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DEFORMATIONS OF HOLOMORPHIC LAGRANGIAN FIBRATIONS

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ABSTRACT. Let $X \to \mathbb{P}^n$ be a $2n$-dimensional projective holomorphic symplectic manifold admitting a Lagrangian fibration over $\mathbb{P}^n$. Matsushita proved that the fibration can be deformed in a codimension one family in the moduli space $\mathrm{Def}(X)$ of deformations of X. We extend his result by proving that if the Lagrangian fibration admits a section, then there is a codimension two family of deformations which also preserve the section.

1. Introduction

Let X be a $2n$-dimensional compact Kähler manifold. We say X is a *holomorphic symplectic manifold* if it admits a closed two-form σ of type $(2,0)$ which is non-degenerate in the sense that $\sigma^{\wedge n}$ trivializes the canonical bundle $K_X = \Omega^{2n}$. Moreover, we call X *irreducible* if $\mathrm{H}^0(X, \Omega^2)$ is one-dimensional and generated by $[\sigma]$. Huybrechts' notes in [7] provide a comprehensive introduction to the standard results on irreducible holomorphic symplectic manifolds.

Let us summarize what is known about fibrations on X. Suppose we have a proper (holomorphic) surjection $f : X \to B$ onto a complex space B such that the general fibre is connected and $0 < \mathrm{dim} B < 2n$. If X is projective and B is a normal variety, then Matsushita [9, 10] showed that

- every irreducible component of a fibre of f is a (holomorphic) Lagrangian submanifold of X; in particular, it is n-dimensional,
- the generic fibre is an abelian variety,
- B is n-dimensional and has only $\mathbb{Q}$-factorial log-terminal singularities,
- K_B^* is ample and B has Picard number one.

Moreover, if B is smooth, then it has the same Hodge numbers as $\mathbb{P}^n$ (see [11]). In particular, for $n = 2$ the base is isomorphic to $\mathbb{P}^2$ (a similar result was obtain by Markushevich [8]). In general it is expected that B should be isomorphic to $\mathbb{P}^n$. Huybrechts [7, Proposition 24.8] extended some of these results by dropping the projectivity assumption: he showed that if X and B are (smooth) Kähler manifolds, then

- every fibre of f is (holomorphic) Lagrangian,
- every smooth fibre is a complex torus,

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- B is n-dimensional and projective,
- K_B^* is ample, and the Picard and second Betti numbers of B both equal one.

In this paper we consider deformations of $f : X \to B$. It is known that the Kuranishi space $\mathrm{Def}(X)$ of deformations of X is a smooth complex manifold of dimension $b_2 - 2$, where b_2 is the second Betti number of X. Under the assumptions that X is projective and $B \cong \mathbb{P}^n$, Matsushita [11] proved that there is a codimension one submanifold of $\mathrm{Def}(X)$ parametrizing deformations of X which are Lagrangian fibrations over $\mathbb{P}^n$. We include a proof of this result in Section 2 in order to set up our notation. Our main result is Theorem 4: we prove that if $f : X \to \mathbb{P}^n$ admits a section, then there is a codimension two submanifold of $\mathrm{Def}(X)$ parametrizing deformations which are Lagrangian fibrations and admit sections.

The Mordell-Weil group $\mathrm{MW}(f)$ is the group of rational sections of $f : X \to B$. Recently Oguiso [14] found a formula for the rank of $\mathrm{MW}(f)$, generalizing the Shioda-Tate formula for elliptic surfaces. Inspired by our techniques, he was able to produce examples with various Mordell-Weil ranks.

Let X' be an arbitrary irreducible holomorphic symplectic manifold. Can X' be deformed to a Lagrangian fibration? The answer in general is unknown. However, the question has been answered in the affirmative for all known examples of irreducible holomorphic symplectic manifolds; see Beauville [2] for the Hilbert schemes of points on a K3 surface, Debarre [5] for the generalized Kummer varieties, and Rapagnetta [15] for O'Grady's examples.

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2. Deforming fibrations

Let X be an irreducible holomorphic symplectic manifold of dimension $2n$. For the following statements, see Section 22 of Huybrechts' notes in [7]. Denote by

$$\mathcal{X} \to (\mathrm{Def}(X), 0)$$

the Kuranishi family parametrizing local deformations of $X = \mathcal{X}_0$. We think of $(\mathrm{Def}(X), 0)$ as the germ of a complex space. It is smooth (deformations are unobstructed) of dimension $b_2 - 2$, where b_2 is the second Betti number of X. Note that when we deform X as a complex manifold it remains holomorphic symplectic and irreducible; for small deformations it also stays Kähler.

The germ $(\mathrm{Def}(X), 0)$ can be represented by a contractible open set, and therefore for each $t \in \mathrm{Def}(X)$ we can choose an isomorphism

$$\psi_t : \mathrm{H}^2(\mathcal{X}_t, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z})$$

known as a *marking*. Let

$$Q_X := \{[\alpha] \in \mathbb{P}(\mathrm{H}^2(X, \mathbb{C})) | q_X(\alpha) = 0 \text{ and } q_X(\alpha + \bar{\alpha}) > 0\}$$

be the *period domain* of X, where q_X is the Beauville-Bogomolov quadratic form on $\mathrm{H}^2(X, \mathbb{Z}) \otimes \mathbb{C}$. Then by the Local Torelli Theorem (see Beauville [1]), the *period map*

$$\mathcal{P}_X : (\mathrm{Def}(X), 0) \to (Q_X, [\sigma])$$

which takes t to $[(\psi_t \otimes \mathbb{C})(\sigma_t)]$ is a local isomorphism.

In this section we prove the following result.

Theorem 1 (Matsushita [11], Corollary 1.7)**.** *Let X be a projective irreducible holomorphic symplectic manifold which admits a Lagrangian fibration $f : X \to \mathbb{P}^n$ over projective space. There is a codimension one submanifold $\Delta^f \subset \mathrm{Def}(X)$ (containing zero) which parametrizes holomorphic symplectic manifolds X' which admit Lagrangian fibrations over $\mathbb{P}^n$.*

Proof. Let $L := f^*\mathcal{O}(1)$ be the pull-back of the hyperplane line bundle; then f is the morphism

$$\phi_L : X \to \mathbb{P}(\mathrm{H}^0(X, L)^*) \cong \mathbb{P}^n$$

induced by the linear system of $L := f^*\mathcal{O}(1)$. Matsushita [11] calculated the direct images of $\mathcal{O}_X$ under f and found that $R^j f_*\mathcal{O}_X \cong \Omega^j_{\mathbb{P}^n}$. Together with the Leray spectral sequence this implies

$$h^0(X, L) := \mathrm{dim}\mathrm{H}^0(X, L) = n + 1$$

and all higher cohomology of L vanishes.

Our aim is to extend L and the map ϕ_L to deformations of X for which the first Chern class $c_1 := c_1(L) \in \mathrm{H}^2(X, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X)$ remains of type $(1, 1)$. Let

$$\Delta^f := \{t \in \mathrm{Def}(X) | \psi_t^{-1}(c_1) \in \mathrm{H}^2(\mathcal{X}_t, \mathbb{Z}) \text{ is of type } (1, 1)\}.$$

If $t \in \Delta^f$, then $\psi_t^{-1}(c_1)$ is orthogonal to σ_t with respect to $q_{\mathcal{X}_t}$, or equivalently, c_1 is orthogonal to $(\psi_t \otimes \mathbb{C})(\sigma_t)$ with respect to q_X. Therefore Δ^f maps isomorphically to a neighbourhood of $[\sigma]$ in

$$Q^f_X := \{[\alpha] \in Q_X | q_X(\alpha, c_1) = 0\}$$

which is of codimension one in Q_X. Hence Δ^f must be of codimension one in $\mathrm{Def}(X)$.

The exponential exact sequence on $\mathcal{X}_t$ gives

$$\ldots \to 0 \to \mathrm{H}^1(\mathcal{X}_t, \mathcal{O}^*) \to \mathrm{H}^2(\mathcal{X}_t, \mathbb{Z}) \stackrel{\beta}{\to} \mathrm{H}^2(\mathcal{X}_t, \mathcal{O}) \to \ldots$$

since $\mathrm{H}^1(\mathcal{X}_t, \mathcal{O})$ vanishes. For $t \in \Delta^f$, $\psi_t^{-1}(c_1)$ is of type $(1, 1)$ and so in the kernel of β. It therefore comes from a unique holomorphic line bundle $\text{Ł}_t \in \mathrm{H}^1(\mathcal{X}_t, \mathcal{O}^*)$. Moreover, we can always choose a representative $(U, 0)$ of the germ $(\mathrm{Def}(X), 0)$ such that $U \cap \Delta^f$ is connected and simply-connected, so there exists a line bundle Ł over $\mathcal{X}|_{\Delta^f}$, restricting to Ł_t on each fibre $\mathcal{X}_t$.

By the theorem on page 210 of Grauert and Remmert [6], $h^i(\mathcal{X}_t, \text{Ł}_t)$ is an upper semi-continuous function of $t \in \Delta^f$. So for t in a neighbourhood of zero

$$h^i(\mathcal{X}_t, \text{Ł}_t) \leq h^i(\mathcal{X}_0, \text{Ł}_0) = h^i(X, L).$$

For $i > 0$ both sides must vanish, as we saw above that $h^i(X, L) = 0$. On the other hand, the Euler characteristic

$$\chi(\mathcal{X}_t, \text{Ł}_t) := \sum_{i=0}^{2n} (-1)^i h^i(\mathcal{X}_t, \text{Ł}_t)$$

is constant (it is given by the Hirzebruch-Riemann-Roch formula) and equal to $\chi(X, L) = n + 1$. Therefore $h^0(\mathcal{X}_t, \text{Ł}_t) = n + 1$ for all t in a neighbourhood of zero. The linear system of Ł_t therefore gives an a priori rational map

$$\phi_t : \mathcal{X}_t \dashrightarrow \mathbb{P}(\mathrm{H}^0(\mathcal{X}_t, \text{Ł}_t)^*) \cong \mathbb{P}^n.$$

The specialization

$$\phi_L = f : X \to \mathbb{P}(\mathrm{H}^0(X, L)^*) \cong \mathbb{P}^n$$

is a morphism and thus has no base-points; therefore the linear system of L_t must also be base-point free for $t \in \Delta^f$ in a neighbourhood of zero. Moreover, $\phi_t : \mathcal{X}_t \to \mathbb{P}^n$ is a Lagrangian fibration by the results of Matsushita (and Huybrechts) cited in the introduction. ☐

3. Fibrations with sections

We wish to look at deformations of a Lagrangian fibration over projective space which admits a section. We first prove that the existence of a section implies that X is projective. The following argument is due to Campana and Oguiso [13].

Lemma 2. *Let $f : X \to B$ be a Lagrangian fibration whose base B is a normal projective variety. If $f : X \to B$ admits a rational multi-valued section (i.e., an n-dimensional subvariety $Y \subset X$ which maps surjectively to B), then X is projective.*

Proof. We know from Huybrechts [7, Proposition 24.8] that the generic fibre X_t is a complex torus and holomorphic Lagrangian. Since X is Kähler, σ generates $\mathrm{H}^0(X, \Omega^2)$, and the restriction $\sigma|_{X_t} \in \mathrm{H}^0(X_t, \Omega^2)$ vanishes, we can conclude that X_t must be projective by a result of Voisin (which appears as Proposition 2.1 in Campana [4]).

Let $a(B)$ and $a(Y)$ be the algebraic dimensions of B and Y respectively. Since B is projective and Y maps surjectively to B, we have

$$n = \mathrm{dim} B = a(B) \leq a(Y) \leq \mathrm{dim} Y = n.$$

Therefore we have equality, which implies that Y is bimeromorphic to a projective variety; in particular, Y is covered by complete algebraic curves. Since the generic fibre X_t is also covered by complete algebraic curves, we conclude that any two generic points of X are joined by a chain of complete algebraic curves. By a result of Campana [3], this implies that X is Moishezon. Since X is also Kähler, it must be projective [12]. ☐

Now suppose that the base B is projective space and that $f : X \to \mathbb{P}^n$ admits a genuine section (i.e., a subvariety $Y \subset X$ which maps isomorphically to $\mathbb{P}^n$). Let $g : Y \hookrightarrow X$ be the inclusion; Y must be holomorphic Lagrangian as

$$\sigma|_Y \in \mathrm{H}^0(Y, \Omega^2) \cong \mathrm{H}^{2,0}(\mathbb{P}^n) = 0.$$

We will need the following result (note that in Voisin's paper X is the Lagrangian submanifold of Y).

Theorem 3 (Voisin [19])**.** *Let X be a holomorphic symplectic manifold and let $g : Y \hookrightarrow X$ be a Lagrangian submanifold. Let g_t^* be the composition*

$$g^* \circ (\psi_t \otimes \mathbb{C}) : \mathrm{H}^2(\mathcal{X}_t, \mathbb{C}) \to \mathrm{H}^2(Y, \mathbb{C})$$

where ψ_t is a marking of $\mathcal{X}_t$. Then the inclusion $Y \hookrightarrow X$ deforms to a Lagrangian submanifold $\mathcal{Y}_t \hookrightarrow \mathcal{X}_t$ if and only if $g_t^*(\sigma_t) = 0$.

We can rephrase this as follows. Let $L_Y \subset \mathrm{H}^2(X, \mathbb{C})$ be the orthogonal complement of $\mathrm{ker} g^*$ with respect to the Beauville-Bogomolov quadratic form q_X. Since $\mathrm{H}^{2,0}(X) \subset \mathrm{ker} g^*$ and L_Y can be defined over $\mathbb{Q}$, L_Y must be of type $(1,1)$. Then

$Y \hookrightarrow X$ deforms to $\mathcal{Y}_t \hookrightarrow \mathcal{X}_t$ if and only if L_Y is preserved under the deformation, i.e., if and only if

$$t \in \Delta := \{t \in \mathrm{Def}(X) | \psi_t^{-1}(L_Y) \in \mathrm{H}^2(\mathcal{X}_t, \mathbb{C}) \textit{ is of type } (1,1)\}.$$

We can now prove our main result.

Theorem 4. *Let X be an irreducible holomorphic symplectic manifold which admits a Lagrangian fibration $f : X \to \mathbb{P}^n$ over projective space and a global section. There is a codimension two submanifold $\Delta^{fs} \subset \mathrm{Def}(X)$ (containing zero) which parametrizes holomorphic symplectic manifolds X' which admit Lagrangian fibrations and global sections.*

Proof. Lemma 2 implies X is projective, and then Theorem 1 implies there exists a codimension one family $\Delta^f \subset \mathrm{Def}(X)$ over which the fibration deforms.

By Theorem 3 we also have

$$\Delta^s := \{t \in \mathrm{Def}(X) | \psi_t^{-1}(L_Y) \in \mathrm{H}^2(\mathcal{X}_t, \mathbb{C}) \text{ is of type } (1,1)\}$$

over which the Lagrangian submanifold $Y \hookrightarrow X$ deforms. Now Δ^s maps isomorphically to a neighbourhood of $[\sigma]$ in

$$Q_X^s := \{[\alpha] \in Q_X | q_X(\alpha, L_Y) = 0\} = \{[\alpha] \in Q_X | \alpha \in \ker g^*\}.$$

Since $Y \cong \mathbb{P}^n$, $\mathrm{H}^2(Y, \mathbb{C})$ and L_Y are one-dimensional, so Q_X^s and Δ^s are codimension one in Q_X and $\mathrm{Def}(X)$ respectively.

Let $\Delta^{fs} := \Delta^f \cap \Delta^s$. It maps isomorphically to a neighbourhood of $[\sigma]$ in

$$Q_X^{fs} := Q_X^f \cap Q_X^s = \{[\alpha] \in Q_X | q_X(\alpha, c_1) = 0 \text{ and } \alpha \in \ker g^*\}.$$

Observe that $L := f^*\mathcal{O}(1)$ must satisfy

$$c_1(L)^{n+1} = 0 \in \mathrm{H}^{2n+2}(X, \mathbb{Z}),$$

which implies $q_X(c_1) = 0$ (see [7]) and thus $[c_1] \in Q_X^f$. On the other hand

$$g^*(c_1) = c_1(\mathcal{O}(1)) \neq 0,$$

so $c_1 \notin \ker g^*$ and $[c_1] \notin Q_X^s$. This suffices to show that Q_X^f and Q_X^s intersect transversely, and thus Q_X^{fs} and Δ^{fs} are codimension two in Q_X and $\mathrm{Def}(X)$ respectively.

Finally, observe that if $t \in \Delta^{fs}$, then $\mathcal{X}_t$ is a Lagrangian fibration over $\mathbb{P}^n$ and it contains a deformation $\mathcal{Y}_t$ of Y. By specialization, $\mathcal{Y}_t$ is a section of the fibration. More precisely, the set of values t for which $\mathcal{Y}_t$ maps isomorphically to $\mathbb{P}^n$ is open in Δ^{fs} and contains zero. This completes the proof. □

Example. Let S be a K3 surface which contains a smooth genus g curve C, $g \geq 2$, such that Pic$S$ is generated by $[C]$. Let $\mathcal{C} \to |C| \cong \mathbb{P}^g$ be the family of curves linearly equivalent to C. Then the compactified Jacobian

$$J^0 := \overline{\mathrm{Jac}}^0(\mathcal{C}/\mathbb{P}^g)$$

is a deformation of the Hilbert scheme of g points on S [2]. Moreover, J^0 is a Lagrangian fibration over $\mathbb{P}^g$ which admits a section. Observe that we can deform S in a 19-dimensional family, while keeping the genus g curve. These produce deformations of J^0 keeping the Lagrangian fibration and section. On the other hand, the space of deformations of the Hilbert scheme of points on S is 21-dimensional. This is in agreement with Theorem 4.

We find similar agreement for Lagrangian fibrations which are deformations of the generalized Kummer varieties (see Debarre [5] or the author's paper [16]).

Remark. Let $f : X \to B$ be a Lagrangian fibration. If the fibres of X are all reduced and irreducible, then there exists the (Altman-Kleiman) compactified relative Picard scheme

$$P := \overline{\mathrm{Pic}}^0(X/B).$$

If the fibres of P are reduced and irreducible, we also have

$$X^0 := \overline{\mathrm{Pic}}^0(P/B).$$

Both P and X^0 admit global sections. Let U and U^0 denote the open subsets of X and X^0, respectively, given by removing all singular fibres; then U is a torsor over U^0. In fact in cases where the singular fibres of X are not too complicated, X itself is a torsor over X^0 (by which we mean they are compactified group schemes over B and locally isomorphic as fibrations); moreover P and X^0 are holomorphic symplectic manifolds (see [18]). In [17] it was shown that torsors X over X^0 are parametrized by the one-dimensional space $\mathrm{H}^2(P, \mathcal{O}^*)$ of gerbes on P, with X^0 being the unique fibration which admits a global section. This agrees with Theorem 1 and Theorem 4, which together imply that fibrations which admit sections must be codimension one inside the family of all fibrations.

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