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# UPPER BOUNDS FOR FINITE ADDITIVE 2-BASES

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ABSTRACT. For a positive integer $N$, a set $\mathcal{A} \subset [0, N] \cap \mathbb{Z}$ is called a 2-basis for $N$ if every integer $n \in [0, N]$ can be represented as $n = a + b$, where $a, b \in \mathcal{A}$. In this paper, we give a lower bound estimate for the cardinality of an additive 2-basis for $N$, as $N \to \infty$, which improves the existing results on this topic.

## 1. INTRODUCTION

For a positive integer $N$, an *additive* 2-*basis* (or simply a 2-*basis*) of size $k$ for $N$ is a set $\{0 = a_1 < a_2 < \cdots < a_k\}$ of integers such that every positive integer up to $N$ can be represented as a sum of two elements of the set. A 2-basis $\mathcal{A}$ for $N$ is called *restricted* if $\mathcal{A} \subset [0, N/2]$.

For a positive integer $N$, we denote by $k = k(N)$ (resp. $k_r = k_r(N)$) the smallest number of elements that can form a 2-basis (resp. a restricted 2-basis) for $N$. In this paper, we are interested in a lower bound estimation for $k$ and $k_r$, as $N \to \infty$. For this purpose, let

$$\sigma := \lim \sup_{N\to\infty} \frac{N}{(k(N))^2}, \qquad \sigma_r := \lim \sup_{N\to\infty} \frac{N}{(k_r(N))^2}.$$

Rohrbach [9] conjectured that $\sigma = \frac{1}{4}$, but this was disproved by many authors based on various constructions of thin 2-bases (cf. [2], [7]). In particular, Mrose [7] constructed a 2-basis $\mathcal{A} \subset [0, N]$ for every large $N$ with

$$\frac{N}{|\mathcal{A}|^2} \geq \frac{2}{7} = 0.285714\ldots.$$

The exact value of $\sigma$ is still a mystery. Even a heuristic argument is still to be found in the literature that would suggest what the true value of $\sigma$ should be. Nevertheless, besides various constructions of thin 2-bases which yield lower bounds, there have been a number of results giving upper bounds for $\sigma$. For any 2-basis $\mathcal{A}$ for a positive integer $N$, a simple counting argument implies that

$$N + 1 \leq \binom{|\mathcal{A}| + 1}{2},$$

which yields $\sigma \leq \frac{1}{2}$. This trivial bound for $\sigma$ has been improved as follows:

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$$\sigma \le 0.4992 \qquad \text{(Rohrbach [9])}$$
$$\sigma \le 0.4903 \qquad \text{(Moser [5])}$$
$$\sigma \le 0.4867 \qquad \text{(Riddell [8])}$$
$$\sigma \le 0.4847 \qquad \text{(Moser, Pounder and Riddell [6])}$$
$$\sigma \le 0.4802 \qquad \text{(Klotz [4])}$$
$$\sigma \le 0.4789 \qquad \text{(Güntürk and Nathanson [1])}$$

Among all these results, Rohrbach [9] attained his bound with a combinatorial argument, and all others used the Fourier series. In particular, Klotz [4] also appealed to Rohrbach's combinatorial method; Güntürk and Nathanson introduced the Fourier series for functions of two variables into the problem to obtain their improvement.

The restricted case has also been studied by several authors. Rohrbach [9] proved that $\sigma_r \le 0.4654$. This was later improved by Riddell [8] to 0.43356, and subsequently by Moser, Pounder and Riddell [6] to 0.42435.

For a large integer $N$ and a 2-basis $\mathcal{A}$ for $N$, from the definition we know that $\mathcal{A}+\mathcal{A}$ covers all integers in the interval $[0, N]$. This determines asymptotically $2N$ elements of $\mathcal{A}+\mathcal{A}$, yet gives a trivial estimate $|\mathcal{A}| \ge (\sqrt{2}+o(1))\sqrt{N}$. Any better estimate for $|\mathcal{A}|$, no matter what method is used, essentially depends on the study of the irregular distribution of the elements of the sumset $\mathcal{A}+\mathcal{A}$ over the interval $[0, 2N]$ (or $[0, N]$ if $\mathcal{A}$ is restricted). Such irregularity is often captured by studying the values of the exponential sums over the set. Inspired by an idea involved in the author's recent work [10] on upper bounds for generalized Sidon sets, we shall study the distribution of the sumset $\mathcal{A}+\mathcal{A}$ over some subintervals of $[0, 2N]$ and derive from that a better upper bound for $\sigma$.

**Theorem 1.1.** *We have*

$$\sigma \le 0.46972.$$

With a similar method, we also get an improvement for $\sigma_r$.

**Theorem 1.2.** *We have*

$$\sigma_r \le \frac{7+\sqrt{5}}{22} = 0.41982\ldots.$$

*Notation.* Throughout this paper, $A \gtrsim B$ (resp. $A \lesssim B$) means that $A \ge (1+o(1))B$ (resp. $A \le (1+o(1))B$) as $N \to \infty$.

## 2. Preliminaries

For a finite set $\mathcal{B} \subset \mathbb{Z}$, we denote by $f_{\mathcal{B}}(\beta)$ the generating function of $\mathcal{B}$ as

$$f_{\mathcal{B}}(\beta) = \sum_{b\in\mathcal{B}} e(\beta b), \quad \text{where} \quad e(t) = \exp(2\pi i t).$$

We also define, for any $n \in \mathbb{Z}$, that

$$r_{\mathcal{B}}(n) := \#\{(a,b) \in \mathcal{B}\times\mathcal{B} : a+b = n\}$$

and

$$d_{\mathcal{B}}(n) := \#\{(a,b) \in \mathcal{B}\times\mathcal{B} : a-b = n\}.$$

The following lemma (Lemma 2.1) is fundamental in our proofs of Theorems 1.1 and 1.2. It is a more general form of Lemma 2 in [10]. We believe that such a result is of independent interest and will be useful in other places. Thus we give the lemma in a more general setting even though the proofs of the theorems will only need it with a very special weight function.

Let $u(x)$ be a non-negative function supported on $[0,1]$, with piecewise continuous derivative and bounded total variation, and

$$\int_0^1 u(t)dt = 1.$$

Let $W(x)$ be the even function on $[-1,1]$ defined by

$$W(x) := \int_0^{1-|x|} u(t)u(t+|x|)dt.$$

For given real numbers $p$, $\delta$ with $p > 2\delta > 0$, let $w_{p,\delta}(x)$ be the periodic function on $(-\infty, +\infty)$ which, on a period $[-p/2, p/2]$, is defined by

$$w_{p,\delta}(x) = \begin{cases} W(x/\delta) & \text{if } |x| \le \delta, \\ 0 & \text{if } \delta < |x| \le \frac{1}{2}p. \end{cases} \tag{2.1}$$

**Lemma 2.1.** *Suppose $\mathcal{B} \subset [0,N] \cap \mathbb{Z}$, and $w_{p,\delta}(x)$ is given by (2.1). Let*

$$D_{p,\delta}(\mathcal{B}) = \sum_{m=-N}^{N} w_{p,\delta}(m/N) d_{\mathcal{B}}(m)$$

*and*

$$R_{p,\delta,\kappa}(\mathcal{B}) = \sum_{m=0}^{2N} w_{p,\delta}(m/N + \kappa) r_{\mathcal{B}}(m).$$

*Then for any fixed real number $\kappa$, we have*

$$D_{p,\delta}(\mathcal{B}) + R_{p,\delta,\kappa}(\mathcal{B}) \ge \frac{2\delta}{p}|\mathcal{B}|^2 \tag{2.2}$$

*and*

$$D_{p,\delta}(\mathcal{B}) \ge R_{p,\delta,\kappa}(\mathcal{B}). \tag{2.3}$$

*Proof.* We first note that $w_{p,\delta}(x)$ has a formal Fourier expansion into the cosine series

$$w_{p,\delta}(x) = \frac{a_{p,\delta}(0)}{2} + \sum_{n=1}^{\infty} a_{p,\delta}(n) \cos(2n\pi x/p), \tag{2.4}$$

where, for the integer $n \ge 0$,

$$a_{p,\delta}(n) = \frac{2}{p} \int_{-\frac{p}{2}}^{\frac{p}{2}} w_{p,\delta}(x) \cos(2n\pi x/p) dx.$$

By a straightforward calculation, we get

$$a_{p,\delta}(n) = \frac{2\delta}{p} \left\{ \left( \int_0^1 u(t) \cos(2n\pi\delta t/p) dt \right)^2 + \left( \int_0^1 u(t) \sin(2n\pi\delta t/p) dt \right)^2 \right\},$$

from which we see that

$$\frac{a_{p,\delta}(0)}{2} = \frac{\delta}{p}, \quad a_{p,\delta}(n) \ge 0 \quad \text{for} \quad n \ge 1 \tag{2.5}$$

and that the Fourier series (2.4) converges uniformly to $w_{p,\delta}(x)$. From (2.4) and (2.5), we thus have

$$\begin{aligned} D_{p,\delta}(\mathcal{B}) &= \sum_{m=-N}^{N} d_{\mathcal{B}}(m)\Big(\frac{\delta}{p}+\sum_{n=1}^{\infty} a_{p,\delta}(n)\cos\big(\frac{2nm\pi}{pN}\big)\Big) \\ &= \frac{\delta}{p}\sum_{m=-N}^{N} d_{\mathcal{B}}(m)+\sum_{n=1}^{\infty} a_{p,\delta}(n)\sum_{m=-N}^{N} d_{\mathcal{B}}(m)\cos\big(\frac{2nm\pi}{pN}\big) \\ &= \frac{\delta}{p}|\mathcal{B}|^2+\sum_{n=1}^{\infty} a_{p,\delta}(n)\Big|f_{\mathcal{B}}\big(\frac{n}{pN}\big)\Big|^2 \qquad (2.6) \end{aligned}$$

and

$$\begin{aligned} R_{p,\delta,\kappa}(\mathcal{B}) &= \sum_{m=0}^{2N} r_{\mathcal{B}}(m)\Big(\frac{\delta}{p}+\sum_{n=1}^{\infty} a_{p,\delta}(n)\cos\big(\frac{2n(m+\kappa N)\pi}{pN}\big)\Big) \\ &= \frac{\delta}{p}\sum_{m=0}^{2N} r_{\mathcal{B}}(m)+\sum_{n=1}^{\infty} a_{p,\delta}(n)\sum_{m=0}^{2N} r_{\mathcal{B}}(m)\cos\big(\frac{2n(m+\kappa N)\pi}{pN}\big) \\ &= \frac{\delta}{p}|\mathcal{B}|^2+\sum_{n=1}^{\infty} a_{p,\delta}(n)\Re\Big({f_{\mathcal{B}}}^2\big(\frac{n}{pN}\big)e\big(\frac{\kappa n}{p}\big)\Big). \qquad (2.7) \end{aligned}$$

From (2.6), (2.7), we thus have

(2.8)
$$D_{p,\delta}(\mathcal{B})+R_{p,\delta,\kappa}(\mathcal{B})=\frac{2\delta}{p}|\mathcal{B}|^2+\sum_{n=1}^{\infty} a_{p,\delta}(n)\left(\Big|f_{\mathcal{B}}\big(\frac{n}{pN}\big)\Big|^2+\Re\Big({f_{\mathcal{B}}}^2\big(\frac{n}{pN}\big)e\big(\frac{\kappa n}{p}\big)\Big)\right).$$

Note that, for each $n\ge 1$, we have $a_{p,\delta}(n)\ge 0$ and

$$\Big|f_{\mathcal{B}}\big(\frac{n}{pN}\big)\Big|^2+\Re\Big({f_{\mathcal{B}}}^2\big(\frac{n}{pN}\big)e\big(\frac{\kappa n}{p}\big)\Big)\ge 0;$$

(2.2) then follows. It is also clear that (2.3) holds for any $\kappa$ from (2.6), (2.7), and the fact that $a_{p,\delta}(n)\ge 0$ for all positive integers $n$. □

**Lemma 2.2.** *Suppose $\mathcal{A}$ is a 2-basis for a sufficiently large integer $N$. Then for any integer $n$ not divisible by $N$, we have*

$$\Big|f_{\mathcal{A}}(n/N)\Big| \lesssim \sqrt{|\mathcal{A}|^2-2N}. \qquad (2.9)$$

*Proof.* When $N\nmid n$, we have

$$\sum_{m=0}^{N-1} e(nm/N)=0.$$

From this we get

$${f_{\mathcal{A}}}^2(n/N)=\sum_{m=0}^{2N} r_{\mathcal{A}}(m)e(mn/N)=\sum_{m=0}^{2N} {r_{\mathcal{A}}}^*(m)e(mn/N),$$

where

$${r_{\mathcal{A}}}^*(m)=\begin{cases} r_{\mathcal{A}}(m)-2 & \text{if } \ 0\le m\le N-1,\\ r_{\mathcal{A}}(m) & \text{otherwise.}\end{cases}$$

Since $\mathcal{A}$ is a 2-basis for $N$, we have $r_{\mathcal{A}}{}^*(m) \geq 0$ for all but at most $O(|\mathcal{A}|)$ integers $m \in [0, N-1]$. Thus

$$\left|f_{\mathcal{A}}^2(n/N)\right|^2 \leq \sum_{m=0}^{2N} r_{\mathcal{A}}{}^*(m) + O(|\mathcal{A}|) \leq |\mathcal{A}|^2 - 2N + O(|\mathcal{A}|).$$

The lemma then follows from the fact that $|\mathcal{A}|^2 > (2+\varepsilon)N$ for some $\epsilon > 0$ as $N \to \infty$. □

In our application of Lemma 2.1 to the proofs of Theorems 1.1 and 1.2, we take $u(t) = 1$ on $[0,1]$. Thus

$$w_{p,\delta}(x) = \begin{cases} 1 - |x|/\delta & \text{if } |x| \leq \delta, \\ 0 & \text{if } \delta < |x| \leq \frac{1}{2}p. \end{cases} \tag{2.10}$$

In the next lemma and henceforth, we shall suppose $w_{p,\delta}(x)$ given by (2.10). We note that in the cosine series (2.4) we then have the following, which is more precise than (2.5):

(2.11)

$$a_{p,\delta}(n) = \frac{p}{\delta(n\pi)^2}(1 - \cos(2n\pi\delta/p)) \geq 0 \quad \text{for} \quad n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} a_{p,\delta}(n) = 1 - \frac{\delta}{p}.$$

**Lemma 2.3.** *Suppose* $0 < \delta \leq \frac{1}{2}$. *Suppose* $\mathcal{A}$ *is a* 2*-basis for a sufficiently large integer* $N$*, and* $D_{p,\delta}(\mathcal{A})$ *is as defined in Lemma* 2.1*, with* $w_{p,\delta}(x)$ *given by* (2.10)*. If* $p \geq 1$*, then we have*

$$D_{p,\delta}(\mathcal{A}) \leq D_{1,\delta}(\mathcal{A}) \lesssim |\mathcal{A}|^2 - 2(1-\delta)N. \tag{2.12}$$

*Furthermore, if* $\mathcal{A}$ *is a restricted* 2*-basis for* $N$*, and* $p \geq \frac{1}{2} + \delta$*, then* (2.12) *holds as well.*

*Proof.* Note that if $p \geq 1$, then

$$w_{p,\delta}(x) \leq w_{1,\delta}(x) \quad \text{for} \quad x \in [-1,1],$$

which yields

$$D_{p,\delta}(\mathcal{A}) \leq D_{1,\delta}(\mathcal{A}) \quad \text{if} \quad p \geq 1. \tag{2.13}$$

Now from (2.6), (2.11), and Lemma 2.2, we have

$$\begin{aligned} D_{1,\delta}(\mathcal{A}) &= \delta|\mathcal{A}|^2 + \sum_{\substack{n=1 \\ N \nmid n}}^{\infty} a_{1,\delta}(n) \left| f_{\mathcal{A}}\left(\frac{n}{N}\right)\right|^2 + \sum_{n=1}^{\infty} a_{1,\delta}(nN)|\mathcal{A}|^2 \\ &\lesssim \delta|\mathcal{A}|^2 + (1-\delta)(|\mathcal{A}|^2 - 2N) + O\left(\frac{|\mathcal{A}|^2}{N^2\delta}\right) \\ &\lesssim_\delta |\mathcal{A}|^2 - 2(1-\delta)N, \end{aligned} \tag{2.14}$$

which, along with (2.13), proves (2.12).

If $\mathcal{A}$ is a restricted 2-basis for $N$, and $p \geq \frac{1}{2} + \delta$, then we have $D_{p,\delta}(\mathcal{A}) = D_{1,\delta}(\mathcal{A})$ since in both sums $m \in [-N/2, N/2]$ and $w_{p,\delta}(x) = w_{1,\delta}(x)$ for $x \in [-1/2, 1/2]$. □

## 3. Proof of Theorem 1.1

In this section, we shall use the lemmas to prove Theorem 1.1. Suppose $\mathcal{A}$ is a 2-basis for a sufficiently large integer $N$. Without loss of generality, we shall assume $|\mathcal{A}| \asymp \sqrt{N}$. Let $\delta, \varepsilon$ be real numbers satisfying $0 < \varepsilon < \delta < \frac{1}{2}$, the actual values to be determined later, and $w_{p,\delta}(x)$ be given by (2.10). From Lemma 2.1, we have

$$D_{1+2(\delta-\varepsilon),\delta}(\mathcal{A}) + R_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) \geq \frac{2\delta}{1+2(\delta-\varepsilon)}|\mathcal{A}|^2$$

and

$$D_{1+3\delta-\varepsilon,\delta}(\mathcal{A}) + R_{1+3\delta-\varepsilon,\delta,\delta}(\mathcal{A}) \geq \frac{2\delta}{1+3\delta-\varepsilon}|\mathcal{A}|^2.$$

Combining these with Lemma 2.3, we see that $R_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) + R_{1+3\delta-\varepsilon,\delta,\delta}(\mathcal{A})$ is

$$\begin{aligned} &\gtrsim \frac{2\delta}{1+2(\delta-\varepsilon)}|\mathcal{A}|^2 + \frac{2\delta}{1+3\delta-\varepsilon}|\mathcal{A}|^2 - 2D_{1,\delta}(\mathcal{A}) \\ &\gtrsim \left(\frac{2\delta}{1+2(\delta-\varepsilon)} + \frac{2\delta}{1+3\delta-\varepsilon} - 2\right)|\mathcal{A}|^2 + 4(1-\delta)N. \end{aligned} \tag{3.1}$$

Now we notice that

$$R_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) = R^{(1)}_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) + R^{(2)}_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}),$$

where

$$R^{(1)}_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) = \sum_{m \leq \varepsilon N} \frac{\varepsilon}{\delta}\left(1 - \frac{m}{\varepsilon N}\right) r_{\mathcal{A}}(m) + \sum_{(1-\varepsilon)N < m \leq N} \frac{\varepsilon}{\delta}\left(1 - \frac{N-m}{\varepsilon N}\right) r_{\mathcal{A}}(m)$$

and

$$R^{(2)}_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) = \sum_{N < m \leq (1+2\delta-\varepsilon)N} \left(1 - \frac{|m-(1+\delta-\varepsilon)N|}{\delta N}\right) r_{\mathcal{A}}(m).$$

From these and the fact that

$$R_{1+3\delta-\varepsilon,\delta,\delta}(\mathcal{A}) = \sum_{(1+\delta-\varepsilon)N < m \leq (1+3\delta-\varepsilon)N} \left(1 - \frac{|m-(1+2\delta-\varepsilon)N|}{N}\right) r_{\mathcal{A}}(m),$$

we have

$$R_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) + R_{1+3\delta-\varepsilon,\delta,\delta}(\mathcal{A}) \leq R^{(1)}_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) + \sum_{N < m \leq 2N} r_{\mathcal{A}}(m). \tag{3.2}$$

Since $\mathcal{A}$ is a 2-basis for $N$ and since the coefficient of $r_{\mathcal{A}}(m)$ for each $m$ in $R^{(1)}_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A})$ is less than 1, we see that, apart from an error term of at most $O(|\mathcal{A}|)$,

$$\begin{aligned} &\sum_{m=0}^{N} r_{\mathcal{A}}(m) - R^{(1)}_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) \\ &\quad \gtrsim 2N - 2\left(\sum_{m \leq \varepsilon N} \frac{\varepsilon}{\delta}\left(1 - \frac{m}{\varepsilon N}\right) + \sum_{(1-\varepsilon)N < m \leq N} \frac{\varepsilon}{\delta}\left(1 - \frac{N-m}{\varepsilon N}\right)\right) \\ &\quad \gtrsim \left(2 - \frac{2\varepsilon^2}{\delta}\right)N. \end{aligned} \tag{3.3}$$

Now, from (3.1), (3.2), and (3.3), we get

$$\begin{aligned}|\mathcal{A}|^2 &= \sum_{m=0}^{N} r_{\mathcal{A}}(m) + \sum_{m=N+1}^{2N} r_{\mathcal{A}}(m)\\ &\gtrsim R^{(1)}_{1+2(\delta-\varepsilon),\delta,\delta-\varepsilon}(\mathcal{A}) + \left(2-\frac{2\varepsilon^2}{\delta}\right)N + \sum_{N<m\le 2N} r_{\mathcal{A}}(m)\\ &\gtrsim \left(\frac{2\delta}{1+2(\delta-\varepsilon)} + \frac{2\delta}{1+3\delta-\varepsilon} - 2\right)|\mathcal{A}|^2 + 4(1-\delta)N + \left(2-\frac{2\varepsilon^2}{\delta}\right)N,\end{aligned}$$

which gives

$$\frac{|\mathcal{A}|^2}{N} \gtrsim \frac{6-4\delta-\frac{2\varepsilon^2}{\delta}}{3-\frac{2\delta}{1+2(\delta-\varepsilon)}-\frac{2\delta}{1+3\delta-\varepsilon}}. \tag{3.4}$$

Let $\delta = 0.2257$ and $\varepsilon = 0.0882$ in (3.4), then we have

$$\frac{|\mathcal{A}|^2}{N} \gtrsim 2.12893875 > \frac{1}{0.46972},$$

which proves Theorem 1.1. □

## 4. Proof of Theorem 1.2

Suppose $\mathcal{A}$ is a restricted 2-basis for a sufficiently large integer $N$. (Thus $\mathcal{A} \subset [0, N/2]$.) Let $\delta \in [1/6, 1/2]$. From (2.3) and Lemma 2.3, we have

$$R_{\frac{1}{2}+\delta,\delta,\frac{1}{2}\delta-\frac{1}{4}}(\mathcal{A}) \le D_{\frac{1}{2}+\delta,\delta}(\mathcal{A}) \lesssim |\mathcal{A}|^2 - 2(1-\delta)N. \tag{4.1}$$

On the other hand, since $\frac{1}{6} \le \delta \le \frac{1}{2}$, we see that

$$\begin{aligned}&R_{\frac{1}{2}+\delta,\delta,\frac{1}{2}\delta-\frac{1}{4}}(\mathcal{A})\\ &= \sum_{0\le m\le(\frac{1}{4}+\frac{\delta}{2})N} \left(1-\frac{|m-(\frac{1}{4}-\frac{\delta}{2})N|}{\delta N}\right) r_{\mathcal{A}}(m)\\ &\quad + \sum_{(\frac{3}{4}-\frac{\delta}{2})N\le m\le N} \left(1-\frac{|m-(\frac{3}{4}+\frac{\delta}{2})N|}{\delta N}\right) r_{\mathcal{A}}(m)\\ &\gtrsim 2\sum_{0\le m\le(\frac{1}{4}+\frac{\delta}{2})N} \left(1-\frac{|m-(\frac{1}{4}-\frac{\delta}{2})N|}{\delta N}\right)\\ &\quad + 2\sum_{(\frac{3}{4}-\frac{\delta}{2})N\le m\le N} \left(1-\frac{|m-(\frac{3}{4}+\frac{\delta}{2})N|}{\delta N}\right)\\ &\gtrsim \left(4\delta - \frac{2\left(\frac{3}{2}\delta-\frac{1}{4}\right)^2}{\delta}\right)N.\end{aligned} \tag{4.2}$$

Thus from this and (4.1), we get

$$\frac{|\mathcal{A}|^2}{N} \gtrsim 2 + 2\delta - \frac{2\left(\frac{3}{2}\delta-\frac{1}{4}\right)^2}{\delta} := g(\delta).$$

Theorem 1.2 then follows by noticing that $\frac{\sqrt{5}}{10} \in [\frac{1}{6}, \frac{1}{2}]$ and $\frac{1}{g(\sqrt{5}/10)} = \frac{7+\sqrt{5}}{22}$. □

## 5. A further remark

When estimating the size of a set $\mathcal{A} \subset [0, N] \cap \mathbb{Z}$ in an order 2 additive problem, such as a 2-basis for $N$ or a $B_2[g]$ set (a type of generalized Sidon set), it seems more natural to use the Fourier series of functions of two variables rather than a single variable function, at least in some special cases. For instance, to give an explicit upper bound for a weighted sum like

$$\sum_{m \in \mathcal{I}(N)} w(m/N) r_{\mathcal{A}}(m) = \sum_{\substack{a, b \in \mathcal{A} \\ a+b \in \mathcal{I}(N)}} w\left(\frac{a+b}{N}\right)$$

with $\mathcal{I}(N)$ being a short interval close to 0 or $2N$, we can express the sum as a trigonometric series in accordance with the Fourier expansion of the function $f(x, y) = w(x + y)$ on $[0, 1] \times [0, 1]$ or that of $w(x)$ on $[0, 2]$. Comparing the sum with the Fourier expansion of $w(x)$, the trigonometric sum resulting from the Fourier series of the 2 variable function $f(x, y)$ has a smaller "constant term", and its cosine (and/or sine) terms have period 1 and are thus related to $f_{\mathcal{A}}\left(\frac{n}{N}\right)$ (rather than $f_{\mathcal{A}}\left(\frac{n}{2N}\right)$). Such sums (with $N \nmid n$) have presumably smaller sizes than $f_{\mathcal{A}}\left(\frac{m}{2N}\right)$ $(2 \nmid m)$ have in the 2-basis problem (and a smaller size on average in the $B_2[g]$ problem). This means that, if one could find a weight function $w$ such that the Fourier series of $w(x + y)$ has a small $L^1$-norm, then presumably a better lower bound should be obtained for 2-bases of integers; and a better upper bound for $B_2[g]$-sets would be achieved if the $L^{\frac{4}{3}}$-norm of the Fourier series is small. The difficulty of this approach is to find a (more or less) optimal weight function satisfying the properties we want. This surely requires a better understanding of the Fourier coefficients of 2-variable functions.

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