subset $X$ of $L$ we denote by $\langle X \rangle$ and $\langle X \rangle_p$ the Lie subalgebra and the restricted Lie subalgebra generated by $X$, respectively. We consider left-normed commutators, that is, $[x_1, \ldots, x_n] = [[x_1, x_2], x_3, \ldots, x_n]$.

An element $x \in L$ is called $p$-nilpotent if there exists some non-negative integer $t$ such that $x^{p^t} = 0$; the exponent of $x$, denoted by $\exp(x)$, is the least integer $s$ such that $x^{p^s} = 0$. A $p$-polynomial in $x$ has the form $c_0 x + c_1 x^p + \cdots + c_s x^{p^s}$, where each $c_i \in \mathbb{F}$. Also, recall that $L$ is $p$-nilpotent if there exists a positive integer $k$ such that $L^{p^k} = 0$; the exponent of $L$, denoted by $\exp(L)$, is the minimal integer $s$ such that $L^{p^s} = 0$. We say $L \in \mathcal{F}_p$ if $L$ is finite-dimensional and $p$-nilpotent. Note that if $L \in \mathcal{F}_p$, then $L$ is nilpotent by Engel’s Theorem. It is known that $L \in \mathcal{F}_p$ if and only if $\omega(L)$ is nilpotent as an associative ideal of $u(L)$; see [5].

The $n^{th}$ dimension subalgebra of $L$ is

$$D_n(L) = L \cap \omega^n(L) = \sum_{\nu p^\prime \geq n} \gamma_{\nu}(L)^{p^\prime},$$
Lemma 2.1. If \( u(L) \cong u(H) \), then the following statements hold.

1. If \( L \in \mathcal{F}_p \), then \( |\text{cl}(L) - \text{cl}(H)| \leq 1 \).
2. \( D_i(L)/D_{i+1}(L) \cong D_i(H)/D_{i+1}(H) \), for every \( i \geq 1 \).

We remark that whether or not the nilpotence class of \( G \) is determined by \( \mathbb{F}_pG \) has been considered in recent years. However no major result is reported up-to-date; see [3].

We note that \( u(L) \) has the invariant dimension property: that is, the rank of every \( u(L) \)-module is uniquely defined. Now let \( I \) be a restricted ideal of \( L \) and denote by \( I^n \) the vector space spanned by all \( z_1z_2\ldots z_n \), where \( z_i \in I \). We know that the kernel of the natural map \( u(L) \to u(L/I) \) is equal to \( Iu(L) = u(LI) \). It follows that \( \omega^n(I) u(L) = I^n u(L) = (Iu(L))^n \), for every \( n \geq 1 \). Furthermore, we observe that each quotient \( I^n u(L)/I^{n+1} u(L) \) has a natural right \( u(L/I) \cong u(L)/Iu(L) \)-module structure given by

\[
(u + I^{n+1} u(L)) \cdot (z + Iu(L)) = uz + I^{n+1} u(L),
\]

for every \( u \in I^n u(L) \) and \( z \in u(L) \). The proofs of the following two lemmas are analogous to the corresponding lemmas in [3].

Lemma 2.2. Let \( I \) be an ideal of \( L \). Then each factor \( I^n u(L)/I^{n+1} u(L) \) is a free right \( u(L/I) \)-module of rank \( \dim_{\mathbb{F}_p} \omega^n(I)/\omega^{n+1}(I) \).

Lemma 2.3. Suppose that \( \dim_{\mathbb{F}_p} I/D_2(I) \) is finite. Suppose further that \( J \) is a restricted ideal of \( H \) such that \( \varphi(Iu(L)) = Ju(H) \). Then \( D_n(I)/D_{n+1}(I) \cong D_n(J)/D_{n+1}(J) \), for every \( n \geq 1 \).

Using the identity \([ab,c] = a[b,c] + [a,c]b\), which holds in any associative algebra, we can see that \( L'_p u(L) = [\omega(L), \omega(L)] u(L) \). Thus the ideal \( L'_p u(L) \) is preserved by \( \varphi \). So we may apply the previous lemma to the case \( I = L'_p \) and \( J = H'_p \).

Corollary 2.4. Suppose that \( L \) and \( H \) are finite-dimensional restricted Lie algebras such that \( u(L) \cong u(H) \). Then, for every positive integer \( n \), we have

\[
D_n(L'_p)/D_{n+1}(L'_p) \cong D_n(H'_p)/D_{n+1}(H'_p).
\]

Lemma 2.5. Let \( L \in \mathcal{F}_p \) such that \( \text{cl}(L) = 2 \). Then, \( \dim_{\mathbb{F}_p} L' \) is determined, for every \( t \geq 0 \). In particular, the exponent of \( L' \) is determined.

Proof. Note that, by Lemma 2.1, \( H \) is nilpotent of class at most 3. So, \( H' \) is abelian. Note that \( D_{p^t+1}(H'_p) = \cdots = D_{p^t}(H'_p) = H'^{p^t} \), for every \( t \geq 1 \). So, by Corollary 2.4 we have \( L'_p/L'^{p^t} \cong H'_p/H'^{p^t} \) and

\[
L'^{p^t}/L'^{p^{t+1}} \cong H'^{p^t}/H'^{p^{t+1}},
\]
for every $t \geq 1$. Now let $s = \exp(L')$. It follows that

$$H^{t^s} = H^{t^{s+1}}.$$  

But $H$ is $p$-nilpotent. Hence, $H^{t^s} = 0$. Thus $\exp(H') \leq \exp(L')$. It follows then by symmetry that the exponent of $H'$ is $s$. A reverse induction on $t$ shows that $\dim s L_p^{t^s}$ is determined, for every $t \geq 0$. □

Lemma 2.6. Let $L \in \mathcal{F}_p$ such that $L_p'$ is cyclic. The following statements hold.

1. $cl(L) \leq 3$.
2. We have $L^{t^s} u(L) = (L_p'u(L))^{t^s}$, for every $t \geq 1$.

Proof. Let $u \in L_p'$ such that $L_p' = \langle u \rangle_p'$. Let $z \in L$. There exist coefficients $\alpha_0, \ldots, \alpha_t \in F$ such that $[u, z] = \sum_{k=0}^t \alpha_k u^k$. So, $[z, u^k] = 0$, for every $k \geq 1$. It follows that

$$[u, z^k] = \alpha_0^{k-1}[u, z],$$

for every $k \geq 1$. But $z$ is $p$-nilpotent. Hence, either $\alpha_0 = 0$ or $[u, z] = 0$. Thus $[u, z] \in \langle u^m \rangle_p'$. Hence, $\gamma_3(L) \leq L_p$. Now let $v \in L$. Note that $[v, u^k] = 0$, for every $k \geq 1$. Since $[u, z] \in \langle u^m \rangle_p'$, we deduce that $[u, z, v] = 0$. It follows that $cl(L) \leq 3$. To prove the second assertion note that $L^{t^s} u(L) \subseteq (L_p'u(L))^{t^s}$. So we need to prove that $(L_p'u(L))^{t^s} \subseteq L^{t^s} u(L)$. Note that $[vz, u] = v[z, u] + [v, u]z \in \langle u^m \rangle_p' u(L)$, for every $v, z \in L$. It follows by the PBW Theorem that $[u(L), L_p'] \subseteq \langle u^m \rangle_p' u(L)$. Hence, for every $t \geq 1$, we have

$$(L_p'u(L))^{t^s} = \langle (u) u'(L) \rangle^{t^s} \subseteq \langle (u) u'(L) \rangle^{t^s} u(L) = L^{t^s} u(L),$$

as required. □

3. Metacyclic restricted Lie algebras

A restricted Lie algebra $L$ is called metacyclic if $L$ has a cyclic restricted ideal $I$ such that $L/I$ is cyclic. In other words, there exist generators $x, y \in L$ and some $p$-polynomials $g$ and $h$ such that

$$h(x) \in \langle y \rangle_p, \quad [y, x] = g(y).$$

In this section $L$ is a non-abelian metacyclic restricted Lie algebra in the class $\mathcal{F}_p$.

Let $g(y) = \beta_0 y + \beta_1 y^p + \beta_2 y^{p^2} + \cdots$. We claim that $\beta_0 = 0$. Suppose otherwise. Then

$$y = \beta [y, x] + \beta_1 y^p + \beta_2 y^{p^2} + \cdots,$$

where $\beta \neq 0$. Note that $[x, y^p] = 0$, for every $t \geq 1$. Since $x$ is $p$-nilpotent, there exists an integer $k$ such that

$$[y, x] = \beta [y, x, x] = \cdots = \beta^{p^k-1}[y, x^k] = 0,$$

contradicting the fact that $L$ is non-abelian.

Let $m$ be the largest integer such that $x, x^p, \ldots, x^{p^{m-1}}$ are linearly independent modulo $\langle y \rangle_p'$. So there exist coefficients $c_1, \ldots, c_m \in F$ such that $\sum_{i=1}^m c_i x^p \in \langle y \rangle_p'$. Let $k$ be the smallest integer such that $c_k \neq 0$. Then

$$x^{p^k} = - \sum_{i=k+1}^m \frac{c_i}{c_k} x^{p^i} \text{ modulo } \langle y \rangle_p'.$
It follows from the equation above that $x^{p^k} \in (y, x^{p^{k+1}})_p$ and hence $x^{p^k} \in (y, x^{p^l})_p$, for every $j \geq k$. Since $x$ is $p$-nilpotent, we get $x^{p^k} \in (y)_p$. So, by our choice of $m$, we should have $k = m$. We conclude that there exist another $p$-polynomial $f$ and positive integers $m, n$ such that the following relations hold in $L$:

\[ x^{p^m} = f(y) = y^{p^r} + \cdots, \]

\[ y^{p^n} = 0, \]

\[ [y, x] = g(y) = \beta_s y^{p^{s'}} + \cdots, \beta_s \neq 0. \]

Since $L$ is not abelian, we have $1 \leq r \leq n$ and $1 \leq s \leq n - 1$. Now it is easy to see that the commutator subalgebra $L'$ is one-dimensional and $\text{cl}(L) = 2$. So, $L' = \mathbb{F}[x, y] \leq L^p$. Since $x^{p^{n+r}} = y^{p^n} = 0$, we have $\exp(x) = m + n - r$. Also, $[y, x]^{p^{n+s}} = \beta_s^{n+s} y^{p^n} = 0$. Thus, the exponent of $L'$ is equal to $n - s$ which is determined by Lemma 2.5.

Since $L' \subseteq L^p$, we have $D_{p}(L) = L^p + L^{p^{k-1}} = L^p$ for every $k \geq 1$. But each quotient $D_{p^k}(L)/D_{p^{k+1}}(L)$ is determined by Lemma 2.1. We can then use a similar method as in Lemma 2.5 to show that $\dim L^{p^k}$ is determined, for every $k \geq 0$. In particular, the exponent of $L$ is determined. So we have:

**Lemma 3.1.** Let $L \in \mathcal{F}_p$ be a non-abelian metacyclic restricted Lie algebra. Then the exponent of $L$ is determined.

We need the following technical lemma in the case $p = 2$.

**Lemma 3.2.** Let $L$ and $H$ be non-abelian restricted Lie algebras over a perfect field of characteristic 2. Suppose that $L$ and $H$ are generated by $x, y$ and $u, v$, respectively, subject to the following relations:

\[ x^2 = f(y^2) = \alpha_r y^{2^r} + \cdots, \quad u^2 = f(v^2) = \alpha v^{2^n}, \]

\[ y^{2^n} = 0, \quad v^{2^n} = 0, \]

\[ [x, y] = g(y^2) = \beta_s y^{2^s} + \cdots, \beta_s \neq 0, \quad [u, v] = g(v^2) + \beta v^{2^n}, \]

where $f$ and $g$ are some 2-polynomials, $s \leq r \leq n$, and $\alpha, \beta \in \mathbb{F}$. If $u(L) \cong u(H)$, then $L \cong H$.

**Proof.** Note that $n \geq 2$, since $L$ is not abelian. Suppose first that $n \geq 3$ and $s < n - 1$. We first replace $v$ by $v_1 = v + \beta v^{2^n - 1}$, where $\beta' = (\beta/\beta_s)^{2^{-s}}$. So we get the following relations in $H$:

\[ u^2 = f(v_1^{2^n}) + \alpha' v_1^{2^n - 1}, \quad v_1^{2^n} = 0, \quad [u, v_1] = g(v_1^{2^n}). \]

We can now replace $u$ by $u_1 = u + \alpha^{1/2} v_1^{2^n - 2}$ and note that the map induced by $x \mapsto v_1$, $y \mapsto v_1$ is an isomorphism between $L$ and $H$. Similarly, if $n \geq 3$ and $r = s = n - 1$, we can find generators $u_1, v_1$ of $H$ such that the map induced by $x \mapsto u_1$, $y \mapsto v_1$ is an isomorphism. Now we consider the case $n = 2$. We have the following relations:

\[ x^2 = a_1 y^2, \quad u^2 = (a_1 + a_2) v^2, \]

\[ y^4 = 0, \quad v^4 = 0, \]

\[ [x, y] = b_1 y^2, \quad [u, v] = (b_1 + b_2) v^2, \]
where \( a_1, a_2, b_1, b_2 \in \mathbb{F} \). Suppose that \( \varphi : u(L) \to u(H) \) is an isomorphism and let \( f = \varphi(x) \) and \( g = \varphi(y) \). So, \( f \) and \( g \) generate \( u(H) \), \( f^2 = a_1 g^2, g^4 = 0 \), and \( [f, g] = b_1 g^2 \). Let \( w = [u, v] \) and take \( \{ u, v, w \} \), \( u < v < w \), as a basis for \( H \). Let \( E \) be the vector space spanned by PBW monomials \( E \). Since \( H \) is a perfect field, \( E \) is an \( H \)-module.

Similarly we deduce that \( \bar{\omega} \). So \([ \bar{\omega} \rangle = \bar{\omega} \). Let \( \bar{\omega} \) generate \( u(H) \), it follows that \( \bar{\omega} \) generate \( u(H)/\omega^2(H) \). Hence, \( \bar{\omega} \) generate \( u(H) \) since \( \omega(H) \) is nilpotent. Thus, the homomorphism induced by \( x \mapsto \bar{f}, y \mapsto \bar{g} \) is an isomorphism between \( L \) and \( H \).

4. Proof of the theorem

Suppose that \( L \in \mathcal{F}_p \) is a non-abelian metacyclic restricted Lie algebra over a perfect field \( \mathbb{F} \). Let \( \varphi : u(L) \to u(H) \) be an algebra isomorphism. Note that \( H \in \mathcal{F}_p \), since \( \omega(L) \) and \( \omega(H) \) are nilpotent.

Let \( x, y \) be some generators of \( L \). By our discussion from Section 3 there exist \( p \)-polynomials \( f, g \) and positive integers \( m, n \) such that:

\[
x^p = f(y) = y^n + \cdots, \quad r \geq 1,
\]

\[
y^p = 0,
\]

\[
[y, x] = g(y) = \beta_s y^n + \cdots, \quad s \geq 1.
\]

We fix the generators \( x, y \) such that \( \exp(L/\langle y \rangle_p) = m \) is minimum. Note that if \( \exp(x) \leq \exp(y) \), that is, \( m \leq r \), we can assume that \( s \leq r \). Indeed, if \( r < s \), then we replace \( x \) by \( x' = x - y^{p-1} \) and continue this process if necessary.

Let \( t = n - s - 1 \). Since \( cl(L) = 2 \), we have \( \exp(L') = \exp(H') = t + 1 \), by Lemma 2.5. Also, \( cl(H) \leq 3 \), by Lemma 2.4. In particular, \( H' \) is abelian. Hence, by Corollary 2.4 we have

\[
L_p'/L_p'' \cong H'_p/H_p'' = (H' + H_p'')/H_p''.
\]

Since \( L'_p \) is cyclic, there exists \( w \in H' \) such that \( H'_p = Fw + H_p'' \). Hence, \( H'_p = \langle w \rangle_p \) because \( H \in \mathcal{F}_p \). Since the ideal \( L'_p/u(L) \) is preserved by \( \varphi \), we use Lemma 2.6 to see that \( \varphi(L''_p u(L)) = H''_p u(H) \). So, \( \varphi \) induces an isomorphism

\[
u(L/L''_p) \to u(H/H''_p).
\]

We may now assume by induction on \( \dim L \) that \( L/L''_p \cong H/H''_p \). We shall find generators \( u, v \in H \) and replace \( x \) and \( y \) by appropriate generators of \( L \), if necessary, so that the map induced by \( x \mapsto u, y \mapsto v \) is an isomorphism between \( L \) and \( H \).

Step 1. \( H \) is metacyclic, too.
Let $u$ and $v$ be some fixed representatives of the images of $x$ and $y$ under the isomorphism $L/L' \rightarrow H/H'$. Thus we have the following relations between $u$ and $v$:

(1) $u^m = f(v) + \alpha_1 v^p$, $v^p = \alpha_2 u^p$, $[v, u] = g(v) + \alpha_3 w^p$,

where $H' = \langle w \rangle$. Note that $c_l(H) \leq 3$, by Lemma 2.1. We observe that $w^{p^{m+1}} \in \langle v \rangle$, $v^{p^{n+1}} = 0$, and $[u, v] \in \langle v \rangle$. So, $H$ is metacyclic.

Since $\exp(H') = t + 1$, we have $v^{p^{n-1}} \neq 0$. Note that $y^{p^{n-1}} = [y, x]^p \in L'$. We deduce that $v^{p^{n-1}} = 0$ modulo $H'$. Hence, $v^p = 0$. So we can replace (1) by the following relations:

\[
\begin{align*}
  u^m &= f(v) + \alpha v^{p^{n-1}}, \\
  v^p &= 0, \\
  [v, u] &= g(v) + \beta v^{p^{n-1}}.
\end{align*}
\]

**Step 2.** Assume $(x)_p \cap (y)_p = 0$.

In other words, $f = 0$. So, we have the following relations in $L$:

\[
\begin{align*}
  x^m &= y^n = 0, \\
  [y, x] &= g(y).
\end{align*}
\]

Thus the following relations hold in $H$:

\[
\begin{align*}
  u^m &= \alpha v^{p^{n-1}}, \\
  v^p &= 0, \\
  [v, u] &= g(v) + \beta v^{p^{n-1}}.
\end{align*}
\]

Suppose $\beta \neq 0$. If $s < n - 1$, then we replace $v$ by $v_1 = v + \beta' v^{p^{n-1}}$, where $\beta' = (\beta/\beta_s)^p$. Suppose $s = n - 1$. Since $L$ and $H$ are non-abelian, we have $\beta + \beta_s \neq 0$. Now we replace $u$ by $u_1 = \beta_s u/(\beta + \beta_s)$. So, without loss of generality, we assume that the following relations hold in $H$:

\[
\begin{align*}
  u^m &= \alpha v^{p^{n-1}}, \\
  v^p &= 0, \\
  [v, u] &= g(v).
\end{align*}
\]

If $m \geq n$, then $\alpha = 0$. Otherwise, $\exp(H) = m + 1$ whereas $\exp(L) = m$, contradicting Lemma 3.1. If $m \leq n - 1$, then we replace $u$ by $u_1 = u - \alpha' v^{p^{n-1}}$, where $\alpha' = \alpha^{p^{-m}}$. Then we have

\[
\begin{align*}
  u_1^p &= u^p - \alpha' p^{p^{n-1}} \text{ modulo } \gamma_p((u, v^{p^{n-1}})).
\end{align*}
\]

So, if $m \geq 2$, then

\[
\begin{align*}
  u_1^m &= u^m - \alpha v^{p^{n-1}} \text{ modulo } \gamma_p((H^p, H')).
\end{align*}
\]

Note that $\gamma_p((H^p, H')) = 0$ since $c_l(H) = 2$. In the case $m = 1$, note that $\gamma_p((u, v^{p^{n-1}})) \subseteq \gamma_2(H) = 0$ unless $p = 2$ and $m = n - 1 = 1$. So, virtually, $u_1^m = 0$ and we would be done. The remaining case $p = 2$ and $m = n - 1 = 1$ satisfies the hypothesis of Lemma 3.2 and so $L \cong H$.

**Step 3.** The general case.
Suppose that \( \exp(x) \leq \exp(y) \), that is, \( m \leq r \). So, as mentioned earlier, we can assume that \( s \leq r \). We replace \( x \) by \( x_1 = x - (f(y))^{p^{-m}} \). So, \( x_1^{p^{m}} = 0 \) unless \( m = r = 1 \) and \( p = 2 \). We observe that \( x_1 \) and \( y \) generate \( L \) and \( \langle x_1 \rangle_p \cap \langle y \rangle_p = 0 \). So we would be done by Step 2. In the special case \( m = r = 1 \) and \( p = 2 \), we note that \( L \cong H \), by Lemma 3.2. Hence we may assume that \( m > r \). Now we have two cases.

Case I. \( r \leq s \). Since \( x^{p^{m}} = f(y) \), we get \( y^{p^{r}} \in \langle x^{p^{m}}, y^{p^{r+1}} \rangle_p \). Thus, \( y^{p^{r}} \in \langle x^{p^{m}} \rangle_p \), because \( y \) is \( p \)-nilpotent. Since \( r \leq s \), it follows that \( [y, x] \in \langle x^{p^{m}} \rangle_p \). Thus, there exists a \( p \)-polynomial \( h \) such that \( y^{p^{r}} = h(x^{p^{m}}) \). Now we replace \( y \) by \( y_1 = y - (h(x^{p^{m}}))^{p^{-r}} \). Note that \( \exp(y_1) = r < m \). So we have found new generators \( x' = y_1, y' = x \) such that \( \exp(L/(y'_p)) < m \). This contradicts the minimality of \( m \). So Case I is not possible and we have to consider the case \( r > s \).

Case II. \( r > s \).

If \( r = n - 1 \), then \( x^{p^{m}} = y^{p^{n-1}} \). We replace \( y \) by \( y_1 = y - x^{p^{m-n+1}} \). Observe that \( \langle x \rangle_p \cap \langle y_1 \rangle_p = 0 \) and so we are done by Step 2. It remains to consider the case \( r < n - 1 \). In this case we replace \( x \) by \( x_1 = x + \alpha' x^{p^{r-s}} \), where \( \alpha' = \alpha^{p^{-m}} \). We get the following relations in \( L \):

\[
x_1^{p^{m}} = f(y) + \alpha y^{p^{n-1}}, \quad y^{p^{n}} = 0, \quad [y, x_1] = g(y).
\]

Note that \( s < n - 1 \). Now we replace \( y \) by \( y_1 = y - \beta' y^{p^{n-1}} \), where \( \beta' = (\beta/\beta_s)^{p^{-r}} \). Note that \( [x_1, y] = [x_1, y_1] \). Also, we observe that \( g(y_1) = g(y) - \beta y^{p^{n-1}} \). Thus,

\[
[y_1, x_1] = [y, x_1] = g(y) = g(y_1) + \beta y_1^{p^{n-1}}.
\]

Furthermore, \( x_1^{p^{m}} = f(y_1) + \alpha y_1^{p^{n-1}} \), since \( r > s \). So, we get the following relations in \( L \):

\[
x_1^{p^{m}} = f(y_1) + \alpha y_1^{p^{n-1}},
\]

\[
y_1^{p^{n}} = 0,
\]

\[
[y_1, x_1] = g(y_1) + \beta y_1^{p^{n-1}}.
\]

We deduce that the map induced by \( x_1 \mapsto u, y_1 \mapsto v \) is an isomorphism between \( L \) and \( H \). \( \square \)

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, 1984 MATHEMATICS ROAD, VANCOUVER, BC, CANADA, V6T 1Z2

E-mail address: usefi@math.ubc.ca