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A VERSION OF FABRY'S THEOREM FOR POWER SERIES WITH REGULARLY VARYING COEFFICIENTS

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ABSTRACT. For real power series whose non-zero coefficients satisfy $|a_m|^{1/m} \to 1$, we prove a stronger version of Fabry's theorem relating the frequency of sign changes in the coefficients and analytic continuation of the sum of the power series.

For a set Λ of non-negative integers, we consider the counting function

$$n(x, \Lambda) = \#\Lambda \cap [0, x].$$

We say that Λ is *measurable* if the limit

$$\lim_{x \to +\infty} n(x, \Lambda)/x$$

exists, and we call this limit the *density* of Λ.

Let $S = \{a_m\}$ be a sequence of real numbers. We say that a *sign change* occurs at the place m if there exists $k < m$ such that $a_m a_k < 0$ while $a_j = 0$ for $k < j < m$.

Theorem A. *Let Δ be a number in $[0, 1]$. The following two properties of a set Λ of positive integers are equivalent:*

(i) *Every power series*

$$f(z) = \sum_{m=0}^{\infty} a_m z^m \tag{1}$$

of radius of convergence 1*, with real coefficients and such that the changes of sign of $\{a_m\}$ occur only for $m \in \Lambda$, has a singularity on the arc*

$$I_\Delta = \{e^{i\theta} : |\theta| \leq \pi\Delta\}.$$

(ii) *For every $\Delta' > \Delta$ there exists a measurable set $\Lambda' \subset \mathbf{N}$ of density Δ' such that $\Lambda \subset \Lambda'$.*

Implication (ii) $\longrightarrow$ (i) is a consequence of Fabry's General Theorem [5, 3], as restated by Pólya. For the implication (i) $\longrightarrow$ (ii) see [9]. Fabry's General Theorem takes into account not only the sign changes of coefficients but also the absolute values of coefficients. It has a rather complicated statement, and the sufficient condition of the existence of a singularity given by this theorem is not the best

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possible. The best possible condition in Fabry's General Theorem is unknown; see, for example, the discussion in [4].

Alan Sokal (private communication) asked what happens if we assume that the power series (1) satisfies the additional regularity condition:

$$\lim_{m\in P, m\to\infty} |a_m|^{1/m} = 1, \tag{2}$$

where $P = \{m : a_m \neq 0\}$. This condition holds for most interesting generating functions. The answer is somewhat surprising:

Theorem 1. *Let Δ be a number in $[0,1]$. The following two properties of a set Λ of positive integers are equivalent:*

a) *Every power series* (1) *satisfying* (2)*, with real coefficients and such that the changes of sign of the coefficients a_m occur only for $m \in \Lambda$, has a singularity on the arc I_Δ.*

b) *All measurable subsets $\Lambda' \subset \Lambda$ have densities at most Δ.*

We recall that the *minimum density*

$$\underline{D}_2(\Lambda) = \lim_{r\to 0+} \liminf_{x\to+\infty} \frac{n((r+1)x,\Lambda) - n(x,\Lambda)}{rx}$$

can be alternatively defined as the sup of the limits

$$\lim_{x\to\infty} n(x,\Lambda')/x \tag{3}$$

over all measurable sets $\Lambda' \subset \Lambda$.

Similarly the *maximum density* of Λ is

$$\overline{D}_2(\Lambda) = \lim_{r\to 0+} \limsup_{x\to\infty} \frac{n((r+1)x,\Lambda) - n(x,\Lambda)}{rx},$$

and it equals the inf of the limits (3) over all measurable sequences of non-negative integers Λ' containing Λ.

For all these properties of minimum and maximum densities, see [12].

Thus condition (ii) is equivalent to $\overline{D}_2(\Lambda) \leq \Delta$, while condition b) is equivalent to $\underline{D}_2(\Lambda) \leq \Delta$.

Here is a gap version of Theorem 1:

Theorem 2. *The following two properties of a set Λ of positive integers are equivalent:*

A. Every power series

$$\sum_{m\in\Lambda} a_m z^m \tag{4}$$

satisfying (2) *has a singularity on I_Δ.*

A'. Every power series (4) *satisfying* (2) *has a singularity on every closed arc of length $2\pi\Delta$ of the unit circle.*

B. $\underline{D}_2(\Lambda) \leq \Delta$.

The equivalence between A and A' is immediate, as all assumptions of the statement A are invariant with respect to the change of the variable $z \mapsto \lambda z$, $|\lambda| = 1$.

Proof of Theorem 1. b) $\longrightarrow$ a).

Proving this by contradiction, we assume that $\underline{D}_2(\Lambda) \leq \Delta$, and that there exists a function f of the form (1) with the property (2) which has an analytic continuation to I_Δ, and such that the sign changes occur only for $m \in \Lambda$.

Without loss of generality we assume that $a_0 = 1$ and $\Delta < 1$.

Lemma 1. *For a function f as in* (1) *to have an immediate analytic continuation from the unit disc to the arc I_Δ it is necessary and sufficient that there exists an entire function F of exponential type with the properties*

$$a_m = (-1)^m F(m), \quad \textit{for all} \quad m \geq 0, \tag{5}$$

and

$$\limsup_{t\to\infty} \frac{\log|F(te^{i\theta})|}{t} \leq \pi b|\sin\theta|, \quad |\theta| < \alpha, \tag{6}$$

with some $b < 1 - \Delta$ and some $\alpha \in (0, \pi)$.

This result can be found in [1]; see also [2, 4].

Consider the sequence of subharmonic functions

$$u_m(z) = \frac{1}{m}\log|F(mz)|, \quad m = 1, 2, 3, \ldots. \tag{7}$$

This sequence is uniformly bounded from above on every compact subset of the plane, because F is of exponential type. Moreover, $u_m(0) = 0$ because of our assumption that $a_0 = F(0) = 1$. The Compactness Principle [8, Th. 4.1.9] implies that from every sequence of integers m one can choose a subsequence such that the limit $u = \lim u_m$ exists. This limit is a subharmonic function in the plane that satisfies, in view of (6),

$$u(re^{i\theta}) \leq \pi br|\sin\theta|, \quad |\theta| < \alpha, \tag{8}$$

with some b satisfying

$$0 < b < 1 - \Delta.$$

We use the following result of Pólya [11, footnote 18, p. 703]:

Lemma 2. *Let f be a power series* (1) *of radius of convergence* 1. *Let $\{a_{m_k}\}$ be a subsequence of coefficients with the property*

$$\lim_{k\to\infty} |a_{m_k}|^{1/m_k} = 1,$$

and assume that for some $r > 0$ the number of non-zero coefficients a_j on the interval $m_k \leq j \leq (1+r)m_k$ is $o(m_k r)$ as $k \to \infty$. Then f has no analytic continuation to any point of the unit circle.

Lemma 2 also follows from the results of [1] or [4].

Now we show that (2) implies the following:

Lemma 3. *Every limit function has the property $u(x) = 0$ for $x \geq 0$.*

Proof of Lemma 3. Let $U = \{x : x \geq 0, u(x) < 0\}$. This set is open because u is upper semi-continuous. Take any closed interval $J = [c, d] \subset U$. Then $u(x) \leq -\epsilon,\ x \in J$, with some $\epsilon > 0$. Let $\{m_k\}$ be the sequence of integers such that $u_{m_k} \to u$. Then from the definition of u_m we see that

$$\log|F(m_k x)| \leq -m_k\epsilon/2 \quad \text{for} \quad x \in J$$

and for all large k. Together with (5) and (2) this implies that $a_j = 0$ for all $j \in m_k J$. Let $a_{m'_k}$ be the last non-zero coefficient before cm_k. Applying Lemma 2 to the sequence $\{m'_k\}$ we conclude that f has no analytic continuation from the unit disc. This is a contradiction, which proves Lemma 3. □

Now we use the following general fact:

Grishin's Lemma. *Let $u \leq v$ be two subharmonic functions, and let μ and ν be their respective Riesz measures. Let E be a Borel set such that $u(z) = v(z) > -\infty$ for $z \in E$. Then the restrictions of the Riesz measures on E satisfy*

$$\mu|_E \leq \nu|_E.$$

The references are [13, 7, 6].

In view of Lemma 2, we can apply Grishin's Lemma to u and $v(z) = \pi b|\mathrm{Im}\, z|$ and $E = [0, \infty) \subset \mathbf{R}$. We obtain that the Riesz measure $d\mu$ of any limit function u of the sequence $\{u_k\}$ satisfies

$$d\mu|_{[0,\infty)} \leq b\, dx. \tag{9}$$

Now we go back to our coefficients and function F. By our assumption, the sign changes occur on a sequence Λ whose minimum density is at most Δ. Choose a number a such that $b < a < 1 - \Delta$. By the first definition of the minimum density, there exist $r > 0$ and a sequence $x_k \to \infty$ such that

$$n((1 + r)x_k, \Lambda) - n(x_k, \Lambda) \leq (1 - a)rx_k.$$

Lemma 4. *Let $(a_0, a_1, \ldots, a_N)$ be a sequence of real numbers, and let f be a real analytic function on the closed interval $[0, N]$, such that $f(n) = (-1)^n a_n$. Then the number of zeros of f on $[0, N]$, counting multiplicities, is at least N minus the number of sign changes of the sequence $\{a_n\}$.*

Proof. Consider first an interval (k, n) such that $a_k a_n \neq 0$ but $a_j = 0$ for $k < j < n$. We claim that f has at least

$$n - k - \#(\text{sign changes in the pair } (a_k, a_n))$$

zeros on the open interval (k, n). Indeed, the number of zeros of f on this interval is at least $n-k-1$ in any case. This proves the claim if there is a sign change in the pair (a_k, a_n). If there is no sign change, that is, if $a_n a_k > 0$, then $f(n)f(k) = (-1)^{n-k}$. So the number of zeros of f on the interval (n, k) is of the same parity as $n - k$. But f has at least $n - k - 1$ zeros on this interval; thus the total number of zeros is at least $n - k$. This proves our claim.

Now let a_k be the first and a_n the last non-zero term of our sequence. As the interval (k, n) is a disjoint union of the intervals to which the above claim applies, we conclude that the number of zeros of f on (k, n) is at least $(n - k)$ minus the number of sign changes of our sequence. On the rest of the interval $[0, N]$ our function has at least $N - n + k$ zeros, so the total number of zeros is at least N minus the number of sign changes. □

Let u be a limit function of the subsequence $\{u_{m_k}\}$ with $m_k = [x_k]$. By Lemma 4, the function F has at least $arx_k - 2$ zeros on each interval $[x_k, (1 + r)x_k]$, which implies that the Riesz measure μ of u satisfies

$$\mu([1, 1 + r]) \geq ar.$$

This contradicts (9) and thus proves the implication b) $\longrightarrow$ a).

Proof of Theorem 2, $B \longrightarrow A$. This is essentially the same as the previous proof. Proving by contradiction, we assume that B holds but that there exists a function f of the form (4) with the property (2) which has an analytic continuation to I_Δ. Applying Lemma 1, we obtain an entire function F with properties (5) and (6). Then we consider subharmonic functions u_m and the limit functions u of this sequence. Using Lemmas 2, 3 and Grishin's lemma, we obtain the inequality (9) for the Riesz measure μ of u, exactly as in the proof of Theorem 1.

Now we notice that condition B of Theorem 2 means that the entire function F has zeros at some sequence of integers of maximum density at least $1-\Delta$. Denoting by $n(x)$ the number of zeros of F on $[0,x]$ and choosing a number $a \in (b, 1-\Delta)$, we obtain that there exist $r > 0$ and a sequence of integers $m_k \to \infty$ such that

$$n((1+r)m_k) - n(m_k) \geq arm_k.$$

This implies that for the limit function u of the sequence u_{m_k}, the Riesz measure μ satisfies $\mu([1, 1+r]) \geq ar$, which contradicts (9). This contradiction proves implication $B \longrightarrow A$ in Theorem 2.

Proof of implications a) $\longrightarrow$ b) *of Theorem* 1 *and* $A \longrightarrow B$ *of Theorem* 2. Suppose that a set Λ of positive integers does not satisfy b), B. We will construct power series f of the form (4) with real coefficients which has an immediate analytic continuation from the unit disc to the arc I_Δ. This will simultaneously prove the implications a) $\longrightarrow$ b) of Theorem 1 and $A \longrightarrow B$ of Theorem 2, as the number of sign changes of any sequence does not exceed the number of its non-zero terms.

Let $\Lambda' \subset \Lambda$ be a measurable set of density $\Delta' > \Delta$. Let S be the complement of Λ' in the set of positive integers. Then S is also measurable and has density $1-\Delta'$.

Consider the infinite product

$$F(z) = \prod_{t \in S} \left(1 - \frac{z^2}{t^2}\right).$$

This is an entire function of exponential type with indicator $\pi(1-\Delta')|\sin\theta|$, and furthermore,

$$\log|F(z)| \geq \pi(1-\Delta')|\mathrm{Im}\, z| + o(|z|), \tag{10}$$

as $z \to \infty$ outside the set $\{z : \mathrm{dist}(z, S) \leq 1/4\}$. (See [10, Ch. II, Thm. 5] for this result.) Now we use the sufficiency part of Lemma 1 and define the coefficients of our power series by $a_m = (-1)^m F(m)$. Then we have all the needed properties; in particular (2) follows from (10). □

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References

[1] N. U. Arakelyan and V. A. Martirosyan, The location of singularities of power series on the boundary of the disk of convergence, Izvestia Akademii Nauk Armyanskoi SSR Ser. Mat. vol. 22, No. 1 (1987) 3-21, 102 (Russian); English translation: Soviet Journal of Contemporary Mathematical Analysis, 22 (1987) 1–19. MR898105 (88k:30003)

[2] N. Arakelyan, W. Luh and J. Müller, On the localization of singularities of lacunar power series, Complex Variables and Elliptic Equations, 52 (2007) 561–573. MR2340942 (2008f:30005)

[3] L. Bieberbach, Analytische Fortsetzung, Springer, Berlin, 1955. MR0068621 (16:913a)
[4] A. Eremenko, Densities in Fabry's theorem, preprint, arXiv:0709.2360.
[5] E. Fabry, Sur les séries de Taylor qui ont une infinité de points singuliers, Acta Math., 22 (1899) 65–87. MR1554901
[6] B. Fuglede, Some properties of the Riesz charge associated with a δ-subharmonic function. Potential Anal., 1 (1992) 355–371. MR1245891 (94m:31003)
[7] A. F. Grishin, Sets of regular growth of entire functions I, Teor. Funktsii Funktsional. Anal. i Prilozhen. No. 40 (1983) 36–47 (Russian). MR738442 (85j:30057)
[8] L. Hörmander, The analysis of linear partial differential operators, vol. I, Springer, Berlin, 1983. MR0717035 (85g:35002a)
[9] P. Koosis, The Logarithmic Integral, vol. II, Cambridge Univ. Press, Cambridge, 1992. MR1195788 (94i:30027)
[10] B. Ya. Levin, Distribution of zeros of entire functions, AMS, Providence, RI, 1980. MR589888 (81k:30011)
[11] G. Pólya, Über gewisse notwendige Determinantenkriterien fur die Fortsetzbarkeit einer Potenzreihe, Math. Ann., 99 (1928) 687–706. MR1512473
[12] G. Pólya, Untersuchungen über Lücken und Singularitäten von Potenzreihen, Math. Z., 29 (1929) 549–640. MR1545027
[13] C. de la Valle-Poussin, Potentiel et problème généralisé de Dirichlet, Math. Gazette, 22 (1938) 17–36.

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