A NOTE ON THE $2k$-TH POWER MEAN OF CHARACTER SUMS OVER THE QUARTER INTERVAL

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Abstract. We obtain an asymptotic formula for the $2k$-th power mean of odd primitive character sums over the interval $[1, \frac{q}{4})$.

1. Introduction

Let $q \geq 3$ be an integer, $\chi$ be a Dirichlet character modulo $q$. The various arithmetical properties of the character sums

$$\sum_{a=N+1}^{N+H} \chi(a)$$

were investigated by many authors; see [1], [2], [3], [4]. D.A. Burgess [5] obtained the mean value estimate

$$\sum_{\chi \mod q}^{*} \sum_{p} \left| \sum_{m=1}^{h} \chi(n+m) \right|^4 \leq 8\tau^7(q)q^2h^2,$$

where $\sum_{\chi \mod q}^{*}$ denotes the sum over all primitive characters modulo $q$ and $\tau(n)$ is the divisor function.

The author and Zhang [6] studied the $2k$-th power mean of the even primitive character sums over the quarter interval $[1, \frac{q}{4})$ by using mean value theorems of Dirichlet $L$-functions, and obtained the asymptotic formula as follows:

$$\sum_{\chi(-1)=1}^{*} \left| \sum_{a<\frac{q}{4}} \chi(a) \right|^{2k} = \frac{J(q)q^k}{16} \left(\frac{\pi}{8}\right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p|2q} \left(1 - \frac{1-C_{2k-2}^{k-1}}{p^2}\right) + O(q^{k+c}),$$

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where $\sum_{\chi(-1)=1}^*$ denotes the sum over all primitive characters modulo $q$ such that $\chi(-1) = 1$, $\epsilon$ is any fixed positive number, $J(q)$ denotes the number of all primitive characters modulo $q$, $\prod_{p \mid q}$ denotes the product over all prime divisors $p$ of $q$, and $C_m^q = \frac{m!}{n!(m-n)!}$. Because Lemma 3 of reference [8] is not correct, which will be explained in our Lemma 2.6, this result is also incorrect. The last factor in the main term $\left(1 - \frac{C_m^q}{p^2} \right)$ should be $A(0, k, p, 2)$, as defined in our theorem below.

However, this result is correct in the case $k = 2$.

In this paper we study odd primitive character sums over the interval $[1, \frac{q}{2}]$ by transforming them to $L$-functions, and we obtain an asymptotic formula for the $2k$-th power mean using the same method as in [6].

**Theorem 1.1.** Let $q \geq 5$ be an odd integer. Then we have the asymptotic formula

$$\sum_{\chi(-1)=-1}^* \left| \sum_{a<\frac{q}{2}} \chi(a) \right|^{2k} = C(k) q^k J(q) \zeta^{2k-1}(2) \prod_{p \mid q} \left(1 - \frac{1}{p^2} \right)^{2k-1} \prod_{p^2 \mid 2q} A(0, k, p, 2) + O(k^{k+\epsilon}),$$

where

$$C(k) = \frac{1}{2^{2k}} \sum_{i=0}^{k} C_k^i (-2)^{k-i} \sum_{j=0}^{i} 6^j \sum_{s=0}^{k-i-j} \sum_{t=0}^{j} \frac{A(|3i + 4s - j - 2t - 2k|, k, 2, 2)}{2^{3i+4s-j-2t-2k}+2k+1},$$

$$A(m, k, p, s) = \sum_{i=0}^{2k-2} \sum_{p \mid m} \frac{1}{p^i} \sum_{j=0}^{s} (-1)^j C_j^{i-k-1} C_{k+m+i-j-1}^{i-j} C_{k+i-j-1}^{i-j},$$

and $\zeta(s)$ denotes the Riemann zeta function.

Taking $k = 2$ in our theorem and noting that

$$\prod_{p \mid q} \left(1 - \frac{1}{p^2} \right)^3 \prod_{p^2 \mid 2q} \left(1 + \frac{1}{p^2} \right) = \prod_{p} \left(1 + \frac{1}{p^2} \right) \prod_{p \mid q} \left(1 - \frac{1}{p^2} \right)^3 \prod_{p^2 \mid 2q} \frac{1}{1 + \frac{1}{p^2}} = \frac{4 \zeta(2)}{5 \zeta(4)} \prod_{p \mid q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)},$$

$\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$, we get

**Corollary 1.2.** Let $q \geq 5$ be an odd integer. Then

$$\sum_{\chi(-1)=-1}^* \left| \sum_{a<\frac{q}{2}} \chi(a) \right|^4 = \frac{9}{128} q^2 J(q) \prod_{p \mid q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} + O(q^{2+\epsilon}).$$

Combining this with Corollary 2 of [6], we have the following
Corollary 1.3. Let \( q \geq 5 \) be an odd integer. Then we have the asymptotic formula
\[
\sum_{\chi \mod q} \left| \sum_{a<q} \chi(a) \right|^4 = \frac{21}{256}q^2J(q)\prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} + O\left(q^{2+\epsilon}\right).
\]

The number \( n > 1 \) is called square-full if in the prime factorization \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) we have \( \alpha_i \geq 2 \) for all \( i = 1, 2, \cdots, r \). Noting that \( J(q) = \phi_2(q) \) if \( q \) is a square-full number, we have

Corollary 1.4. Let \( q \geq 5 \) be a square-full number with \( 2 \nmid q \). Then we have the asymptotic formula
\[
\sum_{\chi \mod q} \left| \sum_{a<q} \chi(a) \right|^4 = \frac{21}{256}q\phi_2(q)\prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} + O\left(q^{2+\epsilon}\right).
\]

Corollary 1.5. Let \( p \geq 5 \) be a prime. Then we have the asymptotic formula
\[
\sum_{\chi \mod p} \left| \sum_{a<p} \chi(a) \right|^4 = \frac{21}{256}p^3 + O\left(p^{2+\epsilon}\right).
\]

Remark 1.6. From our theorem and the theorem in [8], we find that odd and even primitive characters have a very different contribution to the higher moment of character sums over the quarter interval. For the case \( k = 2 \), from Corollaries 1.2 and 1.3, we see that the contribution of odd primitive characters is six times that of the even ones, although (see Remark 2.4 in the next section)
\[
\sum_{a=1}^{\lfloor \frac{q}{4} \rfloor} \chi(a) = 0
\]
for some odd primitive characters, while
\[
\sum_{a=1}^{\lfloor \frac{q}{4} \rfloor} \chi(a) \neq 0
\]
for all even primitive characters.

2. SOME LEMMAS

Lemma 2.1. Let \( \chi \) be a primitive character modulo \( m \) with \( \chi(-1) = -1 \). Then we have
\[
\frac{1}{m} \sum_{b=1}^{m} b\chi(b) = \frac{i}{\pi} \tau(\chi)L(1, \chi),
\]
where \( \tau(\chi) = \sum_{a=1}^{m} \chi(a)e\left(\frac{a}{q}\right) \) is the Gauss sum, \( e(y) = e^{2\pi iy} \), and \( L(s, \chi) \) denotes the Dirichlet \( L \)-function corresponding to \( \chi \).

Proof. This can be easily deduced from Theorem 12.11 and Theorem 12.20 of [7]. \( \square \)
Lemma 2.2. Let \( q \geq 3 \) be an odd number. For any nonprincipal character \( \chi \mod q \), we have
\[
\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1 - 2\chi(2)} \sum_{a=1}^{\frac{q+1}{2}} \chi(a).
\]

Proof. From the properties of Dirichlet characters, we have
\[
\sum_{a=1}^{q} 2a\chi(2a) = \sum_{a=1}^{\frac{q+1}{2}} 2a\chi(2a) + \sum_{a=\frac{q+1}{2}+1}^{q} 2a\chi(2a)
\]
\[
= \sum_{a=1}^{\frac{q+1}{2}} 2a\chi(2a) + \sum_{a=1}^{\frac{q+1}{2}} (2a - 1)\chi(q + 2a - 1) + q \sum_{a=1}^{\frac{q+1}{2}} \chi(2a - 1)
\]
\[
= \sum_{a=1}^{q} a\chi(a) + q \sum_{a=1}^{\frac{q+1}{2}} \chi(2a - 1).
\]
Noting that
\[
\sum_{a=1}^{\frac{q+1}{2}} \chi(2a - 1) + \sum_{a=1}^{\frac{q+1}{2}} \chi(2a) = \sum_{a=1}^{q} \chi(a) = 0,
\]
we can write
\[
(1 - 2\chi(2)) \sum_{a=1}^{q} a\chi(a) = \sum_{a=1}^{q} a\chi(a) - \sum_{a=1}^{q} 2a\chi(2a) = q \sum_{a=1}^{\frac{q+1}{2}} \chi(2a) = \chi(2)q \sum_{a=1}^{\frac{q+1}{2}} \chi(a).
\]
That is,
\[
\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1 - 2\chi(2)} \sum_{a=1}^{\frac{q+1}{2}} \chi(a).
\]
This proves Lemma 2.2. \( \square \)

Lemma 2.3. Let \( q \) be an odd number and \( \chi \) be a primitive Dirichlet character modulo \( q \) such that \( \chi(-1) = -1 \). Then we have
\[
\sum_{a=1}^{[\frac{q}{4}]} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2i\tau} \tau(\chi)L(1, \bar{\chi}).
\]

Remark 2.4. For even primitive characters \( \chi \), by Lemma 2 of [6],
\[
\sum_{a=1}^{[\frac{q}{4}]} \chi(a) = -\frac{i\bar{\chi}(4)}{2\pi} \tau(\chi\chi_4)L(1, \bar{\chi}\chi_4),
\]
where \( \chi_4 \) is the primitive character modulo 4. Since \( \tau(\chi\chi_4)L(1, \bar{\chi}\chi_4) \neq 0 \), we have
\[
\sum_{a=1}^{[\frac{q}{4}]} \chi(a) \neq 0
\]
Lemma 2.3, we know that for all even primitive characters. For odd primitive characters, from the identity in Lemma 2.3, we know that
\[ \sum_{a=1}^{\frac{q}{4}} \chi(a) = 0 \]
if and only if \( \chi(2) = -1 \). For example, if \( p \) is a prime, then the real character sum
\[ \sum_{a=1}^{\frac{p-1}{4}} \left( \frac{a}{p} \right) = 0 \]
if and only if \( p \equiv 3 \pmod{8} \). In this case, the number of quadratic residues and quadratic non-residues are equal in \([1, \frac{p+1}{4}]\).

Proof of Lemma 2.3. We separate \( q \) into two cases: \( q \equiv 1 \pmod{4} \) and \( q \equiv 3 \pmod{4} \). First, we suppose \( q \equiv 1 \pmod{4} \). From the properties of the Dirichlet character modulo \( q \), we can write
\[
4\chi(4) \sum_{a=1}^{q-1} a\chi(a) = \sum_{a=1}^{\frac{q-1}{2}} 4a\chi(4a) + \sum_{a=\frac{q+1}{2}}^{\frac{3q-2}{2}} 4a\chi(4a) + \sum_{a=\frac{3q+1}{2}}^{q-1} 4a\chi(4a)
\]
\[
= \sum_{a=1}^{\frac{q-1}{2}} 4a\chi(4a) + \sum_{a=1}^{\frac{q-1}{2}} (4a + q - 1)\chi(4a - 1)
\]
\[
+ \sum_{a=1}^{q-1} (4a + 2q - 2)\chi(4a - 2) + \sum_{a=1}^{\frac{q-1}{2}} (4a + 3q - 3)\chi(4a - 3)
\]
\[
= \sum_{a=1}^{q-1} a\chi(a) + \chi(4)q \sum_{a=1}^{\frac{q-1}{2}} \chi(a - \frac{q}{4})
\]
(2.1) \[ + 2\chi(4)q \sum_{a=1}^{\frac{q-1}{2}} \chi(a - 2 \cdot \frac{q}{4}) + 3\chi(4)q \sum_{a=1}^{\frac{q-1}{2}} \chi(a - 3 \cdot \frac{q}{4}). \]

Note that \( \frac{q}{4} \equiv \frac{3q+1}{4} \pmod{q} \) if \( q \equiv 1 \pmod{4} \). So from (2.1), we have
\[
4\chi(4) \sum_{a=1}^{q-1} a\chi(a) = \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q \sum_{a=\frac{q+1}{2}}^{\frac{3q-2}{2}} \chi(a)
\]
\[
- 2\chi(4)q \sum_{a=\frac{q+1}{2}}^{\frac{3q-2}{2}} \chi(a) - 3\chi(4)q \sum_{a=1}^{\frac{q-1}{2}} \chi(a)
\]
\[
= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q \sum_{a=\frac{q+1}{2}}^{\frac{3q-2}{2}} \chi(a) - 3\chi(4)q \sum_{a=1}^{\frac{q-1}{2}} \chi(a)
\]
(2.2) \[ = \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q \sum_{a=1}^{\frac{q-1}{2}} \chi(a) - 2\chi(4)q \sum_{a=1}^{\frac{q-1}{2}} \chi(a), \]
where we used the fact \( \chi(-1) = -1 \) and
\[
\sum_{a = \frac{q - 1}{2}}^{\frac{2q - 3}{2}} \chi(a) = - \sum_{a = \frac{3q - 3}{2}}^{\frac{2q - 3}{2}} \chi(a).
\]

Now, from (2.2) and Lemma 2.2, we get
\[
4\chi(4) \sum_{a=1}^{q-1} a \chi(a) = \sum_{a=1}^{q-1} a \chi(a) - (\chi(2) - 2\chi(4)) \sum_{a=1}^{q-1} a \chi(a) - 2\chi(4) \sum_{a=1}^{q} \chi(a).
\]
That is,
\[
\sum_{a=1}^{q-1} \chi(a) = \frac{\chi(4) - \chi(2) - 2}{2q} \sum_{a=1}^{q-1} a \chi(a) = \frac{\chi(4) - \chi(2) - 2}{2q} \sum_{a=1}^{q} a \chi(a).
\]
Then from Lemma 2.1, we have
\[
\sum_{a=1}^{q-1} \chi(a) = \frac{2 + \chi(2) - \chi(4)}{2i\pi} \tau(\chi) L(1, \chi).
\]
This proves Lemma 2.3 in the case of \( q \equiv 1 \) (mod 4). By the same method, we can also prove
\[
\sum_{a=1}^{q-1} \chi(a) = \frac{2 + \chi(2) - \chi(4)}{2i\pi} \tau(\chi) L(1, \chi)
\]
if \( q \equiv 3 \) (mod 4). Combining (2.3) and (2.4), we can immediately get
\[
\sum_{a=1}^{[\frac{q}{2}]} \chi(a) = \frac{2 + \chi(2) - \chi(4)}{2i\pi} \tau(\chi) L(1, \chi).
\]
This completes the proof of Lemma 2.3. \( \square \)

**Lemma 2.5.** Let \( f(x) \) be a polynomial of degree \( k \) with leading coefficient \( a_0 \), and define the difference operator \( \triangle \) by \( (\triangle f)(x) = f(x) - f(x - 1) \). Then we have
\[
\triangle^k f(x) = k!a_0, \quad \triangle^l f(x) = 0 \quad (l \geq k + 1).
\]

**Proof.** This can be easily deduced by the definition of the difference operator and mathematical induction. \( \square \)

**Lemma 2.6.** Let \( q \) be an integer with \( q > 2 \) and let \( \tau_k(n) \) denote the \( k \)-th divisor function (i.e., the number of solutions of the equation \( n_1 n_2 \cdots n_k = n \) in positive integers \( n_1, n_2, \cdots, n_k \)). Then for any complex variable \( s \) with \( \text{Re}(s) > 1 \), we have the identity
\[
\sum_{n=1}^{\infty} \tau_k^2(n) \zeta(2k-1)(s) \prod_{p|q} \left( 1 - \frac{1}{p^s} \right)^{2k-1} \prod_{p|q} A(0, k, p, s).
\]
Proof: This is Lemma 3 of [8]. But the result in [8] is not correct. From the Euler product formula and noting that (see formula 6.4.12 of [9])

\[ \tau_k(n) = \prod_{j=1}^{r} C_{k+\alpha_j - 1}^{\alpha_j} \]

if \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) is the factorization of \( n \) into prime powers, we have

\[ \sum_{n=1}^{\infty} \frac{\tau_k^2(n)}{n^s} = \prod_{p|q} S(k, p, s), \]

where

\[ S(k, p, s) = 1 + \frac{(C_k^1)^2}{p^s} + \frac{(C_k^2)^2}{p^{2s}} + \cdots + \frac{(C_k^{i-1})^2}{p^{is}} + \cdots. \]

Now we calculate \( S(k, p, s) \). For convenience, let \( f(i) = (C_k^{i})^2 \). From the definition of the difference operator \( \triangle \) in Lemma 2.5, we have

\[ S(k, p, s) \left(1 - \frac{1}{p^s}\right)^2 = 1 + \frac{\triangle f(1) - f(0)}{p^s} + \frac{\triangle^2 f(2)}{p^{2s}} + \cdots + \frac{\triangle^{2k} f(i)}{p^{is}} + \cdots. \]

So, from the binomial expansion theorem and comparing the coefficients of \( p^{is} \) for \( 1 \leq i \leq 2k - 2 \), we have

\[ S(k, p, s) \left(1 - \frac{1}{p^s}\right)^{2k-1} = S'(k, p, s) + \frac{\triangle^{2k-1} f(2k-1)}{p^{(2k-1)s}} + \frac{\triangle^{2k} f(2k)}{p^{2ks}} + \cdots; \]

here

\[ S'(k, p, s) = \sum_{i=0}^{2k-2} \frac{1}{p^s} \sum_{j=0}^{i} (-1)^j C_{2k-1}^j f(i-j). \]

Noting that \( f(i) \) is a polynomial in \( i \) of degree \( 2k - 2 \), so from Lemma 2.5 we know that \( \triangle^{2k-1} f(2k-1) = \triangle^{2k} f(2k) = \cdots = 0 \). This yields

\[ S(k, p, s) = \left(1 - \frac{1}{p^s}\right)^{-2k} \sum_{i=0}^{2k-2} \frac{1}{p^s} \sum_{j=0}^{i} (-1)^j C_{2k-1}^j (C_{k+i-j-1}^{i-j})^2 \]

\[ = \left(1 - \frac{1}{p^s}\right)^{-2k} A(0, k, p, s). \]

Hence, Lemma 2.6 follows immediately from the fact \( \zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}. \) \( \Box \)

Lemma 2.7. Let \( q \) be any odd integer with \( q > 2 \) and \( m \geq 0 \) a fixed integer. Then for any complex variable \( s \) with \( \text{Re}(s) > 1 \), we have the identity

\[ \sum_{n=1}^{\infty} \frac{\tau_k(2^m n)^2 \tau_k(n)}{n^s} = A(m, k, 2, s) \zeta^{2k-1}(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right)^{2k-1} \prod_{p|2q} A(0, k, p, s). \]

This lemma is a generalization of Lemma 2.6, which is just the case \( m = 0 \).
Proof. Noting that \( \tau_k(n) \) is a multiplicative function, we can write

\[
\sum_{n=1}^{\infty} \frac{\tau_k(2^m n) \tau_k(n)}{n^s} = \tau_k(2^m) \sum_{n=1}^{\infty} \frac{\tau_k^2(n)}{n^s} + \sum_{n=1}^{\infty} \frac{\tau_k(2^m n) \tau_k(n)}{n^s} \]

\[
= \tau_k(2^m) \sum_{n=1}^{\infty} \frac{\tau_k^2(n)}{n^s} + \sum_{r=1}^{\infty} \sum_{j=1}^{\infty} \frac{\tau_k(2^m + j) \tau_k(2^j) \tau_k^2(r)}{(r \cdot 2^j)^s} \]

\[
= \left( \sum_{j=0}^{\infty} \frac{\tau_k(2^m + j) \tau_k(2^j)}{2^{sj}} \right) \sum_{n=1}^{\infty} \frac{\tau_k^2(n)}{n^s}. \]

For \( n > 1 \), let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) be the factorization of \( n \) into prime powers. Then we have

\[
\sum_{n=1}^{\infty} \frac{\tau_k(2^m n) \tau_k(n)}{n^s} = \left( \sum_{j=0}^{\infty} \frac{C_{k+m+j-1}^j C_{k+j-1}^j}{2^{sj}} \right) \sum_{n=1}^{\infty} \frac{\tau_k^2(n)}{n^s}. \tag{2.5} \]

Let

\[
S = \sum_{j=0}^{\infty} \frac{C_{k+m+j-1}^j C_{k+j-1}^j}{2^{sj}}. \]

Now using the same method as calculating \( S(k, p, s) \) in Lemma 2.6, we can also get

\[
S = \left( 1 - \frac{1}{2^s} \right)^{1-2k} \sum_{j=0}^{2k-2} \frac{1}{2^{sj}} \sum_{j=0}^{i} (-1)^j C_{2k-1}^j C_{k+m+i-j-1}^j C_{k+i-j-1}^{k-j} \]

\[
= \left( 1 - \frac{1}{2^s} \right)^{1-2k} A(m, k, 2, s). \tag{2.6} \]

Noting that

\[
\sum_{n=1}^{\infty} \frac{\tau_k^2(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\tau_k^2(n)}{n^s} \]

if \( q \) is odd, we can easily get Lemma 2.7 from (2.5), (2.6) and Lemma 2.6. \( \square \)

Lemma 2.8. Let \( q \) be any odd integer with \( q > 2 \), \( \chi \) be the Dirichlet character modulo \( q \) and \( m \geq 1 \) be a fixed integer. Then we have the following asymptotic formulas:

\[
\sum_{\chi(-1)=-1} \chi(2^m) |L(1, \chi)|^{2k} = \frac{J(q)}{2m+1} \sum_{n=1}^{\infty} \frac{\tau_k(2^m n) \tau_k(n)}{n^2} + O(q^s) \]
and
\[ \sum_{\chi(-1) = -1}^* \bar{\chi}(2^m)|L(1, \bar{\chi})|^{2k} = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty} \frac{\tau_k(2^m n) \tau_k(n)}{n^2} + O(q^\epsilon). \]

**Proof.** By using the same method as in proving Lemma 5 of [6], we can get these formulas. \(\square\)

### 3. Proof of the theorem

In this section we will complete the proof of the theorem. It is well known that \(|\tau(\chi)| = \sqrt{q}\) if \(\chi\) is a primitive character. So from Lemma 2.3, we can write
\[
\sum_{\chi(-1) = -1}^* \left| \sum_{a < \frac{q}{4}} \chi(a) \right|^{2k} = \frac{q^k}{(2\pi)^{2k}} \sum_{\chi(-1) = -1}^* \left| 2 + \bar{\chi}(2) - \chi(4) \right|^{2k} |L(1, \bar{\chi})|^{2k}
\]
\[
= \frac{q^k}{(2\pi)^{2k}} \sum_{\chi(-1) = -1}^* \left[ 6 + \bar{\chi}(2) + \chi(2) - 2(\bar{\chi}(4) + \chi(4)) \right]|L(1, \bar{\chi})|^{2k}
\]
\[
= \frac{q^k}{(2\pi)^{2k}} \sum_{\chi(-1) = -1}^* \sum_{i=0}^{k} C_k^i \overline{\chi}(2)^i (\bar{\chi}(2) + \chi(2))^{i-1} \bar{\chi}(4)^k |L(1, \bar{\chi})|^{2k}
\]
\[
= \frac{q^k}{(2\pi)^{2k}} \sum_{\chi(-1) = -1}^* \sum_{i=0}^{k} \sum_{j=0}^{k} \sum_{s=0}^{k} \sum_{t=0}^{k} C_k^i C_k^j \bar{\chi}(2)^i \bar{\chi}(2)^j \chi(2)^{k-i-j} |L(1, \bar{\chi})|^{2k},
\]
where \(\chi^{-1}\) has the same meaning as \(\bar{\chi}\), the conjugate of the character \(\chi\). Then from Lemma 2.8 and Lemma 2.7, we easily get
\[
\sum_{\chi(-1) = -1}^* \left| \sum_{a < \frac{q}{4}} \chi(a) \right|^{2k} = C(k) q^k J(q) \zeta^{2k-1}(2) \prod_{p|q} \left( 1 - \frac{1}{p^2} \right)^{2k-1} \prod_{p|2q} A(0, k, p, 2) + O(q^{k+\epsilon}),
\]
where
\[
C(k) = \frac{1}{\pi^{2k}} \sum_{i=0}^{k} C_k^i (-2)^{k-i} \sum_{j=0}^{k} C_k^j \sum_{s=0}^{k-i-j} \sum_{t=0}^{k-i-j} A([3i + 4s - j - 2t - 2k], k, 2, 2)
\]
\[
A(m, k, p, s) = \sum_{i=0}^{2k-2} \sum_{j=0}^{i} (-1)^i C_k^i C_k^{m+i-j} C_k^{s+j} C_k^{t-j}.
\]
This completes the proof of the theorem.

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References


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