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ON THE RELATION BETWEEN THE GENERALIZED POINCARÉ SERIES AND THE STÖHR ZETA FUNCTION

FÉLIX DELGADO DE LA MATA AND JULIO JOSÉ MOYANO FERNÁNDEZ

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ABSTRACT. The aim of this paper is to show the relation between the zeta function introduced by Stöhr and the Poincaré series of a curve singularity introduced by Campillo, Delgado and Gusein-Zade for the complex case. The interpretation of the Stöhr zeta function in terms of integrals with respect to the (generalized) Euler characteristic over suitable subsets of the ring of functions (following the similar construction made by the previously named authors for subsets of the projectivization of the ring) provides the bridge between both subjects.

Let $\mathcal{O}$ be a one-dimensional Cohen-Macaulay Noetherian local ring containing a finite field $\mathbb{F}_q$, with maximal ideal $\mathfrak{m}$. Let $\mathcal{K}$ be the total ring of fractions of $\mathcal{O}$ and let $\overline{\mathcal{O}}$ be the integral closure of $\mathcal{O}$ in $\mathcal{K}$. Assume the degree $\rho := [\mathcal{O}/\mathfrak{m} : \mathbb{F}_q]$ to be finite.

Let R be a ring having a large Jacobson radical (i.e., every prime ideal of R containing the Jacobson radical of R is maximal) and which is its own ring of fractions. A subring $V \neq R$ of R having R as a ring of fractions is called a *Manis valuation ring* of R if, for every regular element $x \in R$, we have either $x \in V$ or $x^{-1} \in V$. A *discrete Manis valuation* of R is a surjective map $v : R \to \mathbb{Z} \cup \{\infty\}$ such that $v(1) = 0$, $v(0) = \infty$ and, for all $a, b \in R$, $v(ab) = v(a) + v(b)$ and $v(a+b) \geq \min(\{v(a), v(b)\})$. Discrete Manis valuations give rise to discrete Manis valuation rings and conversely (see [Ki-Vi], I-(2.2)).

The integral closure $\overline{\mathcal{O}}$ decomposes as a finite intersection of discrete Manis valuation rings (see [Ki-Vi] for more details) $\overline{\mathcal{O}} = V_1 \cap \ldots \cap V_r$, with associated discrete Manis valuations $v_1, \ldots, v_r$. For every $i \in \{1, \ldots, r\}$, we denote $\mathfrak{m}_i := \mathfrak{m}(V_i) \cap \overline{\mathcal{O}}$, where $\mathfrak{m}(V_i)$ is the maximal ideal of V_i.

1. Stöhr zeta function.

(1.1) In [St] the so-called *Stöhr zeta function* is defined as:

$$\zeta(\mathcal{O}, s) := \sum_{\mathfrak{a} \supseteq \mathcal{O}} \sharp\, (\mathfrak{a}/\mathcal{O})^{-s}\,,$$

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where the summation runs through all fractional ideals containing $\mathcal{O}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Following [St], putting $t = q^{-s}$ and writing $Z(\mathcal{O}, t)$ instead of $\zeta(\mathcal{O}, s)$ (just to mark the difference with the parameter's dependency), the Stöhr zeta function splits into a finite sum of partial zeta functions $Z(\mathcal{O}, t) = \sum_{(\mathfrak{b})} Z(\mathcal{O}, \mathfrak{b}, t)$, where $\mathfrak{b}$ is a fractional $\mathcal{O}$-ideal satisfying $\overline{\mathcal{O}} \cdot \mathfrak{b} = \overline{\mathcal{O}}$. The summation varies over a complete system of representatives of the ideal class semigroup of $\mathcal{O}$. For each partial zeta function one has

$$Z(\mathcal{O}, \mathfrak{b}, t) = \sum_{\mathfrak{a} \supseteq \mathcal{O}}^{\mathfrak{a} \sim \mathfrak{b}} t^{\dim_{\mathbb{F}_q}(\mathfrak{a}/\mathcal{O})}. \tag{1}$$

The notation $\mathfrak{a} \sim \mathfrak{b}$ means that $\mathfrak{a} = z^{-1}\mathfrak{b}$ for a non-zero divisor $z \in \mathcal{K}$. In fact Stöhr proves (see [St], Theorem 3.1) that this zeta function can be expressed in terms of the set $S(\mathfrak{b}) := \{(v_1(g), \dots, v_r(g)) \mid g \in \mathfrak{b},\ g \text{ non-zero divisor}\}$ and the *degree* of $\mathfrak{b}$, denoted by $\deg(\mathfrak{b})$, which is the function such that $\deg(\mathcal{O}) = 0$ and $\dim_{\mathbb{F}_q}(\mathfrak{a}/\mathfrak{b}) = \deg(\mathfrak{a}) - \deg(\mathfrak{b})$ for two fractional $\mathcal{O}$-ideals $\mathfrak{a} \supseteq \mathfrak{b}$. More precisely, if $d_i := [\overline{\mathcal{O}}/\mathfrak{m}_i : \mathbb{F}_q]$, he shows that

$$Z(\mathcal{O}, \mathfrak{b}, t) = \frac{t^{\deg(\mathfrak{b})}}{(U_{\mathfrak{b}} : U_{\mathcal{O}})} \frac{q^\rho}{q^\rho - 1} \sum_{\underline{v} \in S(\mathfrak{b})} \sum_{\underline{i} = \underline{0}}^{\underline{1}} (-1)^{|\underline{i}|} q^{\underline{v} \cdot \underline{d} + \deg(\mathfrak{b} \cap \mathfrak{m}^{\underline{v} + \underline{i}})} t^{\underline{v} \cdot \underline{d}}, \tag{2}$$

where $\underline{i} = (i_1, \dots, i_r)$, $|\underline{i}| = i_1 + \dots + i_r$, $\underline{v} = (v_1, \dots, v_r) \in \mathbb{Z}^r$, $\underline{v} \cdot \underline{d} := v_1 d_1 + \dots + v_r d_r$, $\underline{0} := (0, \dots, 0)$ and $\underline{1} := (1, \dots, 1)$. Moreover, $(U_{\mathfrak{b}} : U_{\mathcal{O}})$ is the index of the subgroup $U_{\mathcal{O}}$ of units of $\mathcal{O}$ over the units of $\mathfrak{b}$.

(1.2) Through the proof of Theorem 3.1 in [St], Stöhr introduces the measure of a fractional $\mathcal{O}$-ideal $\mathfrak{a}$ as $\mu(\mathfrak{a}) := q^{\deg(\mathfrak{a})}$. Notice that if $\mathfrak{a} \subset \mathcal{O}$ is an ideal, then $\mu(\mathfrak{a}) = \frac{1}{\sharp(\mathcal{O}/\mathfrak{a})}$. We can extend μ to the system of subsets $\mathcal{M}$ of $\mathcal{K}$ which are obtained from the set of fractional ideals by adding an element of $\mathcal{K}$ (endowed with the operations union and intersection). Now, μ is uniquely determined by the following three properties:

(1) $\mu(\mathcal{O}) = 1$.
(2) $\mu(z + M) = \mu(M)$, for every $z \in \mathcal{K}$ and $M \in \mathcal{M}$.
(3) $\mu(M \cup N) = \mu(M) + \mu(N) - \mu(M \cap N)$ for every $M, N \in \mathcal{M}$.

2. The generalized Euler characteristic.

(2.1) Let $K_0(\mathcal{V}_K)$ be the Grothendieck ring of the set $\mathcal{V}_K$ of reduced quasi-projective varieties X defined over a perfect field K, that is, the ring consisting of symbols $[X]$ of such varieties subject to the following relations:

- $[X_1] = [X_2]$ if X_1 is isomorphic to X_2, for $X_1, X_2 \in \mathcal{V}_K$.
- $[X] = [Y] + [X \setminus Y]$ if Y is a Zariski closed set in X, for $X \in \mathcal{V}_K$.
- $[X_1 \times_K X_2] = [X_1][X_2]$ for $X_1, X_2 \in \mathcal{V}_K$.

Note that $X_1 \times_K X_2$ is again a reduced quasi-projective variety if and only if K is perfect. The symbol of the affine line $[\mathbb{A}^1_K]$ will be denoted by $\mathbb{L}$, and it is different from 0. We also consider the localization $K_0(\mathcal{V}_K)_{\mathbb{L}}$.

(2.2) Let V be a vector space of finite dimension d over an arbitrary field K. Let $SV^\vee = \bigoplus_{i \geq 0} S^i V^\vee$ be the symmetric algebra of the dual space $V^\vee$ of V. It is known that if we fix a basis $v_1, \dots, v_d$ of V, then $SV^\vee$ could be identified with the polynomial ring $K[X_1, \dots, X_d]$ (where the set of indeterminates $X_1, \dots, X_d$

corresponds to the dual basis $v_1^\vee, \ldots, v_d^\vee$ of V). Let us denote by $\mathbb{A}[V]$ the affine scheme $\mathrm{Spec}(SV^\vee) = \mathrm{Spec}(K[X_1, \ldots, X_d])$ and by $\mathbb{P}[V] = \mathrm{Proj}(SV^\vee)$ the projective scheme of the graded K-algebra $K[X_1, \ldots, X_d]$. Note that the set of K-rational (closed) points $\mathbb{A}[V](K)$ of $\mathbb{A}[V]$ could be identified with the set of vectors of V (the vector of coordinates $(a_1, \ldots, a_n)$ corresponds to the maximal ideal $(X_1 - a_1, \ldots, X_d - a_d)$). Of course the set of K-rational closed points $\mathbb{P}[V](K)$ of $\mathbb{P}[V]$ could be identified with the set of vector lines of V, i.e., with the points of the projective d-space $\mathbb{P}V = V \setminus \{0\}/\sim$.

(2.3) Let V be a vector space over K and let us consider a filtration $V = V_0 \supset V_1 \supset \ldots \supset V_i \supset \ldots$ by subspaces V_i such that $W_i := V/V_i$ are of finite dimension $d(i)$ for $i \geq 0$. Now one has the increasing sequence of symmetric algebras

$$SW_0^\vee \hookrightarrow SW_1^\vee \hookrightarrow \ldots \hookrightarrow SW_i^\vee \hookrightarrow SW_{i+1}^\vee \hookrightarrow \ldots,$$

and one can fix bases in such a way that

$$SW_i^\vee \simeq K[X_1, \ldots, X_{d(i)}].$$

Thus $S = \bigcup_{i\geq 0} SW_i^\vee$ is the polynomial ring $K[X_1, \ldots, X_i, \ldots]$. As in the finite-dimensional case we consider the affine scheme $\mathbb{A}[S] := \mathrm{Spec}(S)$ (resp. the projective one $\mathbb{P}[S] := \mathrm{Proj}(S)$) together with the natural morphisms $\lambda_i : \mathbb{A}[S] \longrightarrow \mathbb{A}[W_i]$ induced by the inclusion $S(W_i^\vee) \subset S$. Moreover, the set of K-rational closed points $\mathbb{A}[S](K)$ of $\mathbb{A}[S]$ (i.e., maximal ideals $\mathfrak{m}$ of S such that $S/\mathfrak{m} \simeq K$) could be identified with $K^{\mathbb{N}} = \{(a_1, a_2, \ldots, a_j, \ldots) \mid a_j \in K\}$. It is clear that the maps λ_i above give (by restriction) maps $\lambda(K)_i : \mathbb{A}[S](K) \longrightarrow \mathbb{A}[W_i](K)$. In the projective case, one must consider $\mathbb{P}^*[W_i]$ as the disjoint union of $\mathbb{P}[W_i]$ with a point in order to have well-defined projections $\lambda_i : \mathbb{P}[S] \longrightarrow \mathbb{P}^*[W_i]$ and $\lambda(K)_i : \mathbb{P}[S](K) \longrightarrow \mathbb{P}^*[W_i](K)$. The indeterminates $X_1, \ldots, X_{d(i)}$ could be seen as linear maps $X_i : V/V_i \longrightarrow K$ and so (by composing with the projection) as linear maps $X_i : V \longrightarrow K$, i.e as elements of $V^\vee$. Thus, one has a linear map

$$\varphi : V \longrightarrow K^{\mathbb{N}}$$

defined by $\varphi(v) = (X_1(v), \ldots, X_i(v), \ldots)$. It is clear that $\ker(\varphi) = \bigcap_{i\geq 0} V_i$, and so φ gives rise to an embedding if and only if $\bigcap V_i = 0$. In this case we have $V \hookrightarrow K^{\mathbb{N}} = \mathbb{A}[S](K)$ (resp. $\mathbb{P}V \hookrightarrow \mathbb{P}[S](K)$). Notice that the restriction of the map $\lambda(K)_i$ to V is nothing other than the projection map $\pi_i : V \longrightarrow V/V_i \equiv \mathbb{A}[W_i]$. We have the following diagram:

$$\begin{array}{ccccc}
V & \hookrightarrow & \mathbb{A}[S](K) \equiv K^{\mathbb{N}} & \hookrightarrow & \mathbb{A}[S] \\
\downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \lambda_i(K)} & & \downarrow{\scriptstyle \lambda_i} \\
V/V_i & = & \mathbb{A}[W_i](K) & \hookrightarrow & \mathbb{A}[W_i]
\end{array}$$

(2.4) We will say that a subset $X \subset \mathbb{A}[S]$ is cylindric if there exist $i \geq 0$ and a constructible subset $Y \subset \mathbb{A}[W_i]$ such that $X = \lambda_i^{-1}(Y)$. The *generalized Euler characteristic* of X is defined as

$$\chi_g(X) := [Y] \cdot \mathbb{L}^{-d(i)} \in K_0(\mathcal{V}_K)_{\mathbb{L}}.$$

As in the complex case one can see that $[Y] \cdot \mathbb{L}^{-d(i)}$ does not depend on i, and so $\chi_g(X)$ is well-defined. In a similar way one can define cylindric subsets of $\mathbb{P}[S]$ as the inverse images by λ_i of constructible subsets Y of $\mathbb{P}[W_i]$.

(2.5) However, in this paper we are mainly interested in subsets of V (resp. $\mathbb{P}V$). It is easy to see that there are not cylindric subsets X of V according to the above definition (the inverse image by π_i of any point $x_i \in \mathbb{A}[W_i]$ contains in general closed points which are not K-rational). In order to have a reasonable definition (compatible with the above one) for such sets, one can say that a subset $X \subset V$ is cylindric if there exists a cylindric subset Z of $\mathbb{A}[S]$ such that $X = Z \cap V$ (the same definition is valid for subsets X of $\mathbb{A}[S](K)$ instead of V; in this case one has that X is cylindric if it coincides with the set of K-rational points of a cylindric subset of $\mathbb{A}[S]$). Assume $X \subset V$ to be cylindric. Then there exists a constructible subset $Y \subset \mathbb{A}[W_i]$ such that $X = \lambda_i^{-1}(Y) \cap V$. On the other hand it is also true that

$$\begin{aligned} X &= \lambda_i(K)^{-1}\,(Y \cap \mathbb{A}[W_i](K)) \cap V \\ &= \pi_i^{-1}\,(Y \cap \mathbb{A}[W_i](K)) \\ &= \pi_i^{-1}\,(Y(K)) \end{aligned}$$

(where $Y(K) = Y \cap \mathbb{A}[W_i](K)$ denotes the subset of K-rational closed points of the constructible Y). If we assume that $K = \mathbb{F}_q$ is the finite field of q elements, then every subset $Y \subset \mathbb{A}[W_i](K)$ is a constructible subset of $\mathbb{A}[W_i]$ itself (being a finite set of closed points), and therefore a subset $X \subset V$ is cylindric if and only if $X = \pi_i^{-1}(Y)$ for some $Y \subset \mathbb{A}[W_i](K)$. Notice that $[Y] = \sharp(Y) \in K_0(\mathcal{V}_{\mathbb{F}_q})$. The space V is itself a cylindric one. Any possible definition of a generalized Euler characteristic for V, say $\chi(V) = \varrho$, gives to us that $\chi(\pi_i^{-1}(x))q^{d(i)} = \varrho$ for any $i \geq 0$ and $x \in \mathbb{A}[W_i](K)$ and so, if $X = \pi_i^{-1}(Y)$ for $Y \subset \mathbb{A}[W_i](K)$, then one has that

$$\chi(X) = \frac{[Y]\varrho}{q^{d(i)}} = \frac{\sharp(Y)\varrho}{q^{d(i)}}\ . \tag{3}$$

Thus taking $\varrho = 1$ one has that $\chi(X) := \sharp(Y) \cdot q^{-d(i)}$ is (essentially) the only way to define the Euler characteristic of X. Note that if $X = Z \cap V$ with $Z = \lambda_i^{-1}(W)$ for a constructible subset W of $\mathbb{A}[W_i]$, then one has $\chi_g(Z) = [W] \cdot \mathbb{L}^{-d(i)}$, and so $\chi(X) = \chi_g(Z)(K) = [W(K)] \cdot \mathbb{L}(K)^{-d(i)} = \sharp W(K) \cdot q^{-d(i)}$ is the specialization of χ_g by the map which gives the number of K-rational closed points of a variety.

(2.6) For subsets of $\mathbb{P}V$ and of $\mathbb{P}[S](K)$ one can reproduce the above constructions by looking in this case to the diagram

$$\begin{array}{ccccc} \mathbb{P}V & \hookrightarrow & \mathbb{P}[S](K) \equiv \mathbb{P}K^{\mathbb{N}} & \hookrightarrow & \mathbb{P}[S] \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \lambda_i(K)} & & \downarrow{\scriptstyle \lambda_i} \\ \mathbb{P}^*V/V_i & = & \mathbb{P}^*[W_i](K) & \hookrightarrow & \mathbb{P}^*[W_i] \end{array}$$

(2.7) Furthermore, an interesting question is to decide when the property of being cylindric is preserved under projectivization. A subset $X \subset V$ is called a **cone** if for every $\lambda \in K \setminus \{0\}$ and for every $a \in X$, $\lambda a \in X$.

Lemma. *If $Z \subset V \setminus \{0\}$ is a cone, then Z is cylindric if and only if $\mathbb{P}Z$ is cylindric. In this case*

$$\chi(Z) = (q-1)\chi(\mathbb{P}Z).$$

In particular, if $X \subset \mathbb{P}V$ is cylindric, the set

$$CX := \{y \in V \setminus \{0\} \mid y = \lambda a \ \textit{ for all } \ a \in X, \lambda \in K \setminus \{0\}\}$$

is cylindric.

Proof. If $Z \subset V \setminus \{0\}$ is cylindric, then there is a constructible subset $Y(K) \subset (V/V_i) \setminus \{0\}$ such that $Z = \pi_i^{-1}(Y(K))$. As the coset of 0 in the quotient ring does not belong to $Y(K)$, we can take $\mathbb{P}Y(K)$ as a constructible subset of $\mathbb{P}V/V_i$ so that $\mathbb{P}Z$ is cylindric. □

(2.8) We are interested in the case in which $K = \mathbb{F}_q$, $V = \mathcal{O}$ and $V_i = \mathfrak{m}^{i+1}$ for $i \geq 0$. Since $\mathcal{O}$ is assumed to be Noetherian, by a Krull's Theorem it holds that $\bigcap_{i\geq 0} \mathfrak{m}^i = 0$. Let $\mathfrak{a} \subseteq \mathcal{O}$ be an ideal of $\mathcal{O}$. Since $\mathfrak{a}$ is $\mathfrak{m}$-primary, then $\mathfrak{m}^{i+1} \subseteq \mathfrak{a}$. Let A be the ideal $\mathfrak{a}/\mathfrak{m}^{i+1}$ of $\mathcal{O}/\mathfrak{m}^{i+1}$ so that $\pi_i^{-1}(A) = \mathfrak{a}$. As $\mathcal{O}/\mathfrak{m}^{i+1}$ is a finite-dimensional vector space over $\mathbb{F}_q$, all subsets are constructible, in particular the ideal A. Then $\mathfrak{a}$ is cylindric and its generalized Euler characteristic is

$$\chi(\mathfrak{a}) = \sharp(A) \cdot q^{-d(i)}.$$

In particular, $\chi(\mathfrak{m}^{i+1}) = q^{-d(i)}$.

(2.9) Note that $\mathbb{P}\mathfrak{a} = \mathbb{P}(\mathfrak{a} \setminus \{0\})$ is not a cylindric subset of $\mathbb{P}\mathcal{O}$. Moreover, if H is a subset of $\mathfrak{a}$ such that $(\mathfrak{a} \setminus H) \cap \mathfrak{m}^{i+1} = \emptyset$ for some $i \geq 0$, then $\mathfrak{a} \setminus H$ is cylindric and its projectivization is cylindric too. In particular, for $H = \mathfrak{m}^{i+1}$ one has

$$(q-1)\chi\left(\mathbb{P}(\mathfrak{a} \setminus \mathfrak{m}^{i+1})\right) = \chi(\mathfrak{a}) - q^{-d(i)}.$$

This is a consequence of the considerations of (2.6) and the additivity of the Euler characteristic.

(2.10) We can extend these definitions to any subset $X \subset \mathcal{K}$: it is called *cylindric* if there exists a non-zero divisor element $z \in \mathcal{O}$ such that the set zX is a subset of $\mathcal{O}$ and is cylindric. In this situation, the generalized Euler characteristic is

$$\chi(X) := \frac{\chi(zX)}{\chi(z\mathcal{O})}. \tag{4}$$

Since $\chi_g(\mathcal{O}) = 1$, then for $z = 1$, the latter definition actually extends the definition given by equation (3).

(2.11) Proposition *Let $\mathfrak{a}$ be a fractional $\mathcal{O}$-ideal. Then $\chi(\mathfrak{a}) = \mu(\mathfrak{a})$.*

Proof. Let $\mathfrak{a} \subset \mathcal{O}$ be an ideal of $\mathcal{O}$. For an integer $i \geq 0$, since $\mathfrak{m}^{i+1} \subseteq \mathfrak{a} \subseteq \mathcal{O}$, then $-\dim_{\mathbb{F}_q}(\mathcal{O}/\mathfrak{a}) = \dim_{\mathbb{F}_q}(A) - d(i)$, and therefore

$$\chi(\mathfrak{a}) = \sharp(A) \cdot q^{-d(i)} = q^{\dim_{\mathbb{F}_q}(A) - d(i)} = q^{-\dim_{\mathbb{F}_q}(\mathcal{O}/\mathfrak{a})} = \mu(\mathfrak{a}).$$

Assume now that $\mathfrak{a}$ is a fractional ideal. It means that there exists a non-zero divisor $z \in \mathcal{O}$ such that $z\mathfrak{a} \subseteq \mathcal{O}$. In such a case Stöhr showed that

$$\mu(z\mathfrak{a}) = \mu(z\mathcal{O})\mu(\mathfrak{a}) \ .$$

By using equation (4) the result follows. □

3. Integrals with respect to the generalized Euler characteristic.

(3.1) Let $\psi : \mathbb{P}\mathcal{O} \to G$ be a function with values in an abelian group G with countably many values. It is said to be *cylindric* if, for each $a \in G \setminus \{0\}$, the set $\psi^{-1}(a) \subseteq \mathbb{P}\mathcal{O}$ is cylindric. As was introduced in [CDG-2], the *integral* of a cylindric function ψ over $\mathbb{P}\mathcal{O}$ with respect to the generalized Euler characteristic is

$$\int_{\mathbb{P}\mathcal{O}} \psi d\chi := \sum_{a \in G\setminus\{0\}} \chi\left(\psi^{-1}(a)\right) \cdot a$$

if this sum makes sense in $K_0(\mathcal{V}_{\mathbb{F}_q})_{\mathbb{L}} \otimes_{\mathbb{Z}} G$. In such a case, the function ψ is said to be *integrable*.

(3.2) The definition of a cylindric function can be modified using the corresponding extended notion of the cylindric subset $X \subset \mathcal{K}$: for an abelian group G with countably many values, a function $\psi : X \to G$ is called *cylindric* if the set $\psi^{-1}(a) \subset \mathcal{K}$ is cylindric for all $a \in G \setminus \{0\}$. The integral of ψ over X with respect to the generalized Euler characteristic is

$$\int_X \psi d\chi := \sum_{a \in G\setminus\{0\}} \chi\left(\psi^{-1}(a)\right) \cdot a. \tag{5}$$

For a function $\psi : \mathbb{P}\mathcal{O} \to G$, let us denote by $\widetilde{\psi} : \mathcal{O} \to G$ the function induced by ψ on $\mathcal{O}$ with $\widetilde{\psi}(0) = 0$. Note that any function $f : \mathcal{O} \to G$ such that $f(0) = 0$ and $f(\lambda x) = f(x)$, $\forall x \in \mathcal{O}$ and $\forall \lambda \in \mathbb{F}_q$, is of the form $f = \widetilde{\psi}$ for a suitable $\psi : \mathbb{P}\mathcal{O} \to G$. Notice that functions defined in $X \subset \mathcal{K}$ or $X \subset \mathbb{P}\mathcal{O}$ or $X \subset \mathcal{O}$ could be assumed to be defined in all $\mathcal{K}$ or $\mathbb{P}\mathcal{O}$ or $\mathcal{O}$ by just extending them by 0.

(3.3) Lemma. *The function $\psi : \mathbb{P}\mathcal{O} \to G$ is cylindric if and only if the function $\widetilde{\psi} : \mathcal{O} \to G$ is cylindric. In such a case*

$$(q-1)\int_{\mathbb{P}\mathcal{O}} \psi d\chi = \int_{\mathcal{O}} \widetilde{\psi} d\chi.$$

Proof. This is a straight consequence of (2.7). □

The Stöhr zeta function can be written as an integral in the following terms:

(3.4) Theorem. *Let $\mathfrak{b}$ be a fractional $\mathcal{O}$-ideal containing $\mathcal{O}$. One has*

$$Z(\mathcal{O}, \mathfrak{b}, t) = \frac{q^{\rho} t^{\deg(\mathfrak{b})}}{(q^{\rho}-1)(U_{\mathfrak{b}} : U_{\mathcal{O}})} \int_{\mathfrak{b}} (qt)^{\underline{v}(g)\cdot \underline{d}} d\chi.$$

As a consequence,

$$Z(\mathcal{O}, t) = \frac{q^{\rho}}{q^{\rho}-1} \sum_{\mathfrak{b}} \frac{t^{\deg(\mathfrak{b})}}{(U_{\mathfrak{b}} : U_{\mathcal{O}})} \int_{\mathfrak{b}} (qt)^{\underline{v}(g)\cdot \underline{d}} d\chi,$$

where $\mathfrak{b}$ runs through a complete system of representatives of the ideal class semigroup of $\mathcal{O}$ and satisfies $\overline{\mathcal{O}} \cdot \mathfrak{b} = \overline{\mathcal{O}}$. (In the above formulae we assume that $t^{\infty} = 0$.)

Proof. Stöhr proves that

$$Z(\mathcal{O},\mathfrak{b},t)=\frac{q^{\rho}}{(U_{\mathfrak{b}}:U_{\mathcal{O}})(q^{\rho}-1)}\sum_{\underline{v}\in S(\mathfrak{b})}\sum_{\underline{i}=\underline{0}}^{\underline{1}}(-1)^{|\underline{i}|}q^{\underline{v}\cdot\underline{d}+\deg(\mathfrak{b}\cap\mathfrak{m}^{\underline{v}+\underline{i}})}t^{\underline{v}\cdot\underline{d}}t^{\deg(\mathfrak{b})}.$$

Consider $\mathfrak{b}_{\underline{v}}:=\{g\in\mathfrak{b}\mid \underline{v}(g)=\underline{v}\}$. The set $\mathfrak{b}_{\underline{v}}$ is an affine cylindric cone and $\mathbb{P}\mathfrak{b}_{\underline{v}}$ is cylindric by (2.7), because, denoting $\mathcal{J}(\underline{v}):=\{g\in\mathfrak{b}\mid v_i(g)\geq v_i,\ 1\leq i\leq r\}$, then

$$\mathfrak{b}_{\underline{v}}=\mathcal{J}(\underline{v})\setminus\bigcup_{i=1}^{r}\mathcal{J}(\underline{v}+e_i)$$

for e_i the vector having all entries 0 except for the ith one, which is equal to 1, and so $\mathfrak{m}^{k+1}\subset\mathcal{J}(\underline{v}+\underline{1})\subset\bigcup_{i=1}^{r}\mathcal{J}(\underline{v}+e_i)$. Moreover, Stöhr also proves that:

$$\mu(\mathfrak{b}_{\underline{v}})=\sum_{\underline{i}=\underline{0}}^{\underline{1}}(-1)^{|\underline{i}|}q^{\deg(\mathfrak{b}\cap\mathfrak{m}^{\underline{v}+\underline{i}})}.$$

If we denote $\underline{t}^{\underline{v}(g)}:=(t_1^{v_1(g)},\ldots,t_r^{v_r(g)})$, then $\int_{\mathfrak{b}}\underline{t}^{\underline{v}(g)}d\chi:=\sum_{\underline{v}\in\mathbb{Z}^r}\chi(\mathfrak{b}_{\underline{v}})\underline{t}^{\underline{v}(g)}$ and, as the measures μ and χ agree,

$$\begin{aligned}\int_{\mathfrak{b}}(qt)^{\underline{v}(g)\cdot\underline{d}}d\chi &= \sum_{\underline{v}\in\mathbb{Z}^r}\chi(\mathfrak{b}_{\underline{v}})q^{\underline{v}\cdot\underline{d}}t^{\underline{v}\cdot\underline{d}}=\sum_{\underline{v}\in S(\mathfrak{b})}\mu(\mathfrak{b}_{\underline{v}})q^{\underline{v}\cdot\underline{d}}t^{\underline{v}\cdot\underline{d}}\\ &= \sum_{\underline{v}\in S(\mathfrak{b})}\sum_{\underline{i}=\underline{0}}^{\underline{1}}(-1)^{|\underline{i}|}q^{\underline{v}\cdot\underline{d}+\deg(\mathfrak{b}\cap\mathfrak{m}^{\underline{v}+\underline{i}})}t^{\underline{v}\cdot\underline{d}}.\end{aligned}$$

Hence $Z(\mathcal{O},\mathfrak{b},t)=\frac{t^{\deg(\mathfrak{b})}q^{\rho}}{(U_{\mathfrak{b}}:U_{\mathcal{O}})(q^{\rho}-1)}\int_{\mathfrak{b}}(qt)^{\underline{v}(g)\cdot\underline{d}}d\chi.$ □

(3.5) From (3.3) we can also write the Stöhr zeta function as an integral over the projectivization of a fractional $\mathcal{O}$-ideal:

$$Z(\mathcal{O},\mathfrak{b},t)=\frac{(q-1)q^{\rho}t^{\deg(\mathfrak{b})}}{(q^{\rho}-1)(U_{\mathfrak{b}}:U_{\mathcal{O}})}\int_{\mathbb{P}\mathfrak{b}}(qt)^{\underline{v}(g)\cdot\underline{d}}d\chi.$$

4. The generalized Poincaré series and the Stöhr zeta function.

(4.1) Let us take a multi-index filtration of the ring $\mathcal{O}$ given by the ideals $J(\underline{v}):=\{g\in\mathcal{O}\mid v_i(g)\geq v_i,\ 1\leq i\leq r\}$ for $\underline{v}:=(v_1\ldots,v_r)\in\mathbb{Z}^r$. As in [CDG-2] we introduce the generalized Poincaré series for a perfect field K contained in $\mathcal{O}$ as follows.

Definition. The *generalized Poincaré series* of a multi-index filtration given by the ideals $J(\underline{v})$ is the integral

$$P_g(\underline{t}):=\int_{\mathbb{P}\mathcal{O}}\underline{t}^{\underline{v}(g)}d\chi_g\in K_0(\mathcal{V}_K)_{\mathbb{L}}[[t_1,\ldots,t_r]],$$

where $\underline{t}^{\underline{v}(g)}:=t_1^{v_1(g)}\cdot\ldots\cdot t_r^{v_r(g)}$ is considered as a function on $\mathbb{P}\mathcal{O}$ with values in $\mathbb{Z}[[t_1,\ldots,t_r]]$ (as above, the vector $(t_1^{\infty},\ldots,t_r^{\infty})$ is supposed to be $\underline{0}$).

In [CDG-2] it is also proved (see Proposition 2) that the series $P_g(\underline{t})$ indeed depends on $\mathbb{L}$ in the following terms:

$$P_g(\underline{t};\mathbb{L})=\frac{L_g(\underline{t};\mathbb{L})\cdot\prod_{i=1}^{r}(t_i-1)}{t_1\cdot\ldots\cdot t_r-1},\tag{6}$$

in which $L_g(\underline{t};\mathbb{L}) = \sum_{\underline{v}\in\mathbb{Z}^r} \mathbb{L}^{-\ell(\underline{v})-1} \cdot \frac{1-\mathbb{L}^{-\ell(\underline{v}+\underline{1})+\ell(\underline{v})}}{1-\mathbb{L}^{-1}} \underline{t}^{\underline{v}}$, with $\ell(\underline{v}) := \dim_K (\mathcal{O}/J(\underline{v}))$. Note that for $K = \mathbb{F}_q$, $P_g(\underline{t};q) = \int_{\mathbb{P}\mathcal{O}} \underline{t}^{\underline{v}(g)} d\chi$.

(4.2) The partial zeta function $Z(\mathcal{O},\mathcal{O},t)$ (i.e., the case $\mathfrak{b} = \mathcal{O}$ in equation (1)) is especially interesting. Note that:

$$Z(\mathcal{O},\mathcal{O},t) = \sum_{i=0}^{\infty} \sharp \{\text{principal ideals contained in } \mathcal{O} \text{ of codimension } i\} \cdot t^i.$$

(4.3) We can put $Z(\mathcal{O},\mathcal{O},t)$ in terms of the generalized Poincaré series and, therefore, in terms of the semigroup of $\mathcal{O}$ in an alternative way to those shown by Stöhr (it follows from Theorem 3.1 in [St]) or Zúñiga (see [Zu], Theorem 5.5).

Corollary. One has

$$Z(\mathcal{O},\mathcal{O},t) = \frac{q^\rho(q-1)}{q^\rho - 1} P_g\left((qt)^{d_1}, \ldots, (qt)^{d_r}; q\right).$$

Proof. The set $\mathfrak{b}_{\underline{v}} = \{g \in \mathfrak{b} \mid \underline{v}(g) = \underline{v}\}$ is a cylindric cone and, by (2.7), $\chi_g(\mathfrak{b}_{\underline{v}}) = (\mathbb{L}-1)\chi_g(\mathbb{P}\mathfrak{b}_{\underline{v}})$. In particular, for $\mathfrak{b} = \mathcal{O}$, by applying Theorem (3.4) (see also (3.5)) one has

$$\begin{aligned} Z(\mathcal{O},\mathcal{O},t) &= \frac{q^\rho(q-1)}{q^\rho - 1} \int_{\mathbb{P}\mathcal{O}} (qt)^{\underline{v}(g)\cdot\underline{d}} d\chi \\ &= \frac{q^\rho(q-1)}{q^\rho - 1} P_g\left((qt)^{d_1}, \ldots, (qt)^{d_r}; q\right). \end{aligned}$$

□

(4.4) For the unibranch and two-branch case with $d_i = 1$, for each $1 \leq i \leq r$ (i.e., the so-called *totally rational* case), Stöhr establishes explicit formulae to compute $Z(\mathcal{O},\mathcal{O},t)$ in terms of the semigroup of values (see [St], Theorems 4.1 and 4.3). Nevertheless, for more than two branches such a calculation turns out to be complicated. Our previous corollary (4.3) allows us to compute it via integration in a method as effective as the one for the generalized Poincaré series. Moreover, this corollary (4.3) admits a simple formula in the totally rational case, namely,

$$Z(\mathcal{O},\mathcal{O},t) = q \cdot P_g(qt, \ldots, qt; q).$$

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Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias, Universidad de Valladolid, Valladolid, Spain
E-mail address: fdelgado@agt.uva.es

Institut für Mathematik, Universität Osnabrück, Albrechtstrasse 28a, 49069-Osnabrück, Germany
E-mail address: moyano@agt.uva.es