

## NAKAJIMA'S PROBLEM FOR GENERAL CONVEX BODIES

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ABSTRACT. For a convex body  $K \subset \mathbb{R}^n$ , the  $k$ th projection function of  $K$  assigns to any  $k$ -dimensional linear subspace of  $\mathbb{R}^n$  the  $k$ -volume of the orthogonal projection of  $K$  to that subspace. Let  $K$  and  $K_0$  be convex bodies in  $\mathbb{R}^n$ , and let  $K_0$  be centrally symmetric and satisfy a weak regularity assumption. Let  $i, j \in \mathbb{N}$  be such that  $1 \leq i < j \leq n-2$  with  $(i, j) \neq (1, n-2)$ . Assume that  $K$  and  $K_0$  have proportional  $i$ th projection functions and proportional  $j$ th projection functions. Then we show that  $K$  and  $K_0$  are homothetic. In the particular case where  $K_0$  is a Euclidean ball, we thus obtain characterizations of Euclidean balls as convex bodies having constant  $i$ -brightness and constant  $j$ -brightness. This special case solves Nakajima's problem in arbitrary dimensions and for general convex bodies for most indices  $(i, j)$ .

### 1. INTRODUCTION AND STATEMENT OF RESULTS

Nakajima's problem is concerned with the determination of convex bodies in  $\mathbb{R}^n$  by two projection functions. A convex body in Euclidean space  $\mathbb{R}^n$  is a compact convex set with nonempty interior. Let  $\mathbb{G}(n, k)$  be the Grassmannian of  $k$ -dimensional linear subspaces of  $\mathbb{R}^n$ ,  $k \in \{0, \dots, n\}$ . The  $k$ th projection function  $\pi_k(K)$  of a convex body  $K$  is defined by

$$\pi_k(K) : \mathbb{G}(n, k) \rightarrow \mathbb{R}, \quad L \mapsto V_k(K|L),$$

where  $V_k(K|L)$  is the  $k$ -volume of the orthogonal projection  $K|L$  of  $K$  to  $L$ . For a Euclidean ball, all projection functions are constant functions. A converse statement is true for centrally symmetric convex bodies: A centrally symmetric convex body  $K$  having one constant projection function  $\pi_k(K)$ , for some  $k \in \{1, \dots, n-1\}$ , must be a Euclidean ball.

Examples of nonspherical convex bodies having one constant projection function are well known. For  $k = 1$  these are the bodies of constant width which have been studied extensively. See the surveys [5], [16]. The case  $k = n-1$  of convex bodies of constant brightness was first studied by Blaschke, who constructed smooth bodies of revolution with constant brightness. Further examples, also without rotational symmetry, can be obtained by approximation and convolution arguments applied to surface area measures of convex bodies (cf. [15]). In the intermediate cases, i.e. for  $k \in \{2, \dots, n-2\}$ , the classical examples of smooth convex bodies with a constant

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$k$ th projection function (bodies of constant  $k$ -brightness) are bodies of revolution. The existence of such bodies has been shown by Firey [8] (cf. also [14]). In [13], Goodey and Howard prove the existence of smooth convex bodies with constant  $k$ -brightness not having rotational symmetry and obtain a parametric description of these bodies for  $k \in \{2, \dots, n-3\}$ . They also provide new examples of smooth bodies of revolution having constant  $k$ -brightness, for some  $k \in \{2, \dots, n-2\}$ , by a perturbation argument.

In its original form, Nakajima's problem asks whether a smooth convex body having *two* constant projection functions must be a ball [23]; see [4], [5], [7], [9], [11], [16]. In dimension three and for smooth convex bodies the affirmative answer was given by Nakajima himself in the early 20th century. Recently, in  $\mathbb{R}^3$  the smoothness hypothesis was removed by Ralph Howard [17]. By a well-known reduction argument, this implies also an affirmative answer for the case of constant width and constant 2-brightness in arbitrary dimensions. For smooth convex bodies in general dimensions, Nakajima's problem has been settled in [18] with the exception of two cases. Further results for general convex bodies have subsequently been obtained in [19]. The approach developed in [19] and based on [18] is crucial for the present work and will be refined further.

In the present paper, we consider two convex bodies  $K$  and  $K_0$ , where  $K_0$  is centrally symmetric. We do not assume a priori smoothness of  $K$ . The weak regularity assumption, which is needed for  $K_0$ , can be described in geometric terms. Let  $\|\cdot\|$  denote a Euclidean norm and  $\mathbb{S}^{n-1}$  the unit sphere of  $\mathbb{R}^n$ . Let  $\text{expn}^*(K_0)$  denote the set of all unit vectors  $u \in \mathbb{S}^{n-1}$  such that there exists a point  $y$  in the interior of  $K_0$  and a point  $x$  in the boundary  $\partial K_0$  of  $K_0$  such that  $\|x - y\| = \min\{\|z - y\| : z \in \partial K_0\}$  and  $x - y = \|x - y\|u$ . We call  $\text{expn}^*(K_0)$  the set of directions of the nearest boundary points of  $K_0$  (cf. [21]). Clearly,  $u \in \text{expn}^*(K_0)$  if and only if there is a boundary point  $x \in \partial K_0$  with exterior unit normal vector  $u$  and a Euclidean ball  $B$  with  $x \in B \subset K_0$ ; i.e.  $K_0$  is supported from inside by  $B$  at  $x$ . It is known that any convex body is supported from inside by a ball at almost all of its boundary points (cf. [22]). In Theorem 1.1 we assume that  $\text{expn}^*(K_0)$  has positive spherical Lebesgue measure. This assumption is satisfied, e.g., if  $K_0$  has a small boundary part which is  $C^2$ -smooth with positive Gauss curvature and is arbitrary otherwise. In particular,  $K_0$  may have all kinds of singularities. For polytopes, however, this assumption is not satisfied.

**Theorem 1.1.** *Let  $K, K_0$  be convex bodies in  $\mathbb{R}^n$ , and let  $K_0$  be centrally symmetric. Let  $1 \leq i < j \leq n-2$  be integers with  $(i, j) \neq (1, n-2)$ . Assume that the set of directions of the nearest boundary points of  $K_0$  has positive spherical Lebesgue measure. If there are positive constants  $\alpha, \beta$  such that*

$$\pi_i(K) = \alpha \pi_i(K_0) \quad \text{and} \quad \pi_j(K) = \beta \pi_j(K_0),$$

*then  $K$  and  $K_0$  are homothetic.*

Remarkably, this result also applies to nonsmooth convex bodies  $K_0$ , and therefore yields nor requires smoothness of  $K$ .

In [19], where general convex bodies are considered, the cases  $(i, j)$  with  $1 = i < j < (n+1)/2$  and the case  $(i, j) = (1, 3)$  (for  $n = 5$ ) have been settled. The present Theorem 1.1 thus considerably extends the range of indices  $(i, j)$  for which Nakajima's problem is known to have a positive resolution for general convex bodies. The case  $(i, j) = (1, 2)$ , which is not covered by the present result, has been

established by Howard [17] for  $n \geq 3$ . The cases which still remain open, for general convex bodies, thus are  $(1, n-2)$  for  $n \geq 6$ , and  $(i, n-1)$  for  $n \geq 4$  and  $1 \leq i \leq n-2$ . For smooth convex bodies, where additional continuity arguments can be applied, the unresolved cases are  $(1, n-1)$  and  $(n-2, n-1)$  (see [18]).

The special case where  $K_0$  is a Euclidean ball is the following corollary.

**Corollary 1.2.** *Let  $K$  be a convex body in  $\mathbb{R}^n$ . Let  $1 \leq i < j \leq n-2$  be integers with  $(i, j) \neq (1, n-2)$ . Assume that  $K$  has constant  $i$ -brightness and constant  $j$ -brightness. Then  $K$  is a Euclidean ball.*

Nakajima's problem is closely related to the problem of determining a convex body from its projection functions. We say that a convex body  $K_0$  is determined by its  $k$ th projection function  $\pi_k(K_0)$ , if for any convex body  $K$  with  $\pi_k(K) = \pi_k(K_0)$ ,  $K_0$  is a translate of  $K$  or a translate of  $K^*$  (the reflection of  $K$  in the origin). Continuing the work of Schneider [25], Christina Bauer [2] showed that special convex polytopes  $P$  are determined by just one projection function  $\pi_k(P)$ , where  $k \in \{2, \dots, n-2\}$ . From this she deduced that most (in the sense of Baire categories) convex bodies are determined by their  $k$ th projection function. For  $k = 1$  and  $k = n-1$ , convex bodies which are determined by their  $k$ th projection function must be centrally symmetric. In striking contrast to these results, Campi [3], Gardner and Volčič [10], and Goodey, Schneider and Weil [12], [11] constructed examples of noncongruent pairs of convex bodies for which  $\pi_k(K) = \pi_k(K_0)$  holds for all  $k = 1, \dots, n$ . The latter authors even exhibit a dense set of convex bodies not determined by all of their projection functions. This clearly shows that in general an additional assumption on  $K_0$  such as central symmetry cannot be avoided. A result by Chakerian and Lutwak [6] implies that a centrally symmetric convex body  $K_0$  is determined by any two of its projection functions. This does not solve Nakajima's problem, since here the assumption is more restrictive. However, we will use the result from [6] as an important ingredient in our proof.

## 2. PREPARATIONS

Let  $\langle \cdot, \cdot \rangle$  denote the scalar product and  $\mathbb{S}^{n-1}$  the unit sphere of  $\mathbb{R}^n$ . The support function  $h_K$  of a convex body  $K$  is  $h_K(u) = \max\{\langle x, u \rangle : x \in K\}$ ,  $u \in \mathbb{R}^n$ . It is positively homogeneous of degree one and is convex. By Aleksandrov's theorem [1] on second order differentiability of convex functions, the second differential  $d^2h_K(x)$  in Aleksandrov's sense exists for almost all (with respect to Lebesgue measure)  $x \in \mathbb{R}^n$ . Here  $d^2h_K(x)$  is considered as a linear map of  $\mathbb{R}^n$ . Homogeneity implies that the second order differential also exists for almost all (with respect to spherical Lebesgue measure) unit vectors  $u \in \mathbb{S}^{n-1}$ . For further explanations and definitions we refer to [24] and [19].

In the following, we write  $h$  for the support function of  $K$  and  $h_0$  for the support function of a convex body  $K_0$ . Let  $h$  be second order differentiable at  $u \in \mathbb{S}^{n-1}$ . The restriction  $d^2h(u)|_{u^\perp}$  of the linear map  $d^2h(u)$  to the orthogonal complement  $u^\perp$  of  $u$  is a selfadjoint linear map with respect to the Euclidean structure induced on  $u^\perp$ . The  $n-1$  real and nonnegative eigenvalues of  $d^2h(u)|_{u^\perp}$  are the principal radii of curvature of  $K$ . By Aleksandrov's theorem, these radii of curvature are defined for almost all unit vectors  $u$ . The product of the radii of curvature of  $K$  in direction  $u$  is just  $\det(d^2h(u)|_{u^\perp})$ , the determinant of the Hessian of the support function of  $K$ , whenever the second differential exists.

The determinant of the Hessian of the support function  $h$  of a convex body  $K$  is related to the surface area measure  $S_{n-1}(K; \cdot)$  of  $K$  of order  $n - 1$ . This is a measure on the Borel sets of the unit sphere. It can be defined via a local Steiner formula or as the  $(n - 1)$ -dimensional Hausdorff measure of the reverse spherical image of Borel sets on the unit sphere (cf. [24, (4.2.9) and (4.2.24)]). The Radon-Nikodym derivative of  $S_{n-1}(K; \cdot)$  with respect to the spherical Lebesgue measure  $\sigma_{n-1}$  on  $\mathbb{S}^{n-1}$  can be expressed in terms of the support function  $h$  of  $K$  at points of second order differentiability of  $h$ . For a fixed unit vector  $u \in \mathbb{S}^{n-1}$  and  $i \in \mathbb{N}$ , we put  $\omega_i := \{v \in \mathbb{S}^{n-1} : \langle v, u \rangle \geq 1 - (2i^2)^{-1}\}$ . Hence  $\omega_i \downarrow \{u\}$ , as  $i \rightarrow \infty$ , in the sense of Hausdorff convergence of closed sets. The following lemma provides the required connection to the determinant of the Hessian of  $h$ .

**Lemma 2.1.** *Let  $K \subset \mathbb{R}^n$  be a convex body. If  $u \in \mathbb{S}^{n-1}$  is a point of second order differentiability of the support function  $h$  of  $K$ , then*

$$\lim_{i \rightarrow \infty} \frac{S_{n-1}(K; \omega_i)}{\sigma_{n-1}(\omega_i)} = \det(d^2h(u)|_{u^\perp}).$$

This result goes back to Aleksandrov [1, §8], who remarks that the proof is similar to the argument for a dual assertion concerning the Gauss curvature measure (cf. [19] for further references).

The connection of the surface area measure  $S_{n-1}(K; \cdot)$  of a convex body  $K$  to the projection function of order  $n - 1$  of  $K$  is given by the following well-known equation. If  $u$  is any unit vector, then

$$\pi_{n-1}(K)(u^\perp) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| S_{n-1}(K; dv).$$

We will apply this relation with respect to subspaces of  $\mathbb{R}^n$  as ambient spaces.

For nonnegative numbers  $x_1, \dots, x_{n-1}$  and  $I \subset \{1, \dots, n - 1\}$  we define

$$x_I := \prod_{i \in I} x_i,$$

which is defined as 1 if  $I = \emptyset$ .

The following lemma is proved in [13] in a sequence of lemmata. For the reader's convenience we give a condensed argument for the assertion needed in the sequel.

**Lemma 2.2.** *Let  $2 \leq k \leq n - 3$ ,  $b > 0$ , and let  $x_1, \dots, x_{n-1}$  and  $y_1, \dots, y_{n-1}$  be positive numbers. Assume that  $x_I + y_I = 2b$  whenever  $I \subset \{1, \dots, n - 1\}$  with  $|I| = k$ . Then there is a subset  $R \subset \{1, \dots, n - 1\}$  with  $|R| = n - 2$ , and there are numbers  $x, y > 0$  such that  $x_i = x$  and  $y_i = y$  for  $i \in R$ .*

*Proof.* First, we consider the case  $k = n - 3$ .

If  $x_1 = \dots = x_{n-1}$  and  $y_1 = \dots = y_{n-1}$ , there is nothing to prove.

Assume that  $x_1 \neq x_2$ . Then, for  $i \in \{3, \dots, n - 1\}$ ,

$$\begin{aligned} x_1 \cdot x_3 \cdots \check{x}_i \cdots x_{n-1} + y_1 \cdot y_3 \cdots \check{y}_i \cdots y_{n-1} &= 2b, \\ x_2 \cdot x_3 \cdots \check{x}_i \cdots x_{n-1} + y_2 \cdot y_3 \cdots \check{y}_i \cdots y_{n-1} &= 2b. \end{aligned}$$

The notation  $\check{x}_i$  means that  $x_i$  is omitted from the product. Subtracting the first equation from the second, we get

$$(x_2 - x_1) \cdot x_3 \cdots \check{x}_i \cdots x_{n-1} + (y_2 - y_1) \cdot y_3 \cdots \check{y}_i \cdots y_{n-1} = 0.$$

Hence  $y_1 \neq y_2$  and

$$\frac{x_2 - x_1}{y_1 - y_2} = \frac{y_3 \cdots \check{y}_i \cdots y_{n-1}}{x_3 \cdots \check{x}_i \cdots x_{n-1}},$$

for  $i = 3, \dots, n-1$ . This implies that there is a constant  $c > 0$  such that  $y_i = c \cdot x_i$  for  $i = 3, \dots, n-1$ . Since

$$x_3 \cdots \check{x}_i \cdots x_{n-1} (x_1 + y_1 c^{n-4}) = 2b$$

for  $i = 3, \dots, n-1$  and  $n-1 \geq 4$ , we conclude that  $x := x_3 = \dots = x_{n-1}$ , and therefore also  $y := y_3 = \dots = y_{n-1}$ .

If  $x_2 = x_3$ , then

$$x^{n-3} + y_2 y^{n-4} = 2b.$$

Moreover, from  $x_3 \cdots x_{n-1} + y_3 \cdots y_{n-1} = 2b$  we obtain

$$x^{n-3} + y^{n-3} = 2b.$$

We conclude that  $y_2 = y$ . Thus we can choose  $R = \{2, \dots, n-1\}$ .

Finally, if  $x_2 \neq x_3$ , then the first part of the proof shows that necessarily

$$x_1 = x_4 = \dots = x_{n-1} \quad \text{and} \quad y_1 = y_4 = \dots = y_{n-1}.$$

We know that  $x_3 = x_4$  and  $y_3 = y_4$ . Hence we can choose  $R = \{1, \dots, n-1\} \setminus \{2\}$ .

This completes the proof of the special case  $k = n-3$ .

Now let  $2 \leq k \leq n-3$ . Then the cardinality of  $\{x_1, \dots, x_{n-1}\}$  is at most 2. In fact, for any numbers  $1 \leq i_1 < i_2 < i_3 \leq n-1$  there is a set  $J \subset \{1, \dots, n-1\}$  with  $\{i_1, i_2, i_3\} \subset J$  and  $|J| = k+2$ . Then the numbers  $x_i, i \in J$ , and  $y_i, i \in J$ , satisfy  $x_I + y_I = 2b$  whenever  $I \subset J$  and  $k = |I| = |J| - 2$ . By the special case, the cardinality of the set  $\{x_{i_1}, x_{i_2}, x_{i_3}\}$  is at most 2.

Thus (after a permutation of indices) we can assume that there is some  $l \in \{0, \dots, n-3\}$  such that  $x_1 = \dots = x_l =: x \neq \bar{x} := x_{l+1} = \dots = x_{n-1}$ . If  $l \in \{0, 1\}$  there is nothing to prove. So let  $l \in \{2, \dots, n-3\}$ . Then  $x_1 = x_2 \neq x_{n-2} = x_{n-1}$ . Choosing  $J$  such that  $\{1, 2, n-2, n-1\} \subset J \subset \{1, \dots, n-1\}$  and  $|J| = k+2 \geq 4$ , we obtain a contradiction by applying the special case already established to the numbers  $x_i, i \in J$ , and  $y_i, i \in J$ , with  $k = |J| - 2$ . Hence there is a set  $R \subset \{1, \dots, n-1\}$  with  $|R| = n-2$  and a number  $x > 0$  such that  $x_i = x$  for  $i \in R$ .

By what we have shown and by the assumption,  $x^k + y_I = 2b$  for  $I \subset R$  with  $|I| = k$ . Hence  $y_I = y_{I'}$  for all  $I, I' \subset R$  with  $|I| = |I'| = k$ . Since  $2 \leq k \leq n-3$ , there is some  $y > 0$  such that  $y_i = y$  for all  $i \in R$ .  $\square$

### 3. PROOF OF THEOREM 1.1

The assumption of Theorem 1.1 implies that

$$V_i(K|U) = V_i(K'_0|U) \quad \text{and} \quad V_j(K|L) = \beta' V_j(K'_0|L)$$

for  $U \in \mathbb{G}(n, i)$  and  $L \in \mathbb{G}(n, j)$ , where  $K'_0 := \alpha^{1/i} K_0$  and  $\beta' := \beta/\alpha^{j/i}$ . Thus we can assume that

$$V_i(K|U) = V_i(K_0|U) \quad \text{and} \quad V_j(K|L) = \beta V_j(K_0|L)$$

for  $U \in \mathbb{G}(n, i)$  and  $L \in \mathbb{G}(n, j)$ . Fix  $L$  and consider  $U \subset L$ . Then  $K|L$  and  $K_0|L$  belong to the same  $i$ th projection class with respect to  $L$ , and  $K_0|L$  is centrally symmetric. By a result of Chakerian and Lutwak [6]

$$V_j(K_0|L) \geq V_j(K|L)$$

with equality if and only if  $K_0|L$  is a translate of  $K|L$ . Hence, for  $L \in \mathbb{G}(n, j)$ ,

$$V_j(K|L) = \beta V_j(K_0|L) \geq \beta V_j(K|L).$$

Thus  $\beta \leq 1$  with equality if and only if  $K_0|L$  is a translate of  $K|L$  for all  $L \in \mathbb{G}(n, j)$ .

The remaining part of the proof is devoted to showing that also  $\beta \geq 1$ . Once this has been established, it follows that  $K_0|L$  is a translate of  $K|L$  for all  $L \in \mathbb{G}(n, j)$ , and hence  $K_0$  is a translate of  $K$ .

Let  $\mathbf{P}$  be the set of all  $u \in \mathbb{S}^{n-1}$  such that  $h$  and  $h_0$  (the support functions of  $K$  and  $K_0$ ) are second order differentiable at  $u$  and at  $-u$  and  $\det(d^2h_0(u)|u^\perp) \neq 0$ . Lemma 2.7 in [20] and the assumption  $\sigma_{n-1}(\text{expn}^*(K_0)) > 0$  of Theorem 1.1 imply that the set of unit vectors  $u$  for which  $\det(d^2h_0(u)|u^\perp) \neq 0$  has positive spherical Lebesgue measure. Hence, by Aleksandrov's theorem the set  $\mathbf{P}$  has positive spherical Lebesgue measure. In particular,  $\mathbf{P} \neq \emptyset$ . Hence we can choose a vector  $u \in \mathbf{P}$  which will be fixed for the rest of the proof.

Next we choose  $W \in \mathbb{G}(n, j + 1)$  with  $u \in W$  and put  $L := u^\perp \cap W \in \mathbb{G}(n, j)$ . From

$$V_j((K|W)|L) = \beta V_j((K_0|W)|L)$$

we get

$$\begin{aligned} & \int_{\mathbb{S}^{n-1} \cap W} |\langle u, v \rangle| S_j^W(K|W; dv) + \int_{\mathbb{S}^{n-1} \cap W} |\langle u, v \rangle| S_j^W(K^*|W; dv) \\ &= 2\beta \int_{\mathbb{S}^{n-1} \cap W} |\langle u, v \rangle| S_j^W(K_0|W; dv), \end{aligned}$$

where the upper index  $W$  indicates that the measure is considered with respect to  $W$  as the ambient space. The injectivity of the cosine transform on even measures yields

$$S_j^W(K|W; \cdot) + S_j^W(K^*|W; \cdot) = 2\beta S_j^W(K_0|W; \cdot).$$

Since  $u \in \mathbf{P}$ ,  $h_{K|W}$ ,  $h_{K^*|W}$  and  $h_{K_0|W}$  are second order differentiable at  $u$  with respect to  $W$  as the ambient space. Hence Lemma 2.1 applied in  $W$  yields

$$\begin{aligned} & \det(d^2h_{K|W}(u)|u^\perp \cap W) + \det(d^2h_{K^*|W}(u)|u^\perp \cap W) \\ &= 2\beta \det(d^2h_{K_0|W}(u)|u^\perp \cap W). \end{aligned}$$

To rewrite this relation, we define the linear maps

$$\begin{aligned} L(h)(u) &:= d^2h(u)|u^\perp : u^\perp \rightarrow u^\perp, \\ L(h_0)(u) &:= d^2h_0(u)|u^\perp : u^\perp \rightarrow u^\perp. \end{aligned}$$

Using the exterior calculus as in [18], [19], we arrive at

$$\bigwedge^j L(h)(u) + \bigwedge^j L(h)(-u) = 2\beta \bigwedge^j L(h_0)(u).$$

Since  $L(h_0)(u)$  is an isomorphism, due to our choice of  $u \in \mathbf{P}$ , we can define

$$L_{h_0}(h)(u) := L(h_0)(u)^{-1/2} \circ L(h)(u) \circ L(h_0)(u)^{-1/2}.$$

As in [18], [19] we thus obtain

$$(3.1) \quad \bigwedge^j L_{h_0}(h)(u) + \bigwedge^j L_{h_0}(h)(-u) = 2\beta \bigwedge^j \text{id},$$

where  $\text{id}$  is the identity map on  $u^\perp$ . In the same way, we also get

$$(3.2) \quad \bigwedge^i L_{h_0}(h)(u) + \bigwedge^i L_{h_0}(h)(-u) = 2 \bigwedge^i \text{id}.$$

In this situation, Lemma 3.4 in [18] implies that  $L_{h_0}(h)(u)$  and  $L_{h_0}(h)(-u)$  have a common orthonormal basis of eigenvectors  $e_1, \dots, e_{n-1}$ , where  $x_1, \dots, x_{n-1}$  denote the corresponding eigenvalues (relative principal radii of curvature) of  $L_{h_0}(h)(u)$  and  $y_1, \dots, y_{n-1}$  are the eigenvalues of  $L_{h_0}(h)(-u)$ . Applying these basis vectors to (3.2) and (3.1), we get the polynomial equations

$$(3.3) \quad \begin{cases} x_I + y_I = 2, & |I| = i, \\ x_J + y_J = 2\beta, & |J| = j, \end{cases}$$

where  $I, J \subset \{1, \dots, n-1\}$ .

We distinguish three cases.

*Case 1.*  $x_{i'} = 0$  for some  $i' \in \{1, \dots, n-1\}$ . Assume that  $x_1 = 0$ . Then  $y_1 \cdot y_{J'} = 2\beta$  whenever  $J' \subset \{2, \dots, n-1\}$  with  $|J'| = j-1$ . This immediately implies that  $y_\iota > 0$  for  $\iota = 1, \dots, n-1$ , and then  $y_{J'} = y_{J''}$  for all  $J', J'' \subset \{2, \dots, n-1\}$  with  $|J'| = |J''| = j-1$ . Thus we conclude that  $y_2 = \dots = y_{n-1} =: y > 0$ , since  $j-1 \leq n-3$ .

If also  $x_{i'} = 0$  for some  $i' \neq 1$ , say  $x_2 = 0$ , then, in the same way, we get  $y_1 = y_3 = \dots = y_{n-1} = y$ . Hence  $y_1 = \dots = y_{n-1} = y$ . But then  $y^i = 2$  and  $y^j = 2\beta$  by (3.3), which implies that  $\beta \geq 1$ .

If  $x_2, \dots, x_{n-1} > 0$ , then we infer that  $x_I + y^i = 2$  whenever  $I \subset \{2, \dots, n-1\}$  with  $|I| = i$ . This shows that  $x_I = x_{I'}$  for  $I, I' \subset \{2, \dots, n-1\}$  with  $1 \leq |I| = |I'| = i \leq n-3$ . Clearly, this implies that  $x_2 = \dots = x_{n-1}$ . Therefore by (3.3) we have

$$x^i + y^i = 2, \quad x^j + y^j = 2\beta.$$

By Jensen's inequality,

$$1 = \frac{x^i + y^i}{2} = \left( \frac{x^i + y^i}{2} \right)^{j/i} \leq \frac{(x^i)^{j/i} + (y^i)^{j/i}}{2} = \beta,$$

i.e.  $\beta \geq 1$ .

*Case 2.*  $y_{i'} = 0$  for some  $i' \in \{1, \dots, n-1\}$ . This is treated as in Case 1.

*Case 3.*  $x_1, \dots, x_{n-1} > 0$  and  $y_1, \dots, y_{n-1} > 0$ . We can apply Lemma 2.2 with  $k = i$  or  $k = j$  (as appropriate) and combine this information with (3.3) to get

$$x^i + y^i = 2, \quad x^j + y^j = 2\beta.$$

Here again we use the fact that  $j \leq n-2 = |R|$ , where  $R$  is as in Lemma 2.2. By Jensen's inequality, we deduce that  $\beta \geq 1$ .

As described before, this concludes the proof in all cases that can occur.

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