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ON CLOSED δ-PINCHED MANIFOLDS WITH DISCRETE ABELIAN GROUP ACTIONS

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Abstract. Let M^n be a closed odd n-manifold with sectional curvature $\delta < \sec_M \leq 1$, and let M admit an effective isometric $\mathbb{Z}_p^k$-action with p prime. The main results in the paper are: (1) if $\delta > 0$ and $n \geq 5$, then there exists a constant $p(n,\delta)$, depending only on n and δ, such that $p \geq p(n,\delta)$ implies that (i) $k \leq \frac{n+1}{2}$, (ii) the universal covering space of M is homeomorphic to S^n if $k > \frac{3}{8}n+1$, (iii) the fundamental group $\pi_1(M)$ is cyclic if $k > \frac{n+1}{4}+1$; (2) if $\delta = 0$ and $n = 3$, then $k \leq 4$ for $p = 2$ and $k \leq 2$ for $p \geq 3$, and $\pi_1(M)$ is cyclic if $p \geq 5$ and $k = 2$.

0. Introduction

In the past decade, the closed positively curved n-manifold, M^n, with symmetry has been researched. Much investigation focuses on the case that M^n admits an effective isometric torus T^k-action with k large ([8], [3], [5], [13]–[16], [18]). In this field, the original work is by Grove and Searle [8], and a dramatic improvement after it is the work by Wilking [18]. Roughly, their work shows that the manifold M is constrained to have the cohomology of a symmetry space if k is large enough. (The present paper was inspired by and coincides with this line.) A natural step is to further replace the T^k-action by a disconnected group action. Fang and Rong [4] studied M^n which admits an effective isometric $\mathbb{Z}_p^k$-action or a $T^1 \oplus \mathbb{Z}_p^k$-action with p prime for n even or odd respectively, where $p \geq p(n)$, a constant depending only on n. A natural problem [4] is: *to research M^n with n odd which admits an effective isometric $\mathbb{Z}_p^k$-action.*

Due to the problem above, the present paper obtains the following result.

Theorem A. *Let M^n be a closed odd n-manifold with sectional curvature $0 < \delta < \sec_M \leq 1$ and $n \geq 5$. Suppose M admits an effective isometric $\mathbb{Z}_p^k$-action. Then there exists a constant $p(n,\delta)$, depending only on n and δ, such that $p \geq p(n,\delta)$ implies:*

1. $k \leq \frac{n+1}{2}$.
2. *The universal covering space of M is homeomorphic to S^n if $k > \frac{3}{8}n+1$.*
3. *The fundamental group $\pi_1(M)$ is cyclic if $k > \frac{n+1}{4}+1$.*

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Remark 0.1. When $n \geq 5$, it is easy to check that '$k = \frac{n+1}{2}$' implies '$k > \frac{3}{8}n + 1$', and that '$k > \frac{3}{8}n + 1$' implies '$k > \frac{n+1}{4} + 1$'. Thus if $k = \frac{n+1}{2}$, or if $k > \frac{3}{8}n + 1$ in Theorem A, then M is homeomorphic to $S^n/\mathbb{Z}_h$ (but we cannot make sure whether $\mathbb{Z}_h$ is conjugate to a linear action).

Remark 0.2. Theorem A was originally inspired by the results in [8] and [4]. Let M^n be a closed n-manifold of positive sectional curvature. The main result in [8] asserts that if M^n admits an effective isometric torus T^k-action, then $k \leq [\frac{n+1}{2}]$ and '=' implies that M is diffeomorphic to a sphere, a lens space or a complex projective space. The main results in [4] are: there exists a constant $p(n)$, depending only on n, such that if a simply connected M^n admits an effective isometric $\mathbb{Z}_p^k$-action or $T^1 \oplus \mathbb{Z}_p^k$-action with prime $p \geq p(n)$ for $n = 2m$ or $2m + 1$ respectively, then $k \leq m$; and that if $k = m$ in addition or if $m \geq 7$ and $k \geq [\frac{3m}{4}] + 2$, then M is homeomorphic to a sphere or a complex projective space.

When $n = 3$, we get the following result (mainly due to the Hamilton's work. See Theorem 5.1 below).

Theorem B. *Let M^3 be a closed 3-manifold of positive sectional curvature. If M admits an effective isometric $\mathbb{Z}_q^k$-action with q prime, then $k \leq 4$ for $q = 2$ and $k \leq 2$ for $q \geq 3$. In addition, $\pi_1(M)$ is cyclic if $q \geq 5$ and $k = 2$.*

Remark 0.3. In Theorem B, '$q \geq 5$ and $k = 2$' is optimal for '$\pi_1(M)$ is cyclic'. One can check that space forms S^3/D_8^* and S^3/T^* admit effective isometric $T^1 \oplus \mathbb{Z}_3$- and $T^1 \oplus \mathbb{Z}_2$-actions respectively (ref. [12]), where D_8^* and T^* are groups in the proof of the lemma in the Appendix.

The rest of the paper is organized as follows:
In Section 1, we show the choice of $p(n, \delta)$ in Theorem A.
In Sections 2-4, we will prove parts 1-3 of Theorem A respectively.
In Section 5, we will give the proof of Theorem B.

1. The choice of $p(n, \delta)$ in Theorem A

The choice of $p(n, \delta)$ in Theorem A is due to the following two results.

Theorem 1.1 [14]**.** *Let M^n be a closed n-manifold with sectional curvature $0 < \delta < \sec_M \leq 1$. Then $\pi_1(M)$ has a finite normal cyclic subgroup with index less than $\omega(n, \delta)$, a constant depending only on n and δ.*

Remark 1.2. X. Rong supplied a conjecture [14]: *Let M^n be a closed n-manifold of positive sectional curvature. Then $\pi_1(M)$ has a finite normal cyclic subgroup with index less than $\omega(n)$, a constant depending only on n.* It should be pointed out that, according to the proofs of the present paper, the conclusions in Theorem A will hold for manifolds of positive sectional curvature (i.e., $\delta = 0$) once the conjecture is verified.

Theorem 1.3 [7]**.** *Let M^n be a closed n-manifold with non-negative sectional curvature. Then the total Betti number of M, with respect to any coefficient field, is less than $c(n)$, a constant depending only on n.*

Assertion. *We choose the constant $p(n, \delta)$ in Theorem A satisfying that $p(n, \delta) \geq \max\{\omega(n, \delta), c(n)\}$ and $p(n + 1, \delta) \geq p(n, \delta) > 2$.*

Remark 1.4. Weinstein's theorem [2] asserts that a closed positively curved manifold M of odd dimension is orientable. Thus if $\mathbb{Z}_p^l$ acts isometrically on M with prime $p > 2$, then $\mathbb{Z}_p^l$ preserves the orientation of M. In addition, if $\mathbb{Z}_p^l$ has a non-empty fixed point set, which is totally geodesic, then it is of even codimension.

2. The proof of Part 1 of Theorem A

Lemma 2.1. *Let M^n be a closed n-manifold with sectional curvature $0 < \delta \leq \sec_M \leq 1$. Then the $\mathbb{Z}_p^2$ group with prime $p \geq p(n, \delta)$ cannot act on M freely and isometrically.*

Proof. We argue by contradiction. Assume that $\mathbb{Z}_p^2$ $(p \geq p(n, \delta))$ acts on M freely and isometrically. Note that $\hat{M} = M/\mathbb{Z}_p^2$ is also a Riemannian manifold with $\delta < \sec_{\hat{M}} \leq 1$. Let $\pi : \tilde{M} \to M$ be the universal covering map. Then $\hat{M} = \tilde{M}/\pi_1(\hat{M})$, and there is the following exact sequence ([1], p. 66):

$$0 \longrightarrow \pi_1(M) \longrightarrow \pi_1(\hat{M}) \xrightarrow{f} \mathbb{Z}_p \oplus \mathbb{Z}_p \longrightarrow 0.$$

Note that we can assume $\pi_1(\hat{M}) = \langle \pi_1(M), \alpha, \beta \rangle$ with $f(\alpha)$ and $f(\beta)$ being generators of $\mathbb{Z}_p \oplus \mathbb{Z}_p$. According to the choice of $p(n, \delta)$ and Theorem 1.1, $\pi_1(\hat{M})$ contains a normal cyclic subgroup, say $\langle \gamma \rangle$, such that $[\pi_1(\hat{M}) : \langle \gamma \rangle] < p$. Then $\alpha^h, \beta^j \in \langle \gamma \rangle$ for some h, j with $0 < h < p$ and $0 < j < p$, so $\langle \alpha^h, \beta^j \rangle$ is a cyclic subgroup. Hence

$$\langle f(\alpha^h), f(\beta^j) \rangle = \langle (f(\alpha))^h, (f(\beta))^j \rangle = \langle f(\alpha), f(\beta) \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$$

is a cyclic group, a contradiction. □

Remark 2.2. Using Lemma 2.1, we can get that $\mathbb{Z}_p^k$ with $p \geq p(n, \delta)$ has an isotropy subgroup of rank $k-1$ if M admits an effective isometric $\mathbb{Z}_p^k$ action (see the following proof). According to [4], parts 1 and 2 of Theorem A can be derived from the proofs in [4] if M is simply connected.

Proof of Part 1 of Theorem A. By Lemma 2.1, $\mathbb{Z}_p^k$ cannot act freely on M for $k \geq 2$, i.e., we can find an isotropy subgroup $\mathbb{Z}_p^l$ with $l \geq 1$. Take a component N of $F(\mathbb{Z}_p^l, M)$, the fixed point set of $\mathbb{Z}_p^l$.

Claim. $N = F(\mathbb{Z}_p^l, M)$. Note that $\alpha(N)$ is also a component of $F(\mathbb{Z}_p^l, M)$ for any $\alpha \in \mathbb{Z}_p^k$. If the claim is not true, then $F(\mathbb{Z}_p^l, M)$ contains at least p components and thus $\sum_{i=0}^n \operatorname{rank}(H_i(F(\mathbb{Z}_p^l, M); \mathbb{Z}_p)) \geq 2p$. On the other hand, by Theorem 2.3 below and Theorem 1.3

$$\sum_{i=0}^n \operatorname{rank}(H_i(F(\mathbb{Z}_p^l, M); \mathbb{Z}_p)) \leq \sum_{i=0}^n \operatorname{rank}(H_i(M; \mathbb{Z}_p)) \leq p.$$

Thus we get a contradiction.

By Remark 1.4, N is a totally geodesic submanifold of even codimension. Consider the induced action $\mathbb{Z}_p^k|_N = \mathbb{Z}_p^k/\mathbb{Z}_p^l \cong \mathbb{Z}_p^{k-l}$ on N. Repeating the process above, we can find a $\mathbb{Z}_p^{k-1}$-fixed point component N_0 of codimension $2m$.

Note that $\mathbb{Z}_p^{k-1}$ can act on the normal space of N_0 as a subgroup of $SO(2m)$. Then $2m \geq 2(k-1)$, i.e., $k \leq m + 1 \leq \frac{n-1}{2} + 1 = \frac{n+1}{2}$. □

Theorem 2.3 [1, p. 163]**.** *Let the group $G \cong \mathbb{Z}_q$ with q prime act on a closed n-manifold M^n. Then* $\sum_{i=0}^n \operatorname{rank}(H_i(F(G, M); \mathbb{Z}_q)) \leq \sum_{i=0}^n \operatorname{rank}(H_i(M; \mathbb{Z}_q))$.

3. The proof of Part 2 of Theorem A

We will use the following connectedness theorem by B. Wilking.

Theorem 3.1 [18]. *Let N^n be a closed n-manifold of positive sectional curvature, and let $L^l \subset N^n$ be a closed totally geodesic embedded submanifold. Then the inclusion map $L^l \hookrightarrow N^n$ is $(2l-n+1)$-connected.*

Remark 3.2. We say that the inclusion map $L \hookrightarrow N$ is *i-connected* if the homotopy groups $\pi_j(N, L) = 0$ for $0 \leq j \leq i$. Then $\pi_1(N) \cong \pi_1(L)$ if $i \geq 2$, and $H_j(N, L; \mathbb{Z}) = 0$ for $0 \leq j \leq i$ (the Hurewicz theorem).

Corollary 3.3 [18]. *Let N^n and L^l be the manifolds in Theorem 3.1. If n is odd and $l = n-2$, then the universal covering space of N is an integer homology sphere.*

Remark 3.4. In Theorem A, to prove that the universal covering space of M, $\tilde{M}$ is homeomorphic to S^n, one only needs to verify that $\tilde{M}$ is an integer homology sphere ([5], [17]).

Proof of Part 2 of Theorem A. By Remark 2.2, there is a $\mathbb{Z}_p^{k-1}$-fixed point component N_0. Analyzing the representation of $\mathbb{Z}_p^{k-1}$ on the normal space of N_0, we can take a $\mathbb{Z}_p$-fixed point component N such that the effective part of $\mathbb{Z}_p^k|_N$ is isomorphic to a $\mathbb{Z}_p^{k-1}$ group.

If $5 \leq n \leq 11$, then $k > \frac{3}{8}n + 1$ implies $k = \frac{n+1}{2}$ (see part 1 of Theorem A), and thus $\dim(N) = n-2$. Hence the universal covering space of M is homeomorphic to S^n by Corollary 3.3 (see Remark 3.4).

Assume that $n \geq 13$. It is not hard to check that $\dim(N) \geq \frac{3}{4}(n-1)$ by part 1 of Theorem A (note that n is odd). Then the inclusion map $i: N \hookrightarrow M$ is at least $\frac{n-1}{2}$-connected by Theorem 3.1, so $i_*(\pi_1(N)) = \pi_1(M)$ (see Remark 3.2). Let $\pi: \tilde{M} \to M$ be the universal covering map. Then $\pi^{-1}(N)$ is simply connected, and so $i: \pi^{-1}(N) \hookrightarrow \tilde{M}$ is also $\frac{n-1}{2}$-connected.

On the other hand, we can assume $\dim(N) \leq n-4$ by Corollary 3.3; then the effective $\mathbb{Z}_p^{k-1}$-action on N satisfies $k-1 > \frac{3}{8}\dim(N) + 1$. By induction we can assume that $\pi^{-1}(N)$, the universal covering space of N, is homeomorphic to a sphere. Then $\tilde{M}$ is an integer homology sphere because $i: \pi^{-1}(N) \hookrightarrow \tilde{M}$ is $\frac{n-1}{2}$-connected (see Remark 3.2). Hence $\tilde{M}$ is homeomorphic to S^n indeed by Remark 3.4. □

4. The proof of Part 3 of Theorem A

In the proof of Part 3 of Theorem A, we will use:

Lemma 4.1 [5]. *Let N^n $(n \geq 5)$ be a closed positively curved n-manifold, and let L be a closed totally geodesic embedded submanifold of codimension 2. Then $\pi_1(N)$ is cyclic.*

Proof of Part 3 of Theorem A. As the proof of Part 2 of Theorem A, take a $\mathbb{Z}_p$-fixed point component N which admits an effective $\mathbb{Z}_p^{k-1}$-action.

When $n = 5$ and 7, '$k > \frac{n+1}{4} + 1$' implies $k = \frac{n+1}{2}$. Then as in the proof of Part 2 of Theorem A, $\dim(N) = n-2$, so $\pi_1(M)$ is cyclic by Lemma 4.1.

Assume that $n \geq 9$. One can check that $\dim(N) \geq \frac{n+1}{2}$ by Part 1 of Theorem A. Then the inclusion map $i: N \hookrightarrow M$ is at least 2-connected by Theorem 3.1, so $i_*(\pi_1(N)) = \pi_1(M)$ (see Remark 3.2). On the other hand, we can assume

$\dim(N) \leq n-4$ by Lemma 4.1; then the effective $\mathbb{Z}_p^{k-1}$-action on N satisfies $k-1 > \frac{\dim(N)+1}{4}+1$. By induction we can get that $\pi_1(N)$ is cyclic, so $\pi_1(M)$ is cyclic because $\pi_1(N) \cong \pi_1(M)$. $\square$

5. The proof of Theorem B

In this section, the following remarkable result by R. Hamilton will be a basis.

Theorem 5.1 [9]. *Let M^3 be a closed 3-manifold of positive sectional curvature. If a group G acts isometrically on M, then M admits a metric of positive constant sectional curvature for which G acts isometrically.*

Using Theorem 5.1, we give the following lemma.

Lemma 5.2. *Let M^3 be a closed 3-manifold of positive sectional curvature. If the isometry group of M, $\mathrm{Iso}(M)$, contains a $\mathbb{Z}_q^k$ subgroup with q prime, then $k \leq 4$ for $q=2$ and $k \leq 2$ for $q \geq 3$.*

Proof. We first prove that $k \leq 2$ for $q \geq 3$. We claim that $\mathbb{Z}_q^2$ cannot act freely on M. Assuming the claim, we will prove that $k \leq 2$. Note that we can assume that $k \geq 2$. By the claim, we can find $e \neq \alpha \in \mathbb{Z}_q^k$ such that α has a non-empty fixed point set and that $\dim(F(\alpha, M)) = 1$ (see Remark 1.4). Note that M is a positive 3-space form by Theorem 5.1; then by Theorem 2.3 and the lemma in the Appendix,

$$\sum_{i=0}^{1} \mathrm{rank}(H_i(F(\alpha, M); \mathbb{Z}_q)) \leq \sum_{i=0}^{3} \mathrm{rank}(H_i(M; \mathbb{Z}_q)) \leq 4.$$

Thus $F(\alpha, M)$ contains at most 2 components; then $\mathbb{Z}_q^k$ preserves each component of $F(\alpha, M)$. That is, the isotropy group of $\mathcal{O}$, a component of $F(\alpha, M)$, contains a $\mathbb{Z}_q^{k-1}$ subgroup. Therefore $k \leq 2$ because $\mathbb{Z}_q^{k-1}$ can act faithfully on the normal space of $\mathcal{O}$ (note that $\mathcal{O}$ is of codimension 2).

Now we prove the above claim. Assume that $\mathbb{Z}_q^2$ acts freely on M. Note that $M/\mathbb{Z}_q^2$ is also a 3-manifold of positive sectional curvature, and thus it is also a positive 3-space form by Theorem 5.1. Because $\pi_1(M/\mathbb{Z}_q^2)/\pi_1(M) \cong \mathbb{Z}_q^2$, $H_1(M/\mathbb{Z}_q^2; \mathbb{Z})$ contains a $\mathbb{Z}_q^2$ subgroup (recall that $H_1(M/\mathbb{Z}_q^2; \mathbb{Z}) \cong \pi_1(M/\mathbb{Z}_q^2)/C$, where C is the commutator subgroup of $\pi_1(M/\mathbb{Z}_q^2)$). This is a contradiction to the lemma in the Appendix.

The proof for $q=2$ is similar to the above (the difference is that there exists a $\mathbb{Z}_2^3$ subgroup (note that we can assume that $k \geq 4$) which preserves the orientation of M such that it cannot act freely on M). $\square$

Next we will give Lemma 5.3. Note that Lemmas 5.2 and 5.3 together imply Theorem B.

Lemma 5.3. *Let M^3 be a closed 3-manifold of positive sectional curvature. If $\mathrm{Iso}(M)$ contains a $\mathbb{Z}_q^2$ subgroup with prime $q \geq 5$, then $\pi_1(M)$ is cyclic.*

Before proving Lemma 5.3, we first observe the following lemma.

Lemma 5.4. *Let M^3 be a closed 3-manifold of positive sectional curvature. If $\mathrm{Iso}(M)$ contains a $\mathbb{Z}_q^2$ subgroup with prime $q \geq 5$, then $\mathrm{Iso}(M)$ contains a $T^1 \oplus \mathbb{Z}_q$ subgroup, and $\mathbb{Z}_q$ preserves every exceptional T^1-orbit.*

Proof. We first prove that $\mathrm{Iso}(M)$ contains a $T^1 \oplus \mathbb{Z}_q$ subgroup. According to p. 108 in [12], $k = \mathrm{rank}(\mathrm{Iso}(M)) \geq 1$; i.e., $\mathrm{Iso}(M)$ contains a torus T^k subgroup with $k \geq 1$, because M is a space form of positive constant sectional curvature by Theorem 5.1. If $\mathrm{rank}(\mathrm{Iso}(M)) \geq 2$, the conclusion is obvious. If $\mathrm{rank}(\mathrm{Iso}(M)) = 1$, recall that the identity component $\mathrm{Iso}_0(M)$ of $\mathrm{Iso}(M)$ is T^1, or $SO(3)$, or $SU(2)$.

Endow $\mathrm{Iso}(M)$ with a bi-invariant metric. Then the conjugate map

$$\mathrm{Iso}(M) \times \mathrm{Iso}_0(M) \longrightarrow \mathrm{Iso}_0(M), (g, g_0) \longmapsto g g_0 g^{-1}$$

induces an $\mathrm{Iso}(M) \subset O(n)$ action on $T_e(\mathrm{Iso}_0(M))$, the tangent space at e, where $n = \dim(\mathrm{Iso}(M))$. Note that $n = 1$ or 3; then the $\mathbb{Z}_q^2 \subset \mathrm{Iso}(M) \subseteq O(n)$ action on $\mathbb{R}^n$ has a non-empty fixed point set. Take a line tX fixed by $\mathbb{Z}_q^2$, where $X \in T_e(\mathrm{Iso}_0(M))$. The subgroup $\overline{\exp(tX)}$, the closure of $\exp(tX)$, commutes with the subgroup $\mathbb{Z}_q^2$ ($\subset \mathrm{Iso}(M)$), where $\exp : T_e(\mathrm{Iso}_0(M)) \to \mathrm{Iso}_0(M)$ is the exceptional map. Note that $\overline{\exp(tX)}$ contains a T^1-subgroup; i.e., we find a $T^1 \oplus \mathbb{Z}_q$ subgroup.

Assume that there is $\mathbb{Z}_r \subset T^1$ with r prime such that $F(\mathbb{Z}_r, M) \neq \emptyset$. Note that T^1 preserves the orientation of M; then $\dim(F(\mathbb{Z}_r, M)) = 1$ (see Remark 1.4). By Theorem 2.3 and the lemma in the Appendix

$$\sum_{i=0}^{1} \mathrm{rank}(H_i(F(\mathbb{Z}_r, M); \mathbb{Z}_r)) \leq \sum_{i=0}^{3} \mathrm{rank}(H_i(M; \mathbb{Z}_r)) \leq 6.$$

Thus $F(\mathbb{Z}_r, M)$ contains at most 3 components. Therefore if $q \geq 5$, then the $\mathbb{Z}_q$ in $T^1 \oplus \mathbb{Z}_q$ preserves every component of $F(\mathbb{Z}_r, M)$ (note that $\mathbb{Z}_q$ preserves $F(\mathbb{Z}_r, M)$). □

In the rest, we will only need to give the proof of Lemma 5.3, in which the following results will be used.

Lemma 5.5 [5]**.** *Let M be a closed Riemannian manifold on which T^1 acts isometrically. If there is an isometry ϕ on M^*, then $\chi(F(\phi, M^*)) = \mathrm{Lef}(\phi; M^*)$, where $M^* = M/T^1$.*

Recall that the Lefschetz number $\mathrm{Lef}(\phi; M^*) = \sum_i (-1)^i \mathrm{trace}(\phi_{i*})$, where $\mathrm{trace}(\phi_{i*})$ is the trace of the induced map by ϕ on $H_i(M^*; \mathbb{Q})$. Lemma 5.5 generalizes the result ([10], p. 63): *any isometry ϕ on a closed Riemannian manifold M satisfies $\chi(F(\phi, M)) = Lef(\phi; M)$.*

Lemma 5.6 [8]**.** *Let M be a closed manifold of positive sectional curvature on which T^1 acts isometrically. If T^1 has fixed point set of codimension 2, then $\pi_1(M)$ is cyclic.*

Note that Lemma 4.1 is an extending version of Lemma 5.6.

Lemma 5.7 [1, p. 91]**.** *Let G be a connected compact Lie group, and let X be an arcwise connected G-space. Then the projectional map $p : X \to X/G$ induces an onto map on fundamental groups.*

Proof of Lemma 5.3. By Lemma 5.4, $\mathrm{Iso}(M)$ contains a $T^1 \oplus \mathbb{Z}_q$ subgroup, and $\mathbb{Z}_q$ preserves every exceptional T^1-orbit.

Claim. we can assume that M^* is homeomorphic to S^2, where $M^* = M/T^1$. Let α be the generator of $\mathbb{Z}_q$, and let $\hat{\alpha}$ denote the induced action by α on M^*. Assuming the claim, by Lemma 5.5 we can get

$$\chi(F(\hat{\alpha}, M^*)) = \mathrm{Lef}(\hat{\alpha}; M^*) = 2$$

(note that α preserves the orientation of M. See Remark 1.4). Then $F(\hat{\alpha}, M^*)$ is M^* or two points. If $F(\hat{\alpha}, M^*) = M^*$, then α preserves every T^1-orbit on M, and thus $T^1 \oplus \mathbb{Z}_q$-action is not effective, a contradiction. Then $F(\hat{\alpha}, M^*)$ contains only two points. In other words, α preserves only two T^1-orbits on M. Since α preserves every exceptional T^1-orbit on M, there are at most two exceptional orbits on M; i.e., M^* contains at most two singular points. Hence M is a gluing of two solid tori, so M is homeomorphic to a lens space, and thus $\pi_1(M)$ is cyclic.

Next we will prove the claim above. By Lemma 5.6, we can assume that the T^1-action on M has an empty fixed point set. Then M^* is an orientable manifold of dimension 2. On the other hand, $\pi_1(M^*)$ is finite by Lemma 5.7. According to the classification of closed orientable surfaces, M^* is homeomorphic to S^2. □

Appendix

Lemma. *Let M^3 be a positive 3-space form. Then its first homology group $H_1(M;\mathbb{Z}) \cong 0$, or $\mathbb{Z}_h$ with $h \geq 2$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_{2l}$ with l odd.*

Proof. According to p. 111 in [12] (cf. [11]), $\pi_1(M)$ is one group of the following list: C_m, D^*_{4m}, $D'_{2^k(2n+1)}$, T^*, $T'_{8\cdot 3^k}$, O^*, I^* and the direct product of any of these groups with a cyclic group of relatively prime order. In this list,

C_m is a cyclic group of order m,

$D^*_{4m} = \{x, y | x^2 = (xy)^2 = y^m\}$,

$T^*, O^*, I^* = \{x, y | x^2 = (xy)^3 = y^n, x^4 = 1\}$ for $n = 3, 4, 5$ respectively,

$D'_{2^k(2n+1)} = \{x, y | x^{2^k} = 1, y^{2n+1} = 1, xy = y^{-1}x\}$ with $k \geq 2$ and $n \geq 1$,

$T'_{8\cdot 3^k} = \{x, y, z | x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1\}$ with $k \geq 1$.

Recall that $H_1(M;\mathbb{Z}) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)]$, where $[\pi_1(M), \pi_1(M)]$ denotes the commutator subgroup of $\pi_1(M)$. One can check that $H_1(M;\mathbb{Z}) \cong \mathbb{Z}_m$, $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, 0, $\mathbb{Z}_{2^k}$ and $\mathbb{Z}_{3^k}$ when $\pi_1(M) \cong C_m$, D^*_{4m} with m odd or even, T^*, O^*, I^*, $D'_{2^k(2n+1)}$ and $T'_{8\cdot 3^k}$ respectively (for example, $[D^*_{4m}, D^*_{4m}] = \{xyx^{-1}y^{-1} = xyxx^{-2}y^{-1} = (xy)^2y^{-m-2} = y^{-2}\}$, so $D^*_{4m}/[D^*_{4m}, D^*_{4m}] \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ for m odd or even respectively). □

In the proof above, T^*, O^* and I^* are a binary tetrahedral group of order 24, a binary octahedral group of order 48, and a binary icosahedral group of order 120 respectively.

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