PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 1, January 2009, Pages 19–26
S 0002-9939(08)09437-9
Article electronically published on July 29, 2008

PARAMETRIC DECOMPOSITION OF POWERS OF PARAMETER IDEALS AND SEQUENTIALLY COHEN-MACAULAY MODULES

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(Communicated by Bernd Ulrich)

Abstract. Let M be a finitely generated module of dimension d over a Noetherian local ring $(R, \mathfrak{m})$ and $\mathfrak{q}$ an ideal generated by a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M. For each positive integer n, set

$$\Lambda_{d,n} = \{\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d | \alpha_i \geqslant 1, 1 \leqslant i \leqslant d \text{ and } \sum_{i=1}^{d} \alpha_i = d + n - 1\}$$

and $\mathfrak{q}(\alpha) = (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})$ for each $\alpha \in \Lambda_{d,n}$. Then we prove in this note that M is a sequentially Cohen-Macaulay module if and only if there exists a good system of parameters $\underline{x}$ such that the equality $\mathfrak{q}^n M = \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha) M$ holds true for all $n \geq 1$. As an application, we show that the sequentially Cohen-Macaulayness of a module can be characterized by a very special expression of the Hilbert-Samuel polynomial of a good parameter ideal.

1. Introduction

Throughout this paper we denote by R a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$ and by M a finitely generated R-module with $\dim M = d$. Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of the module M and $\mathfrak{q} = (x_1, \ldots, x_d)$ the parameter ideal of M generated by $\underline{x}$. For each integer $n \geqslant 1$, we set

$$\Lambda_{d,n} = \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d | \alpha_i \geqslant 1 \text{ for all } 1 \leqslant i \leqslant d \text{ and } \sum_{i=1}^{d} \alpha_i = d + n - 1\}.$$

Let $\mathfrak{q}(\alpha) = (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})$ for each $\alpha = (\alpha_1, \ldots, \alpha_d) \in \Lambda_{d,n}$. We say that the system $\underline{x}$ of parameters has the *property of parametric decomposition* if the equality $\mathfrak{q}^n M = \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha) M$ holds true for all $n \geqslant 1$. The main purpose of this paper is to study the question of when a given system of parameters of M has the property of parametric decomposition. Notice that Heinzer, Ratliff and Shah [HRS, Theorem 2.4] proved that an R-regular sequence always has the property of parametric decomposition. Later, Goto and Shimoda [GS1, Theorem 1.1] showed that the converse is also true

Received by the editors November 15, 2006, and, in revised form, September 11, 2007, and November 18, 2007.

2000 *Mathematics Subject Classification.* Primary 13H10; Secondary 13H99.

Key words and phrases. Parametric decomposition, sequentially Cohen-Macaulay module, dimension filtration, good system of parameters.

when each element of the sequence is a non-zerodivisor in R. Moreover, they gave in [GS2, Theorem 1.1] a characterization of R with $\dim R \geqslant 2$, in which every system of parameters of R has the property of parametric decomposition. In order to generalize this result of Goto and Shimoda, let us recall some notions which were defined in [CC]. A filtration $\mathcal{D}: \ H^0_{\mathfrak{m}}(M) = D_0 \subset D_1 \subset \ldots \subset D_t = M$ of submodules of M is said to be a *dimension filtration* if D_{i-1} is the largest submodule of D_i with $\dim D_{i-1} < \dim D_i$ for all $i = t, t-1, \ldots, 1$. If D_i/D_{i-1} is Cohen-Macaulay for all $1 \leq i \leq t$, M is called a *sequentially Cohen-Macaulay module.* A system $\underline{x} = x_1, \ldots, x_d$ of parameters of M is called a *good system of parameters* of M if $D_i \cap (x_{d_i+1}, \ldots, x_d)M = 0$ for all $0 \leq i \leq t-1$, where $d_i = \dim D_i$. Now, we restrict our interest in the above question to the set of all good systems of parameters of M. It turns out that the property of parametric decomposition of a good system of parameters can be characterized by the sequentially Cohen-Macaulayness of the module. The following theorem is the main result of this paper.

Theorem 1.1. *The following statements are equivalent:*
(i) *M is a sequentially Cohen-Macaulay module.*
(ii) *Every good system of parameters of M has the property of parametric decomposition.*
(iii) *There exists a good system of parameters of M having the property of parametric decomposition.*

As a consequence of Theorem 1.1 we again obtain the main result of Goto-Shimoda [GS2, Theorem 1.1]. It should be noted here that Theorem 1.1 of [GS2] was stated for local rings, but its proof still works in the module case.

Corollary 1.2. *Let $\dim M \geqslant 2$ and $H^0_{\mathfrak{m}}(M)$ the 0^{th} local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$. Then the following statements are equivalent:*
(i) *$M/H^0_{\mathfrak{m}}(M)$ is a Cohen-Macaulay module and $\mathfrak{m}H^0_{\mathfrak{m}}(M) = 0$.*
(ii) *Every system of parameters of M has the property of parametric decomposition.*

Before we give proofs for Theorem 1.1 and its corollary in Section 3, we need some basic facts on good systems of parameters and sequentially Cohen-Macaulay modules, which will be summarized in Section 2. In Section 4 we shall show that the Hilbert-Samuel polynomial of a sequentially Cohen-Macaulay module M with respect to a good parameter ideal can be computed effectively by using the dimension filtration $\mathcal{D}$ of M (Theorem 4.1).

2. Preliminaries

Throughout this paper, R is a Noetherian local commutative ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module with $\dim M = d$. Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of the module M, and we denote by $\mathfrak{q}$ the ideal generated by $x_1, \ldots, x_d$. For positive integers n, we set

$$\Lambda_{d,n} = \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d | \alpha_i \geqslant 1 \text{ for all } 1 \leqslant i \leqslant d \text{ and } \sum_{i=1}^{d} \alpha_i = d+n-1\}.$$

Let $\mathfrak{q}(\alpha) = (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})$ for each $\alpha = (\alpha_1, \ldots, \alpha_d) \in \Lambda_{d,n}$. Then $\mathfrak{q}^n M \subseteq \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M$, and if the equality $\mathfrak{q}^n M = \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M$ holds true for a system of parameters $\underline{x}$ of M, we say that $\underline{x}$ has the *property of parametric decomposition*. Recall that a filtration $\mathcal{D}$: $H^0_{\mathfrak{m}}(M) = D_0 \subset D_1 \subset \ldots \subset D_t = M$ of submodules of M is said to be a *dimension filtration* if D_{i-1} is the largest submodule of D_i with $\dim D_{i-1} < \dim D_i$ for all $i = t, t-1, \ldots, 1$, and a system of parameters $\underline{x} = x_1, \ldots, x_d$ of M is called a *good system of parameters* of M if $D_i \cap (x_{d_i+1}, \ldots, x_d)M = 0$ for all $0 \le i \le t-1$, where $d_i = \dim D_i$.

Now, let us briefly give some facts on the dimension filtration and good systems of parameters (see [CC], [CN]). Because of the Noetherian property of M, the dimension filtration of M exists uniquely. Therefore, in the sequel we always denote by

$$\mathcal{D} : \mathrm{H}^0_{\mathfrak{m}}(M) = D_0 \subset D_1 \subset \ldots \subset D_t = M$$

with $\dim D_i = d_i$ the dimension filtration of M. In this case, we also say that the dimension filtration $\mathcal{D}$ of M has length t. Moreover, let $\bigcap_{\mathfrak{p}\in \mathrm{Ass} M} N(\mathfrak{p}) = 0$ be a reduced primary decomposition of 0 of M; then $D_i = \bigcap_{\dim(R/\mathfrak{p})\geqslant d_{i+1}} N(\mathfrak{p})$. Put $N_i = \bigcap_{\dim(R/\mathfrak{p})\leqslant d_i} N(\mathfrak{p})$. Therefore $D_i \cap N_i = 0$ and $\dim(M/N_i) = d_i$. By the Prime Avoidance there exists a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ such that $x_{d_i+1}, \ldots, x_d \in \mathrm{Ann}\,(M/N_i)$. It follows that $D_i \cap (x_{d_i+1}, \ldots, x_d)M \subseteq N_i \cap D_i = 0$ for all $0 \le i \le t-1$. Thus $\underline{x} = x_1, \ldots, x_d$ is a good system of parameters of M, and therefore the set of good systems of parameters of M is non-empty. Let $\underline{x} = x_1, \ldots, x_d$ be a good system of parameters of M. It easy to see that $x_1, \ldots, x_{d_i}$ is a good system of parameters of D_i and $x_1^{n_1}, \ldots, x_d^{n_d}$ is a good system of parameters of M for any d-tuple of positive integers $n_1, \ldots, n_d$.

Lemma 2.1. *Let $\underline{x} = x_1, \ldots, x_d$ be a good system of parameters of M. Then $D_i = 0 :_M x_j$ for all $d_i < j \le d_{i+1}$ and $0 \le i \le t-1$, and therefore $0 :_M x_i^l = 0 :_M x_i$ for all $l \ge 1$.*

Proof. Since $D_i \cap (x_{d_i+1}, \ldots, x_d)M = 0$, we have $D_i \subseteq 0 :_M x_j$ for all $j \geqslant d_i$. Thus it suffices to prove that $0 :_M x_j \subseteq D_j$ for every $d_i < j \leqslant d_{i+1}$. Assume that $0 :_M x_j \nsubseteq D_i$. Let s be the largest integer such that $0 :_M x_j \nsubseteq D_{s-1}$. Then $t \geqslant s > i$ and $0 :_M x_j = 0 :_{D_s} x_j$. Since $d_s \geqslant d_{i+1} \geqslant j$, x_j is a parameter element of D_s and $\dim(0 :_M x_j) < d_s$. Hence $0 :_M x_j \subseteq D_{s-1}$ by the maximality of D_{s-1}. This contradicts the choice of s. Therefore $0 :_M x_j = D_i$. □

Recall that M is said to be a *sequentially Cohen-Macaulay module* if each quotient D_i/D_{i-1} in the dimension filtration of M is Cohen-Macaulay. Notice that the notion of sequentially Cohen-Macaulay modules was introduced by Stanley [St] for the graded case, and it was studied for the local case in [Sch], [CN]. Also notice that a special type of sequentially Cohen-Macaulay rings called approximately Cohen-Macaulay rings was studied very early by Goto [G].

3. Proof of Theorem 1.1

The following result is an immediate consequence of the definition of a good system of parameters in a sequentially Cohen-Macaulay module.

Lemma 3.1. *Let $\underline{x} = x_1, \ldots, x_d$ be a good system of parameters of a sequentially Cohen-Macaulay module M and $\mathfrak{q} = (x_1, \ldots, x_d)$. Then*

$$\mathfrak{q}^n M \cap D_i = \mathfrak{q}^n D_i$$

for all $n \ge 1$ and $0 \le i \le t-1$.

Proof. Since D_{i+1}/D_i is a Cohen-Macaulay module with $\dim D_{i+1}/D_i = d_{i+1} > d_i = \dim D_i$ and $\mathfrak{q}D_{i+1} = (x_1, \ldots, x_{d_{i+1}})D_{i+1}$ for $0 \le i \le t-1$, it follows from well-known facts in commutative algebra that

$$\mathfrak{q}^n D_{i+1} \cap D_i = \mathfrak{q}^n D_i.$$

Therefore

$$\begin{aligned}\mathfrak{q}^n M \cap D_i &= (\mathfrak{q}^n D_t \cap D_{t-1}) \cap D_i = \mathfrak{q}^n D_{t-1} \cap D_i \\ \ldots &= \mathfrak{q}^n D_{i+1} \cap D_i = \mathfrak{q}^n D_i\end{aligned}$$

for all $n \ge 1$ and $0 \le i \le t-1$ as required. □

Let s be a positive integer and $y_1, \ldots, y_s \in \mathfrak{m}$. For each $1 \leqslant i \leqslant s$ and $\alpha = (\alpha_1, \ldots, \alpha_i) \in \Lambda_{i,n}$, we set $Q_i = (y_1, \ldots, y_i)R$, $Q = (y_1, \ldots, y_s)R$ and $Q_i(\alpha) = (y_1^{\alpha_1}, \ldots, y_i^{\alpha_i})$, $Q(\alpha) = (y_1^{\alpha_1}, \ldots, y_s^{\alpha_s})$. The following result is due to Heinzer-Ratliff-Shah [HRS, Theorem 2.4]. But we give here the module version of this result proved by Goto-Shimoda [GS2, Lemma 2.1].

Lemma 3.2. *Let s be a positive integer and $y_1, \ldots, y_s$ an M-regular sequence. Then*

$$Q^n M = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)M$$

for all $n \geqslant 1$.

Because the techniques and methods of proof of Theorem 1.1 heavily depend on the works of Goto-Shimoda [GS1] and [GS2], let us summarize the auxiliary results in [GS1] into the following.

Lemma 3.3. *With the notation as above the following assertions hold true:*

(i) *Let $y \in R$ and assume that $0 :_M y^\ell \subseteq yM$ for all $\ell \geqslant 1$. Then y is a non-zerodivisor on M.*

(ii) *Suppose that $Q^n M = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)M$ for all $n \geqslant 1$. Then*

$$Q_{s-1}M : y_s^\ell \subseteq QM + (0 :_M y_s^\ell)$$

for all $\ell \geqslant 1$.

(iii) *Suppose that $Q^n M = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)M$ for all $n \geqslant 1$. Then*

$$Q_i^n M = \bigcap_{\alpha \in \Lambda_{i,n}} Q_i(\alpha)M$$

for all $1 \leqslant i \leqslant s$ and $\ell \geqslant 1$.

In the above lemma, the key is assertion (ii), which is given in the proof of Lemma 3.2 in [GS1]. By this lemma one gets the following.

Lemma 3.4. *Suppose that (1) $Q^n M = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)M$ for all $n \ge 1$ and that (2) $0 :_M y_i^\ell = 0 :_M y_i$ for all $1 \le i \le s$ and $\ell \ge 1$. Then for all integers $1 \le i \le j \le s$, the element y_j is a non-zerodivisor on $M/[Q_{i-1}M + (0 :_M y_j)]$, so that one has the equality*

$$Q_{i-1}M : y_j^2 = Q_{i-1}M + (0 :_M y_j).$$

Proof. Since conditions (1) and (2) are independent of the order of $y_1, \ldots, y_s$, we may assume that $i = j$. Then we have

$$Q_{i-1}M : y_i^\ell \subseteq Q_iM + (0 :_M y_i)$$

for all $\ell \geqslant 1$, thanks to (ii) and (iii) of Lemma 3.3. Let $L = M/[Q_{i-1}M + 0 :_M y_i]$ and let $\alpha \in M$ be such that $y_i^\ell \overline{\alpha} = 0$ in L with $\ell \geqslant 1$, where $\overline{\alpha}$ denotes the image of α in L. Then $y_i^{\ell+1}\alpha \in Q_{i-1}M$, so that $\alpha \in Q_{i-1}M : y_i^{\ell+1} \subseteq Q_iM + (0 :_M y_i)$. Hence $\overline{\alpha} \in y_iL$, which shows, by (i) of Lemma 3.3, that y_i is a non-zerodivisor on L. Then the second conclusion is now clear. □

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. (i)⇒(ii). Let $\underline{x} = x_1, \ldots, x_d$ be a good system of parameters of M and $\mathfrak{q} = (x_1, \ldots, x_d)$. We prove by induction on the length t of the dimension filtration $\mathcal{D}$ of M that $\underline{x}$ has the property of parametric decomposition. The case $t = 0$ is trivial. Let $t \geq 1$. Set $\overline{M} = M/D_{t-1}$. Since $\overline{M}$ is a Cohen-Macaulay module, the sequence $x_1, \ldots, x_d$ is $\overline{M}$-regular. Then $\mathfrak{q}^n\overline{M} = \bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)\overline{M}$ by Lemma 3.2. Therefore $\bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M \subseteq \mathfrak{q}^nM + D_{t-1}$. Since $x_1^{\alpha_1}, \ldots, x_d^{\alpha_d}$ is a good system of parameters of M for each $\alpha \in \Lambda_{d,n}$, it follows from Lemma 3.1 that $\mathfrak{q}(\alpha)M \cap D_{t-1} = (x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_{d_{t-1}}^{\alpha_{d_{t-1}}})D_{t-1}$. Now, applying the inductive hypothesis on D_{t-1} we can show with the same method that was used in the proof of Proposition 2.2 of [GS2] that $\bigcap_{\alpha \in \Lambda_{d,n}} \mathfrak{q}(\alpha)M = \mathfrak{q}^nM$.

(ii)⇒(iii) is obvious.

(iii)⇒(i). Let $\underline{x} = x_1, \ldots, x_d$ be a good system of parameters of M having the property of parametric decomposition and $\mathfrak{q} = (x_1, \ldots, x_d)$. Since $0 :_M x_i^l = 0 :_M x_i$ for all $l \geq 1$ and $1 \leq i \leq d$ by Lemma 2.1, we get by Lemma 3.4 that $\mathfrak{q}_iM : x_j^2 = \mathfrak{q}_iM + (0 :_M x_j)$ for all $1 \leq i \leq j \leq d$. Therefore the implication follows by Theorem 3.9 of [CC], and the proof of Theorem 1.1 is complete. □

Proof of Corollary 1.2. (i)⇒(ii). It is easy to see from the hypothesis that M is a sequentially Cohen-Macaulay module with the dimension filtration $\mathcal{D} : H^0_{\mathfrak{m}}(M) \subset M$. Moreover, by Lemma 3.1 we have

$$(x_1, \ldots, x_d)M \cap H^0_{\mathfrak{m}}(M) = (x_1, \ldots, x_d)H^0_{\mathfrak{m}}(M) \subseteq \mathfrak{m}H^0_{\mathfrak{m}}(M) = 0$$

for any system of parameters $x_1, \ldots, x_d$ of M. This means that every system of parameters of M is good; therefore it has the property of parametric decomposition by Theorem 1.1.

(ii)⇒(i). First, it follows by Theorem 1.1 that M is sequentially Cohen-Macaulay. Remember by the definition of the dimension filtration of M that $D_0 = H^0_{\mathfrak{m}}(M)$ and $\dim D_i > 0$ for all $i > 0$. Therefore the implication is proved if we can show that $\mathfrak{m}D_{t-1} = 0$. Suppose the contrary. Then there is an element $x_1 \in \mathfrak{m}$ so that $x_1D_{t-1} \neq 0$ and $\dim M/x_1M = d-1$. Since $d \geqslant 2$, we can choose $x_2 \in \mathfrak{m}$ such that $x_2D_{t-1} = 0$ and $\dim M/(x_1, x_2)M = d - 2$. We observe that the sequence x_1, x_2 and $x_1, x_1 + x_2$ are part of systems of parameters of M. Therefore

$$(x_1^2, x_1 + x_2)M \cap (x_1, (x_1 + x_2)^2)M = (x_1, x_2)^2M = (x_1^2, x_2)M \cap (x_1, x_2^2)M.$$

Since M/D_{t-1} is Cohen-Macaulay, it follows from Lemma 3.1 that

$$\begin{aligned}x_1D_{t-1} &= (x_1^2, x_1+x_2)D_{t-1} \cap (x_1, (x_1+x_2)^2)D_{t-1} \\ &= (x_1^2, x_2)D_{t-1} \cap (x_1, x_2^2)D_{t-1} = x_1^2D_{t-1}.\end{aligned}$$

Thus $x_1D_{t-1} = 0$ by Nakayama's lemma, which is impossible. Hence $\mathfrak{m}D_{t-1} = 0$. □

4. Hilbert-Samuel polynomials

A parameter ideal $\mathfrak{q}$ is called a *good parameter ideal* if it is generated by a good system of parameters. Then, in this section we shall show that for a sequentially Cohen-Macaulay M the Hilbert-Samuel function $H_{\mathfrak{q},M}(n) = \ell(M/\mathfrak{q}^{n+1}M)$ has a special expression with non-negative coefficients, which can be computed by the dimension filtration, and this function coincides with the Hilbert-Samuel polynomial $P_{\mathfrak{q},M}(n)$ for any good parameter ideal $\mathfrak{q}$ of M and all $n \geqslant 1$. Moreover, the sequentially Cohen-Macaulayness of M can be characterized by this expression of the Hilbert-Samuel function.

Theorem 4.1. *Let $\mathcal{D} : D_0 \subset D_1 \subset \ldots \subset D_t = M$ be the dimension filtration of M and set $\mathcal{D}_i = D_i/D_{i-1}$ for all $1 \leq i \leq t$, $\mathcal{D}_0 = D_0$. Then the following statements are equivalent:*

(i) *M is a sequentially Cohen-Macaulay module.*

(ii) *For any good parameter ideal $\mathfrak{q}$ of M, it holds that*

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^{t} \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i)$$

for all $n \geqslant 0$.

(iii) *There exists a good parameter ideal $\mathfrak{q}$ of M such that*

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^{t} \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i)$$

for all $n \geqslant 0$.

Proof. (i)⇒(ii). We argue by the induction on the length t of the dimension filtration $\mathcal{D}$ of M. The case $t = 0$ is obvious. Assume that $t > 0$. By virtue of Lemma 3.1, we have a short exact sequence

$$0 \to D_{t-1}/\mathfrak{q}^{n+1}D_{t-1} \to M/\mathfrak{q}^{n+1}M \to M/\mathfrak{q}^{n+1}M + D_{t-1} \to 0.$$

Therefore, we have $\ell(M/\mathfrak{q}^{n+1}M) = \ell(D_{t-1}/\mathfrak{q}^{n+1}D_{t-1}) + \ell(\mathcal{D}_t/\mathfrak{q}^{n+1}\mathcal{D}_t)$. Since D_{t-1} is a sequentially Cohen-Macaulay module and its dimension filtration is of length $t-1$, it follows from the inductive hypothesis that

$$\ell(D_{t-1}/\mathfrak{q}^{n+1}D_{t-1}) = \sum_{i=0}^{t-1} \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i).$$

Notice that $\mathcal{D}_t$ is Cohen-Macaulay of dimension $d = d_t$, so we have

$$\ell(\mathcal{D}_t/\mathfrak{q}^{n+1}\mathcal{D}_t) = \binom{n+d}{d} \ell(\mathcal{D}_t/\mathfrak{q}\mathcal{D}_t).$$

Hence

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^{t} \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i),$$

for all $n \geqslant 0$.

(ii)⇒(iii) is trivial.

(iii)⇒(i). Since the following sequence is exact:

$$D_{t-1}/\mathfrak{q}^{n+1}D_{t-1} \to M/\mathfrak{q}^{n+1}M \to M/\mathfrak{q}^{n+1}M + D_{t-1} \to 0,$$

we get $\ell(M/\mathfrak{q}^{n+1}M) \leqslant \ell(D_{t-1}/\mathfrak{q}^{n+1}D_{t-1}) + \ell(\mathcal{D}_t/\mathfrak{q}^{n+1}\mathcal{D}_t)$. Therefore, by induction on the length of the dimension filtration we can show that

$$\ell(M/\mathfrak{q}^{n+1}M) \leqslant \sum_{i=0}^{t} \ell(\mathcal{D}_i/\mathfrak{q}^{n+1}\mathcal{D}_i).$$

On the other hand, since

$$\ell(\mathcal{D}_i/\mathfrak{q}^{n+1}\mathcal{D}_i) \leqslant \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i)$$

for all $0 \leq i \leq t$, it follows from the hypothesis that

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^{t} \ell(\mathcal{D}_i/\mathfrak{q}^{n+1}\mathcal{D}_i) = \sum_{i=0}^{t} \binom{n+d_i}{d_i} \ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i).$$

Therefore $\ell(\mathcal{D}_i/\mathfrak{q}^{n+1}\mathcal{D}_i) = \binom{n+d_i}{d_i}\ell(\mathcal{D}_i/\mathfrak{q}\mathcal{D}_i)$ for all $n \geq 0$ and $0 \leq i \leq t$. Thus $\mathcal{D}_i$ is Cohen-Macaulay for all $0 \leq i \leq t$, and this completes the proof. □

Acknowledgment

The authors would like to thank the referee for useful suggestions. Especially, Lemma 3.4 stated by the referee reduces remarkably the proof of Theorem 1.1.

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