SPECTRAL RADIUS ALGEBRAS AND $C_0$ CONTRACTIONS

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Abstract. We consider the spectral radius algebras associated to $C_0$ contractions. If $A$ is such an operator we show that the spectral radius algebra $B_A$ always properly contains the commutant of $A$.

Let $\mathcal{H}$ be a complex, separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T$ is an operator in $\mathcal{L}(\mathcal{H})$, then a subspace $M \subset \mathcal{H}$ is invariant for $T$ if $TM \subset M$, and it is hyperinvariant for $T$ if it is invariant for every operator in the commutant $\{T\}'$ of $T$. A nontrivial invariant subspace (n. i. s.) is one that is neither $\mathcal{H}$ nor the zero subspace. It was shown in [5] that one can associate the so-called spectral radius algebra $B_A$ to each operator $A$. Such an algebra always contains $\{A\}'$, so, when it has an n. i. s. it represents a generalization of the concept of a hyperinvariant subspace. This is the case when $A$ is compact (cf. [5]) or one type of normal operators (cf. [2]). Of course, it is important to establish that the inclusion $\{A\}' \subset B_A$ is proper. Although this has been done for some classes of operators (cf. [2], [3], [5], [6]), the question is, as of this writing, still open.

This paper can be regarded as a sequel to [4] where an extensive investigation of Jordan blocks and $C_0$ contractions (to be defined below) was conducted. A prominent role in this study was played by the so-called extended eigenvalues. (A complex number $\lambda$ is an extended eigenvalue of $A$ if there is a nonzero operator $X$ such that $AX = \lambda XA$.) As we will see, the presence of an eigenvalue or an extended eigenvalue is sufficient to guarantee that $B_A \neq \{A\}'$. Unfortunately, not every Jordan block $S(\theta)$ has either of these. Nevertheless, we will demonstrate that the inclusion under consideration is proper when $A$ belongs to the class $C_0$ (Theorems 3, 5, and 18). Our method utilizes the relationship between $S(\theta)$ and the shift $S$ as well as the quasisimilarity model for $C_0$ contractions.

We briefly review the relevant facts and notation. A contraction $A$ is completely nonunitary if there is no invariant subspace $M$ for $A$ such that $A|M$ is a unitary operator. A completely nonunitary contraction $A$ is said to be of class $C_0$ if there exists a nonzero function $h \in H^\infty$ such that $h(A) = 0$. The inner function $v$ such that $vH^\infty = \{u \in H^\infty : u(A) = 0\}$ is the minimal function of $A$ and is denoted by $m_A$. A very important subclass of $C_0$ contractions are the Jordan blocks. Throughout the paper we will use $S$ to denote the forward unilateral shift of multiplicity 1, and $\{e_n\}_{n=0}^\infty$ the orthonormal basis such that $Se_n = e_{n+1}$, $n \geq 0$. One knows that $S$ can be viewed as multiplication by $z$ on the Hardy space $H^2$.

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From this viewpoint, every invariant subspace of $S$ is of the form $\theta H^2$ for some inner function $\theta$. The compression of $S$ to $H^2 \ominus \theta H^2$ is called a Jordan block and denoted by $S(\theta)$. Also, if $\theta$ is an inner function, then there exists a Blaschke product $b$, a singular inner function $s$, and a constant $\gamma$, $|\gamma| = 1$, such that $\theta = \gamma bs$. We refer to this as the canonical factorization of $\theta$. Furthermore, there is a finite, positive, singular measure $\mu$ on the circle $T$ such that

$$s(z) = \exp \left( -\int_T \frac{\xi + z}{\xi - z} \, d\mu(\xi) \right).$$

For more information one may consult [1].

Given an operator $A \in \mathcal{L}(\mathcal{H})$ with spectral radius $r$ and $m \in \mathbb{N}$, we define $d_m = m/(1 + rm)$ and $R_m = (\sum_{n=0}^{\infty} d_m^n A^n)^{1/2}$. The spectral radius algebra $B_A$ consists of all operators $T \in \mathcal{L}(\mathcal{H})$ such that $\sup_m \|R_m T R_m^{-1}\| < \infty$. The following result from [5] summarizes some of the important properties of $B_A$ (cf. [5, Proposition 2.3, Corollary 2.4]).

**Proposition 1.** Let $A$ be an operator in $\mathcal{L}(\mathcal{H})$. Then $T \in B_A$ if and only if there exists $M > 0$ such that, for all $x \in \mathcal{H}$ and $m \in \mathbb{N}$, $\sum_{n \geq 0} d_m^n \|A^n x\|^2 \leq M \sum_{n \geq 0} d_m^n \|A^n x\|^2$. When $AT = TA$, $|\lambda| \leq 1$, and in particular if $AT = TA$, then $T \in B_A$.

Proposition 1 has a consequence whose verification is straightforward and we leave it to the reader.

**Corollary 2.** If $Au = \lambda u$ for some $|\lambda| < r(A)$, then, for any $v \in \mathcal{H}$, the rank one operator $u \otimes v$ belongs to $B_A$. Furthermore, if $A^* v \neq \lambda v$, then $u \otimes v$ does not commute with $A$. In particular, if $A$ has nontrivial kernel, then $B_A \neq \{A\}'$.

Now we can prove our first result about the inclusion $\{A\}' \subset B_A$.

**Theorem 3.** Let $A \in C_0$ and suppose that $m_A$ is neither a singular inner function nor a Blaschke product with all zeros of the same modulus. Then $B_A \neq \{A\}'$.

**Proof.** The assumption is that $m_A = \gamma bs$ and that $b(\alpha) = 0$ with $|\alpha| < r(A)$. This implies that $\alpha$ is an eigenvalue of $A$, whence the result follows from Corollary 2.

Proposition 1 shows that, in order to establish that an operator $T \in B_A$, it suffices to show that it satisfies $AT = \lambda TA$ for some $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$. The following result (cf. [5, Theorem 4.5]) provides a condition under which a $C_0$ contraction has an extended eigenvalue. Here we use the convention that, if $\mu$ is a measure and $\lambda \in \mathbb{T}$, then $\mu\lambda(E) = \mu(\lambda E)$.

**Theorem 4.** Let $A$ be a $C_0$ contraction and $|\lambda| = 1$. The equation $AX = \lambda XA$ has a solution $X \neq 0$ if and only if the measures $\mu$ and $\mu\lambda$ are not mutually singular or if $\alpha = \lambda \beta$ for some zeros $\alpha, \beta$ of $m_A$.

Theorem 3 allows us to consider the situation when $m_A$ is either a singular inner function or a Blaschke product with all zeros of the same modulus. Now we make a further reduction based on Theorem 4.

**Theorem 5.** Let $A \in C_0$ and suppose that $m_A$ is either a singular inner function such that the support of $\mu$ contains more than one point or that $m_A$ is a Blaschke product with at least 2 different zeros of the same modulus. Then $B_A \neq \{A\}'$. 
Let \( \theta \) be a singular inner function with the singular measure supported at one point. We will first explore both of these possibilities in the special case when \( A = S(\theta) \) and, of course, \( m_A = \theta \). In the former, we will need a fact established in [3].

**Theorem 6.** Let \( A \) be an operator acting on a finite dimensional space. Then \( B_A \neq \{A\}' \).

**Corollary 7.** If \( \theta \) is a Blaschke product with only one zero, then \( S(\theta) \) acts on a finite dimensional Hilbert space. Consequently, \( B_{S(\theta)} \neq \{S(\theta)\}' \)

The case when \( \theta \) is a singular inner function with the singular measure \( \mu \) supported at one point \( \lambda \) on the circle is much more complicated. One knows (cf. [1, p. 22]) that in this case

\[
\theta(z) = \gamma \exp \left\{ \frac{z + \lambda}{z - \lambda} p \right\},
\]

with \( |\gamma| = 1 \) and \( p = \mu(\{\lambda\}) \).

We will exploit the relationship between \( S(\theta) \) and the unilateral shift \( S \). In the rest of the paper, unless specifically noted, \( R_m \) will always mean \( R_m(S^*) \); i.e., it is associated to \( S^* \). We start with a computational result. We leave its verification to the reader.

**Lemma 8.** For all \( m \in \mathbb{N} \) and all \( i \geq 0 \), \( R_m e_i = \alpha_{m,i} e_i \), where \( \alpha_{m,i} \) are complex numbers satisfying \( |\alpha_{m,i}| \leq \sqrt{i+1}, m \in \mathbb{N}, i \geq 0 \).

Before we proceed, we notice that, as a consequence of Lemma 8, \( B_{S^*} \) is quite different from \( B_S = \mathcal{L}(\mathcal{H}) \).

**Theorem 9.** \( B_{S^*} \) is weakly dense in, but properly contained in, \( \mathcal{L}(\mathcal{H}) \).

**Proof.** Since \( S^* \) is not a multiple of an isometry, the fact that \( B_{S^*} \neq \mathcal{L}(\mathcal{H}) \) follows from [2] Theorem 2.7. In order to establish that \( B_{S^*} \) is weakly dense in \( \mathcal{L}(\mathcal{H}) \) it suffices to show that, for any \( i, j \geq 0 \), the rank one operator \( e_i \otimes e_j \) belongs to \( B_{S^*} \). Since \( \|R_m e_i\| \leq \sqrt{i+1} \) and \( R_m^{-1} \) is a contraction, the result follows easily.

**Corollary 10.** If \( f \in H^2 \) and \( f' \in H^2 \), then \( \sup_m \|R_m f\| < \infty \).

Our next goal is to prove that the function \( f \) in Corollary 10 can be found in a specific subspace. Let \( J \) be the antiderivative operator on \( H^2 \), i.e., an operator such that, for all \( f \in H^2 \), \( (Jf)' = f \) and \( (Jf)(0) = 0 \). We will show that there is a function \( g \in H^2 \) such that \( f = J^* g \in H^2 \cap \theta H^2 \). Clearly, \( J^* g \perp \theta H^2 \) if and only if \( g \perp J(\theta H^2) \), so it suffices to establish that \( J(\theta H^2) \) is a subspace of \( H^2 \) whose codimension is infinite.
Theorem 11. If \( \theta \) is a singular inner function such that the associated singular measure is supported at one point, and if \( J \) is the antiderivative operator as above, then the codimension of \( J(\theta H^2) \) in \( H^2 \) is infinite.

Proof. Suppose that \( \theta \) is as in (1), and that the codimension of \( J(\theta H^2) \) is finite. Let \( p_0 = 1 \), and \( p_n = z^{n-1}(2z - 2\lambda - 2p\lambda) + (n - 1)z^{n-2}(z - \lambda)^2 \) for \( n \geq 1 \). It is easy to see that the span of polynomials \( \{p_n\}_{n=0}^\infty \) is dense in \( H^2 \). Therefore, the span of \( \{J(\theta p_n)\}_{n=0}^\infty \) has finite codimension.

On the other hand, a calculation shows that \(-2\lambda^p\theta = (z - \lambda)^2\theta'\), so for all \( f \in H^2 \), \(-2\lambda^p J(\theta f) = J((z - \lambda)^2\theta f) = (z - \lambda)^2\theta f - J[(z - \lambda)^2\theta f' + 2(z - \lambda)\theta f] \). It follows that

\[
J[(2z - 2\lambda - 2p\lambda)\theta f + (z - \lambda)^2\theta f'] = (z - \lambda)^2\theta f.
\]

By choosing \( f = z^{n-1} \), \( n \in \mathbb{N} \), we obtain that \( J(\theta p_n) = (z - \lambda)^2z^{-n+1}\theta \) for all \( n \in \mathbb{N} \). Consequently, \( J(\theta p_n) \in \theta H^2 \) and the span of \( \{J(\theta p_n)\}_{n=1}^\infty \) is a subspace of \( \theta H^2 \). Since the latter has infinite codimension the result follows. □

From this theorem we obtain an important consequence.

Corollary 12. There exists a nonzero function \( f \in H^2 \ominus \theta H^2 \) such that \( f' \in H^2 \) and, consequently, such that \( \sup_m \|R_m f\| < \infty \).

Proof. By Theorem 11 there exists a nonconstant function \( g \in H^2 \ominus J(\theta H^2) \). Let \( f = J^*g \). It is easy to see that \( f \in H^2 \ominus \theta H^2 \) and \( f \neq 0 \). Furthermore, if \( g = \sum g_n z^n \), a straightforward calculation shows that \((J^*g)' = \sum_{n \geq 0} (n + 1)/(n + 2)g_{n+2} z^n \). Thus, \( f' \in H^2 \) and an application of Corollary 10 completes the proof. □

The significance of the membership of \( f \) (in Corollary 12) in \( H^2 \ominus \theta H^2 \) lies in the fact that we can now deduce an analogous result for \( R_m(S(\theta)^*) \).

Corollary 13. Suppose that \( \theta \) is a singular inner function as in (1). There exists a nonzero function \( u \in H^2 \) such that \( \sup_m \|R_m(S(\theta)^*u)\| < \infty \).

Proof. Notice that both \( S(\theta)^* \) and \( S^* \) have the spectral radius 1, so \( d_m(S(\theta)^*) = d_m(S^*) = m/(m + 1) \). Therefore, relative to the decomposition \( H^2 = \theta H^2 \oplus (H^2 \ominus \theta H^2) \), taking into account that \( H^2 \ominus \theta H^2 \) is invariant for \( S^* \), \( R_m^2(S^*) = \left( \begin{array}{cc} * & * \\ * & R_m^2(S(\theta)^*) \end{array} \right) \).

Let \( f \) be the function provided by Corollary 12. Clearly, we can write \( f = 0 + u \) relative to the same decomposition, and it is easy to see that \( \|R_m(S(\theta)^*u)\| = \|R_m(S^*)f\| \). □

Corollary 13 describes a property of \( S(\theta)^* \). Since we are more interested in \( S(\theta) \) it is useful to recall [1 Corollary 3.1.7]. We use the notation \( \bar{\theta}(z) = \overline{\theta(z)} \).

Theorem 14. For every inner function \( \theta \) the adjoint \( S(\theta)^* \) is unitarily equivalent to \( S(\bar{\theta}) \).

Theorem 14 allows us to move the focus of our investigation from the inclusion \( \{A\}' \subset B_A \) to \( \{B\}' \subset B_B \), where \( B \) is unitarily equivalent to \( A \). There is no loss of generality in doing so since the unitary equivalence between \( A \) and \( B \) gives rise to an algebra isomorphism \( \phi \) such that \( \phi(\{A\}') = \{B\}' \) and \( \phi(B_A) = B_B \) (cf. [2 Theorem 2.4]). Finally, we can prove our main result concerning the Jordan blocks.
Theorem 15. $B_{S(\theta)} \neq \{S(\theta)\}'$

Proof. Combining Theorem 3 Theorem 5 and Corollary 7 we see that the only case to consider is when $\theta$ is given by (1). By Theorem 14 it suffices to show that $B_{S(\theta)} \neq \{S(\theta)\}'$. If $u$ is the function supplied by Corollary 13 then $u \otimes v \in B(S(\theta)')$ for any $v \in H^2$. However, $u \otimes v$ commutes with $S(\theta)'$ if and only if $v$ is an eigenvector for $S(\theta)$. □

Next, we return to contraction operators of class $C_0$. In order to obtain a better insight into their structure we rely on the following result (cf. [11 Theorem 3.5.1]).

Theorem 16. Let $A$ be a $C_0$ contraction. Then $A$ is quasisimilar to an infinite direct sum of Jordan blocks $\bigoplus_i S(\theta_i)$.

Since Theorem 16 relates quasisimilar operators we need to establish the relationship between their respective spectral radius algebras as well as between their commutants.

Lemma 17. Suppose that $A$ and $B$ are quasisimilar $C_0$ contractions and let $Y, Z$ be quasi-affinities such that $AY = YB$ and $ZA = BZ$. If $T \in B_B$, then $YTZ \in B_A$. Also $T \in \{B\}'$ if and only if $YTZ \in \{A\}'$.

Proof. $A$ and $B$ have essentially the same quasisimilarity model, so they share the same spectral radius. In particular, $d_m(A) = d_m(B)$ and we will denote both by $d_m$. A calculation shows that $A^n Y TZ = Y B^n T Z$, so, for all $x \in H$, $\sum d^2_m \| A^n Y TZ x \|^2 \leq \| Y \|^2 \sum d^2_m \| B^n T Z x \|^2$. Now, if $T \in B_B$, Proposition 11 shows that there exists $M > 0$ such that, for all $x \in H$, the last expression is dominated by $M^2 \| Y \|^2 \sum d^2_m \| B^n Z x \|^2 = M^2 \| Y \|^2 \sum d^2_m \| Z A^n x \|^2 \leq M^2 \| Y \|^2 \sum d^2_m \| A^n x \|^2$. Consequently, $YTZ \in B_A$. The other assertion is even easier: $AYTZ = YBTZ = YTBZ = YTZA$.

Now we can make the final step in our analysis.

Theorem 18. Let $A$ be a $C_0$ contraction and suppose that $m_A$ is either a Blaschke product with only one zero or a singular inner function such that the support of $\mu$ consists of a single point $\lambda \in T$. Then $B_A \neq \{A\}'$.

Proof. By Theorem 16 $A$ is quasisimilar to a direct sum $S(\Theta) = \bigoplus_i S(\theta_i)$, where each inner function $\theta_i$ is either a Blaschke product with only one zero or a singular inner function of the form (1) for some $p > 0$. Therefore (cf. [11 Theorem 2.4.11]), $r(A) = r(S(\Theta)) = r(S(\theta_1))$. By Theorem 15 there is $X \in B_{S(\theta_1)} \setminus \{S(\theta_1)\}'$ and it is easy to see that the operator $X \oplus 0 \oplus 0 \oplus \cdots \in B_{S(\Theta)} \setminus \{S(\Theta)\}'$. Now the result follows from Lemma 17. □

References


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