ASYMMETRY OF CONVEX SETS WITH ISOLATED EXTREME POINTS

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Abstract. When measuring asymmetry of convex sets \( L \subset \mathbb{R}^n \) in terms of inscribed simplices, the interior of \( L \) naturally splits into regular and singular sets. Based on examples, it may be conjectured that the singular set is empty iff \( L \) is a simplex. In this paper we prove this conjecture with the additional assumption that \( L \) has at least \( n \) isolated extreme points on its boundary.

1. Introduction and statement of results

Throughout, we use standard notation and basic concepts in the theory of convex sets and functions [1, 5]. Let \( E \) be a Euclidean vector space of dimension \( n \). (We usually take \( E = \mathbb{R}^n \).) If \( K \subset E \), then \( \langle K \rangle \) and \( \mathbb{E}[K] \) denote the affine span and convex hull of \( K \), respectively. For \( K = \{B_0, \ldots, B_m\} \) finite, \( \mathbb{E}[K] \) is a convex polytope. This polytope is an \( m \)-simplex if \( B_0, \ldots, B_m \) are affinely independent, or equivalently, if \( \dim \mathbb{E}[K] = \dim \langle K \rangle = m \). A convex set \( L \subset E \) is a convex body if it has nonempty interior. Every convex set is a convex body in its affine span.

Let \( L \) be a compact convex body in \( \mathbb{R}^n \) and \( O \) a point in the interior of \( L \). As in [6] [7], we define a sequence of (affine) invariants \( \{\sigma_m(L, O)\}_{m \geq 1} \). Intuitively, \( \sigma_m(L, O) \) measures how lopsided \( L \) is in dimension \( m \) viewed from \( O \). Since \( L \) is compact and convex, given \( C \in \partial L \), we have \( \langle O, C \rangle \cap \partial L = \{C, C^\circ\} \), where \( C^\circ \) is called the opposite of \( C \) (with respect to \( O \)). We define the distortion function \( \Lambda_L = \Lambda : \partial L \times \text{int} \ L \to \mathbb{R} \) by

\[
\Lambda(C, O) = \frac{d(C, O)}{d(C^\circ, O)}, \quad C \in \partial L, \ O \in \text{int} \ L,
\]

where \( d \) is the Euclidean distance in \( \mathbb{R}^n \). The distortion \( \Lambda \) is a continuous function [6] [7]. By definition, \( \Lambda(C^\circ, O) = 1/\Lambda(C, O) \).

The minimum distortion \( \lambda(O) = \inf_{C \in \partial L} \Lambda(C, O) \), as a function on the interior of \( L \), has been studied by many authors. (See Grünbaum [2] and the extensive references therein.) In particular, there are many lower estimates on Minkowski’s measure of symmetry \( \sup_{O \in \text{int} L} \lambda(O) \) and the derived measures \( \lambda(O_0) \), where \( O_0 \) is the centroid, the centers of circumscribed and inscribed ellipsoids, the centroid of the surface area of \( \partial L \), and the curvature centroid.

Let \( m \geq 1 \). A finite multi-set \( \{C_0, \ldots, C_m\} \) is called an \( m \)-configuration with respect to \( O \) if \( \{C_0, \ldots, C_m\} \subset \partial L \) and \( O \in \mathbb{E}[C_0, \ldots, C_m] \). The set of \( m \)-configurations

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is denoted by $C_m(\mathcal{L}, O)$. We define

$$\sigma_m(\mathcal{L}, O) = \inf_{\{C_0, \ldots, C_m\} \in C_m(\mathcal{L}, O)} \frac{1}{1 + \Lambda(C_i, O)}.$$ 

Since a 1-configuration is an opposite pair of points, we have $\sigma_1(\mathcal{L}, O) = 1$.

An $m$-configuration $\{C_0, \ldots, C_m\}$ for which the infimum is attained is called minimal. Compactness implies that minimal configurations exist. $\sigma_m(\mathcal{L}, \cdot) : \text{int} \mathcal{L} \to \mathbb{R}$ is a continuous function [7], Theorem D. In what follows, we suppress $O$ when no confusion arises. In addition, we also suppress the dimension $n$; in particular, we write $\sigma(\mathcal{L})$ for $\sigma_n(\mathcal{L})$, etc.

In general, we obviously have

$$m \geq 1.$$

For $m = n$, a configuration in $C_n+k(\mathcal{L}, O)$, $k \geq 1$, always contains a subconfiguration in $C(\mathcal{L}, O)$ so that equality holds in (1.1). (See [8] for details.) In other words, the sequence $\{\sigma_m(\mathcal{L})\}_{m \geq 1}$ is arithmetic (with difference $1/(1 + \max_{\partial \mathcal{L}} \Lambda)$) from the $n$-th term onwards.

By [7] (Theorem B), for $m \geq 1$, we have

$$1 \leq \sigma_m(\mathcal{L}) \leq \frac{m + 1}{2}.$$ 

The lower bound $\sigma_m(\mathcal{L}) = 1$ is realized iff there exists an affine subspace $F \subset \mathbb{R}^n$, $O \in F$, of dimension $m$ such that $\mathcal{L} \cap F$ is an $m$-simplex. For $m \geq 2$, the upper bound $\sigma_m(\mathcal{L}) = (m + 1)/2$ is realized iff $\mathcal{L}$ is symmetric (with respect to $O$).

Thus, up to scaling, $\sigma_m(\mathcal{L}, O)$, $m \geq 1$, are measures of symmetry in the sense of Grünbaum [2] since $\sigma_m$ is clearly continuous on the space of compact convex bodies with specified interior points and is also invariant under similarity transformations.

For estimates on the related symmetries of measure

$$\inf_{\{C_0, \ldots, C_m\} \in C_m(\mathcal{L}, O)} \sum_{i=0}^{m} \Lambda(C_i, O) \quad \text{and} \quad \inf_{\{C_0, \ldots, C_m\} \in C_m(\mathcal{L}, O)} \prod_{i=0}^{m} \Lambda(C_i, O)$$

(at least for $m = n$) see also Grünbaum [2].

We define the regular set $\mathcal{R} \subset \text{int} \mathcal{L}$ as

$$\mathcal{R} = \left\{ O \in \text{int} \mathcal{L} \mid \sigma(\mathcal{L}, O) < \sigma_{n-1}(\mathcal{L}, O) + \frac{1}{1 + \max_{\partial \mathcal{L}} \Lambda(\cdot, O)} \right\}.$$

An element of $\mathcal{R}$ is called a regular point. An interior point is called singular if it is not regular. By continuity of the functions in the defining inequality of $\mathcal{R}$, the set $\mathcal{R}$ is open in $\text{int} \mathcal{L}$ (and hence in $\mathbb{R}^n$). The structure of a compact convex body viewed from a regular point is technically much easier to deal with than when viewed from a singular point. For example, as shown below (Lemma 2.1), if $O$ is regular, then there exists a minimal $n$-configuration consisting of extreme points only. (Recall that a point on the boundary of $\mathcal{L}$ is called an extreme point if it is not contained in the interior of a boundary line segment.) This, for $\mathcal{L}$ a convex polytope, reduces the determination of $\sigma(\mathcal{L}, O)$ to a finite enumeration on the vertices of $\mathcal{L}$. Moreover, according to a result in [7], the distortion function $\Lambda(\mathcal{L}, \cdot)$ is concave on $\mathcal{R}$. Irrespective of regularity, concavity of the distortion function holds in 2-dimensions [7]. By contrast, there exists a 4-dimensional cone in which, due to the
existence of singular points near the base, the distortion function is not concave \[6\]. It is therefore important to analyze when and where singularity does occur.

It is easy to show that \( O \in \mathbb{R} \) iff the convex hull of every minimal configuration is an \( n \)-simplex with \( O \) in its interior. (See \[8\] for details.) In addition, \( \Lambda(.,O) \) attains its local maximum at each configuration point.

Because of this, we will need the behavior of the boundary of \( \mathcal{L} \) at a local maximum \( C \) of \( \Lambda(.,O) \) as described in \[7\] (Section 7). For the moment we only use the fact that if \( C \) is a smooth point of \( \partial \mathcal{L} \) at which \( \Lambda(.,O) \) assumes a local maximum, then \( C^o \) is also smooth and the tangent spaces at \( C \) and \( C^o \) to \( \partial \mathcal{L} \) are parallel. If \( C \) is not a smooth point, it is still true that there exist parallel supporting hyperplanes at \( C \) and \( C^o \). In particular, \([C,C^o]\) is an affine diameter in the sense of Grünbaum \[2\]. Thus, if \( O \) is a regular point \( O \) belongs to at least \( n + 1 \) (affinely independent) affine diameters. To determine points with this property is an unsolved problem; in particular, it is not known whether or not the centroid has this property. For further results, see Grünbaum \[2\] and Kosiński \[3,4\].

We denote by \( \mathcal{L}_0 \) the set of extreme points of \( \mathcal{L} \). By a theorem of Minkowski, we have \( \mathcal{L} = [\mathcal{L}_0] \). (See Theorem D in \[3\], p. 84.) We call an extreme point \( C \in \mathcal{L}_0 \) isolated if \( C \) is not a limit point of \( \mathcal{L}_0 \). The following simple example is the motivation for our study:

**Example.** Let \( \mathcal{L} \) have an isolated extreme point \( C \), and assume that, away from \( C \), \( \partial \mathcal{L} \) is smooth. We claim that there are singular points in the interior of \( \mathcal{L} \).

First, since \( C \) is an isolated extreme point, the set of supporting hyperplanes \( H \) at \( C \) such that \( \mathcal{L} \cap H = \{C\} \) is a nonempty open set (in the respective Grassmann manifold). This follows from the conical structure of \( \mathcal{L} \) near \( C \). (In fact, \( \mathcal{L} \) is the convex hull of \( [\mathcal{L}_0 \setminus \{C\}] \) and the single point \( C \); see Lemma 2.2 below.) For each \( H \) in this set, we consider the set of points \( B \in \partial \mathcal{L}, B \neq C \), such that the tangent space of \( \partial \mathcal{L} \) at \( B \) is parallel to \( H \). Since \( \mathcal{L} \) is convex and, away from \( C \), its boundary is smooth, the union \( B \) of these sets is open in \( \partial \mathcal{L} \). (In fact, \( \partial \mathcal{L} \setminus B \) is closed, as follows again from the conical structure of \( \mathcal{L} \) near \( C \).) The convex hull \( [B] \) intersects the interior of \( \mathcal{L} \). Any point \( O \) in this interior must be singular. Indeed, if \( O \) were regular, then at least one point in a mininal configuration \( \{C_0, \ldots, C_n\} \in \mathcal{C}(\mathcal{L},O) \) would be contained in \( B \). This is a contradiction, since the tangent space at that point has no parallel translate tangent to \( \partial \mathcal{L} \) at another smooth point.

The situation in the nonsmooth case is much more complicated. Our first result asserts that regularity of the interior of \( \mathcal{L} \) along with the existence of isolated extreme points impose severe restrictions to the structure of \( \mathcal{L} \).

**Theorem 1.1.** Let \( \mathcal{L} \subset \mathbb{R}^n \) be a compact convex body with all interior points regular. Assume that \( \mathcal{L} \) has (at least) two isolated extreme points \( C_0 \) and \( C_1 \). Then, for any plane \( \tau \) that contains \( C_0 \) and \( C_1 \), the intersection \( \mathcal{L} \cap \tau \) is either \([C_0,C_1]\) or a triangle with \([C_0,C_1]\) as a side.

An illustrative example to Theorem 1.1 to be discussed below is the following:

**Example.** Let \( S \subset \mathbb{R}^3 \) be the unit circle of the coordinate plane spanned by the first and second coordinate axes, and let \( C_+ = (1,0,\pm1) \). Let \( \mathcal{L} \) be the convex hull \([S,C_+,C_-]\). Then \( \mathcal{L}_0 = (S \setminus \{(1,0,0)\}) \cup \{C_\pm\} \). Clearly, \( \mathcal{L}_0 \) is not closed. Due to triangular intersections, we have \( \sigma_2(\mathcal{L},.) = 1 \). Hence \[7\], \( \sigma(\mathcal{L},.) \) is concave on the whole interior of \( \mathcal{L} \).
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Theorem 1.2. Let $\mathcal{L} \subset \mathbb{R}^n$ be as in Theorem 1.1. Assume that $\mathcal{L}$ has at least $n$ isolated extreme points. Then $\mathcal{L}$ is a simplex.

For $\mathcal{L}$ a convex polytope, the extreme points are the vertices and they are all isolated. Theorem 1.2 gives the following:

Theorem 1.3. Let $\mathcal{L} \subset \mathbb{R}^n$ be a convex polytope which is not a simplex. Then there are singular points in the interior of $\mathcal{L}$.

The proof will actually show that, if nonempty, the set of singular points has nonempty interior and its closure contains part of the boundary of $\mathcal{L}$.

2. Proofs

Let $\mathcal{L} \subset \mathbb{R}^n$ be a compact convex body. We first recall the notion of $k$-flat points on $\partial \mathcal{L}$ \cite{7}. Let $C \in \partial \mathcal{L}$. We call an affine subspace $A \subset \mathbb{R}^n$ a supporting flat at $C$ if $C \in A$ and $A$ is contained in a supporting hyperplane of $\mathcal{L}$ at $C$. Consider the set of supporting flats $A$ at $C$ such that $\partial \mathcal{L} \cap A$ is a compact convex body in $A$ and $C$ is contained in its relative interior. Since $\mathcal{L}$ is convex, this set has a unique maximal element denoted by $A_C$. We call $C$ a $k$-(dimensional) flat point if $\dim A_C = k$. Clearly, $C$ is an extreme point iff $k = 0$.

Lemma 2.1. Let $\mathcal{L} \subset \mathbb{R}^n$ be a compact convex body. If $O$ is a regular point of $\mathcal{L}$, then there exists a minimal configuration $\{C_0, \ldots, C_n\} \in \mathcal{C}(\mathcal{L}, O)$ consisting of extreme points.

Proof. Let $\{C_0, \ldots, C_n\} \in \mathcal{C}(\mathcal{L}, O)$ be minimal. Since $O$ is a regular point, $[C_0, \ldots, C_n]$ is an $n$-simplex containing $O$ in its interior and $\Lambda(., O)$ attains a relative maximum at each $C_i$, $i = 0, \ldots, n$. Suppressing the index for simplicity, assume that a configuration point $C$ is not extremal. Then $C$ is a $k$-flat point for some $k > 0$, $k = \dim A_C$. Since $\Lambda(., O)$ attains a relative maximum at $C$, according to a result of \cite{7} (the proposition in Section 7), the antipodal point $C'$ is $l$-flat, $l \geq k$, and $A_C$ is parallel to $A_{C'}$ in the sense that a translate of $A_C$ is contained in $A_{C'}$.

Choose a point $C'$ on the boundary of the compact convex body $\partial \mathcal{L} \cap A_C$ in $A$. Clearly, $C'$ is a lower dimensional flat point than $C$. Since $A_C$ is parallel to $A_{C'}$, $\Lambda(., O)$ is constant on $[C, C']$. Moving $C$ toward $C'$ and replacing $C$ with the moved point, the configuration condition $O \in [C_0, \ldots, C_n]$ stays intact since $O$ is a regular point. Thus, replacing $C$ by $C'$ in the configuration, we arrive at a minimal configuration with $C'$ being a lower dimensional flat point than $C$. Proceeding inductively, we can replace each nonextremal point of the configuration with and extremal point without altering minimality. Lemma 2.1 follows. \hfill $\square$

Corollary. Let $\mathcal{L} \subset \mathbb{R}^n$ be a convex polytope and denote by $V$ the set of vertices. Assume that $O$ is a regular point of $\mathcal{L}$. Then, we have

$$\sigma(\mathcal{L}, O) = \min_{\{V_0, \ldots, V_n\} \in V} \sum_{i=0}^{n} \frac{1}{1 + \Lambda(V_i, O)}.$$ 

Returning to the general setting, as in Section 1, we let $\mathcal{L}_0 \subset \partial \mathcal{L}$ denote the set of extreme points. We have $\mathcal{L} = \overline{\mathcal{L}_0}$. Recall that an extreme point $C$ is isolated if $C$ has an open neighborhood disjoint from $\mathcal{L}_0 \setminus \{C\}$. Our first task is to describe $\mathcal{L}$ near an isolated extreme point.
Lemma 2.2. Let $\mathcal{L} \subset \mathbb{R}^n$ be a compact convex body and $\mathcal{L}_0$ the set of extreme points. Let $C \in \mathcal{L}_0$ be an isolated extremal point. Then

\begin{equation}
U_C = \mathcal{L} \setminus [\mathcal{L}_0 \setminus \{C\}]
\end{equation}

is a relatively open set in $\mathcal{L}$ that contains $C$. For any $C' \in U_C \cap \partial \mathcal{L}$, $C' \neq C$, the line segment $[C, C']$ is on the boundary of $\mathcal{L}$, and it extends to a boundary line segment $[C, C'']$ with $C'' \in [\mathcal{L}_0 \setminus \{C\}]$.

Proof. Let $C$ be an isolated extreme point. For the first statement we need to show that

\begin{equation}
C \notin [\mathcal{L}_0 \setminus \{C\}].
\end{equation}

Assuming the contrary, we can select a sequence $\{C_k\}_{k \geq 1} \subset [\mathcal{L}_0 \setminus \{C\}]$ converging to $C$. For each $k \geq 1$, we can write $C_k$ as a convex linear combination $\sum_{i=0}^{n} \lambda_i C_{ik}$, where $C_{ik} \in \mathcal{L}_0$, $C_{ik} \neq C$. By compactness, we may assume that, for each $0 \leq i \leq n$, $C_{ik} \to C_i$ and $\lambda_i \to \lambda_i$ as $k \to \infty$. Taking the limit, we obtain $C = \sum_{i=0}^{n} \lambda_i C_i$. Since $C$ is an extreme point, the only way this is possible is that this sum reduces to a single term. We obtain that $C_i = C$ for a specific $0 \leq i \leq n$, and so $C_{ik} \to C_i = C$ as $k \to \infty$. Hence $C$ is not isolated. (2.2) follows.

For the second statement, let $C' \notin [\mathcal{L}_0 \setminus \{C\}]$ be a boundary point of $\mathcal{L}$. Since $[\mathcal{L}_0] = \mathcal{L}$, we can certainly write $C'$ as a convex linear combination of $C$ and finitely many points in $\mathcal{L}_0 \setminus \{C\}$. The point $C$ must participate in this linear combination with positive coefficient. Hence $C'$ is in the interior of a segment $[C, C'']$, where $C'' \in [\mathcal{L}_0 \setminus \{C\}]$. Finally, since $C$ and $C'$ are both boundary points of $\mathcal{L}$, the entire line segment $[C, C'']$ is on the boundary of $\mathcal{L}$. Lemma 2.2 follows.

Remark. Consider the second example above. Removing $C_-$ from $\mathcal{L}_0$, we see that $[\mathcal{L}_0 \setminus \{C_\} ]$ is the positive cone $[S, C_+]$ with the half-open segment $[(1,0,0), C_+]$ deleted. Its closure is $[\mathcal{L}_0 \setminus \{C_\} ] = [S, C_+]$, and hence $U_{C_-} = [S, C_-] \setminus [S]$. Notice that, for any $C'$ in the interior of $[C_-, (1,0,0)]$, the line segment $[C_-, C']$ extends beyond $U_{C_-}$ to $[C_-, C_+]$.

Lemma 2.3. Let $C$ be an isolated extreme point of $\mathcal{L}$ with associated open set $U_C$. Then, for every $O \in U_C$, there is a minimal configuration which contains $C$. In particular, if $O$ is regular, then $\Lambda(., O)$ takes a local maximum at $C$.

Proof. Let $O \in U_C$ and, as in Lemma 2.1, choose a minimal configuration consisting of extreme points. If $C$ does not participate in the configuration, then $O$ must be contained in $[\mathcal{L}_0 \setminus \{C\}]$. This contradicts the assumption. Thus, $C$ is a point in the configuration. The last statement is clear.

For the next step we introduce some notation and recall some results in [7] (Section 7). Let $C$ be an isolated extreme point of $\mathcal{L}$. Let $\tau \subset \mathbb{R}^n$ be a plane passing through $C$ and an interior point $O \in U_C$ of $\mathcal{L}$. We consider the planar convex body $\mathcal{L} \cap \tau$ with isolated extreme point $C$. As Lemma 2.3 asserts, $\mathcal{L} \cap \tau$ contains an angular domain with vertex at $C$. We let $[C, P], [C, Q] \subset \partial \mathcal{L} \cap \tau$ denote the maximal side segments of this domain. We orient $\tau$ from $O$ such that the positive orientation corresponds to the sequence $P, C, Q$. As in Section 7 of [7], $\alpha = \alpha_\tau(C)$ is the angle $\angle O C Q$. In a similar vein, we let $\alpha'' = \alpha_\tau(C'')$, where $\alpha''$ is the angle with vertex at $C''$ between the line segment $[C'', O]$ and the right tangent at $C''$ to the boundary of $\mathcal{L} \cap \tau$. ASYMMETRY OF CONVEX SETS 291
Lemma 2.4. Let \( \phi \) be a compact convex body, \( C \) an isolated extreme point, and assume that \( U_C \) consists of regular points. Then, for any plane passing through \( C \) and an interior point of \( \mathcal{L} \), we have

\[
\alpha \leq \alpha^o.
\]

(2.3)

For a boundary point \( B \) of \( \mathcal{L} \cap \tau \), let \( 0 \leq \phi(C) \leq \pi \) denote the angle between the left and right tangent lines at \( B \) to \( \partial \mathcal{L} \cap \tau \). Then we have \( \phi(C) = \angle PCQ \).

For \( O \) close to a fixed interior point of \([C,P]\), the right tangent to \( \mathcal{L} \cap \tau \) at \( C^o \) intersects the extension of the line segment \([C,P]\) beyond \( P \). We let \( R \) denote this intersection point. From the triangle \( \triangle CC^oR \), we obtain

\[
\phi(C) - \alpha + \alpha^o + \beta = \pi,
\]

where \( \beta = \angle C^oRC \). Combining this with (2.3), we get

\[
\phi(C) + \beta \leq \pi.
\]

We now let \( O \) approach a fixed interior point of \([C,P]\). We claim that \( \beta \) approaches \( \phi(P) \). In fact, as \( O \) approaches a fixed interior point of \([C,P]\), the antipodal \( C^o \) approaches \( P \) along the boundary of \( \mathcal{L} \cap \tau \), and the right tangent line at \( C^o \) approaches the left tangent line at \( P \). (See formula (6) in [5], p. 7.) We obtain the following:

**Lemma 2.4.** Let \( \mathcal{L} \subset \mathbb{R}^n \) be a compact convex body, \( C \) an isolated extreme point, and assume that \( U_C \) consists of regular points. Then, for any plane passing through \( C \) and an interior point of \( \mathcal{L} \), we have

\[
\phi(C) + \phi(P) \leq \pi,
\]

where \( \phi(C) \) and \( \phi(P) \) are the tangential angles of \( \mathcal{L} \cap \tau \) at \( C \) and \( P \), and \([C,P]\) is a maximal line segment on the boundary of \( \mathcal{L} \).

In the lemma above, we call \( P \) an adjacent point to the isolated extreme point \( C \). \( P \) is adjacent to \( C \) if \([C,P]\) is a maximal line segment on the boundary of \( \mathcal{L} \).

**Proof of Theorem 1.1.** We may assume that \( \mathcal{L} \cap \tau \) is more than \([C_0,C_1]\), in which case \( \mathcal{L} \cap \tau \) is a compact convex body with isolated extreme points \( C_0 \) and \( C_1 \). Let \( P_0,Q_0 \in \partial \mathcal{L} \cap \tau \) and \( P_1,Q_1 \in \partial \mathcal{L} \cap \tau \) be adjacent to \( C_0 \) and \( C_1 \), respectively. Orient \( \tau \) and choose the labels such that (with respect to any interior point of \( \mathcal{L} \cap \tau \)) \( P_0,C_0,Q_0 \) and \( P_1,C_1,Q_1 \) are positively oriented. Assume first that the adjacent points are all distinct, the right tangent at \( Q_0 \) and the left tangent at \( P_1 \) intersect at a point \( X \), and the left tangent at \( P_0 \) and the right tangent at \( Q_1 \) intersect at a point \( Y \). For the angle sum of the (convex) octagon \([P_0,C_0,Q_0,X,P_1,C_1,Q_1,Y]\) we have

\[
\phi(P_0) + \phi(C_0) + \phi(Q_0) + \beta + \phi(P_1) + \phi(C_1) + \phi(Q_1) + \gamma = 6\pi,
\]

where \( \beta \) and \( \gamma \) are the angles at \( X \) and \( Y \), respectively. On the other hand, by (2.4), we have

\[
\phi(C_0) + \phi(P_0), \phi(C_0) + \phi(Q_0), \phi(C_1) + \phi(P_1), \phi(C_1) + \phi(Q_1) \leq \pi.
\]

Adding these, we obtain

\[
2\phi(C_0) + 2\phi(C_1) + \phi(P_0) + \phi(P_1) + \phi(Q_0) + \phi(Q_1) \leq 4\pi.
\]

Comparing this with (2.5), we get

\[
\phi(C_0) + \phi(C_1) + 2\pi - \beta - \gamma \leq 0.
\]
This is a contradiction. Notice that we get a contradiction even when \( \beta = \pi \) or \( \gamma = \pi \) (the cases when the corresponding tangents coincide), and even when \( P_0 = Q_1 \) but \( P_1 \neq Q_0 \), or when \( P_1 = Q_0 \) but \( P_0 \neq Q_1 \).

If \( X \) or \( Y \) do not exist, we can add additional supporting lines to boundary points of \( L \cap \tau \) and get a contradiction again.

Summarizing, we obtain \( P_0 = Q_1 \) and \( P_1 = Q_0 \). As a byproduct, we also obtain that \( L \cap \tau = [P_0, C_0, P_1, C_1] \). If \( P_0, C_0, P_1, C_1 \) are all distinct, then, by \( (2.4) \), \([P_0, C_0, P_1, C_1]\) is a parallelogram with \([C_0, C_1]\) as a diagonal. Finally, if these points are not distinct, then \( L \cap \tau \) is a triangle with \([C_0, C_1]\) as a side (and \( P_0 \) or \( P_1 \) is the other vertex).

It remains to show that the parallelogram intersection is impossible. As in Lemmas 2.1-2.3, we let \( O \in \text{U}C_n \), and consider a minimal configuration \( \{C_0, C_1, \ldots, C_n\} \in \mathcal{C}(L, O) \) consisting of extreme points only. By the last statement of Lemma 2.2, \( O \) is contained in the interior of the triangle \([P_0, C_0, P_1]\). Thus, the opposite \( P_0^m \) is contained in \([C_0, P_1]\). Any point in the segment \([C_0, P_0^m]\) has the same distortion along \( C_0 \) since \( L \cap \tau \) is a parallelogram. Since \( O \) and \([L_0 \setminus C_0]\) are disjoint, there must be a point \( C_0' \in [C_0, P_1] \) for which \( O \) is on the boundary of \([L_0 \setminus \{C_0\}] \cup \{C_0'\}\).

Thus, \( O \) is on the boundary of \([C_0', C_1, \ldots, C_n]\). Hence \( \{C_0', \ldots, C_n\} \in \mathcal{C}(L, O) \). It must be minimal with \( C_0' \in [C_0, P_0^m] \) since the distortion along \([P_0^m, P_1]\) increases.

This, however, contradicts the regularity of \( O \). Theorem 1.1 follows. \( \square \)

**Proof of Theorem 1.2.** Let \( L \subset \mathbb{R}^n \) be as in Theorem 1.1. For \( 1 \leq m < n \), let \( P_m \) be the following statement: If \( C_0, \ldots, C_m \in L \) are (distinct) isolated extreme points, then they are affinely independent, and, for any \((m+1)\)-dimensional affine subspace \( \tau \subset \mathbb{R}^n \) that contains \( C_0, \ldots, C_m \), the intersection \( L \cap \tau \) is either \([C_0, \ldots, C_m]\) or an \((m+1)\)-simplex with \([C_0, \ldots, C_m]\) as a side.

Notice that \( P_1 \) is Theorem 1.1, and the second statement of \( P_{n-1} \) is Theorem 1.2. Therefore, Theorem 1.2 will follow by proving \( P_m \) by induction with respect to \( m = 1, \ldots, n-1 \). Before the general induction step, it is convenient to have an intermediate step:

**Lemma 2.5.** Let \( L \) be as in Theorem 1.1. Assume that, for a fixed \( 2 \leq m < n \), \( P_i \), \( 1 \leq i < m \), hold. Let \( C_0, \ldots, C_m \) be isolated extreme points of \( L \). Then, \( C_0, \ldots, C_m \) are affinely independent and

\[
L \cap (C_0, \ldots, C_m) = [C_0, \ldots, C_m].
\]

**Proof.** We first show affine independence. Since \( P_{m-1} \) holds, \( C_0, \ldots, C_{m-1} \) are certainly affinely independent. Thus, the affine span \( \tau = \langle C_0, \ldots, C_{m-1} \rangle \subset \mathbb{R}^n \) is \((m-1)\)-dimensional. Applying \( P_{m-2} \) to \( \tau \), we obtain that \( L \cap \tau \) is an \((m-1)\)-simplex with \([C_0, \ldots, C_{m-2}]\) as a side. Since \( C_{m-1} \notin \langle C_0, \ldots, C_{m-2} \rangle \) is an extreme point of \( L \), it is also an extreme point of \( L \cap \tau \). Thus, we have

\[
L \cap \tau = [C_0, \ldots, C_{m-1}].
\]

Now, \( C_m \) cannot be in this set since it is an extreme point of \( L \) and thereby also an extreme point of \( L \cap \tau \). Thus, \( C_0, \ldots, C_m \) are affinely independent. We now add \( C_m \) to \( \tau \) and set \( \tau = \langle C_0, \ldots, C_m \rangle \subset \mathbb{R}^n \), an \( m \) dimensional affine subspace. Applying \( P_{m-1} \), once again, \( L \cap \tau \) must be an \( m \)-simplex with \([C_0, \ldots, C_{m-1}]\) as a side. \( C_m \) is an extreme point in \( L \) and also in \( L \cap \tau \). Equation (2.6) follows. \( \square \)
We now return to the proof of the general induction step. Assume that, for a fixed $2 \leq m < n$, $P_i$, $1 \leq i < m$, hold. The first statement in $P_m$ is contained in Lemma 2.5. To prove the second statement, let $\tau \subset \mathbb{R}^n$ be an $(m+1)$ dimensional affine subspace that contains $C_0, \ldots, C_m$. We may assume that $L \cap \tau \neq [C_0, \ldots, C_m]$, since otherwise we are done. Equation (2.6) and $[L_0] = L$ show that $L$ contains an extreme point $C$ away from $\langle C_0, \ldots, C_m \rangle$. In other words, $C_0, \ldots, C_m, C$ are affinely independent, and the $(m+1)$-simplex $[C_0, \ldots, C_m, C]$ is contained in $L \cap \tau$. It remains to show that

$$L \cap \tau = [C_0, \ldots, C_m, C].$$

To do this, we will show that

$$[C_0, \ldots, \widehat{C_i}, \ldots, C_m, C] \subset \partial L, \quad 0 \leq i \leq m. \tag{2.8}$$

First note that (2.8) implies (2.7). Indeed, (2.8) says that all the faces of $[C_0, \ldots, C_m, C]$ opposite to $C_0, \ldots, C_m$ are on the boundary of $L$. If the face $[C_0, \ldots, C_m]$ were not on the boundary of $L$, then there would be another extreme point of $L$, say $C' \in \partial L \cap \tau$, on the side of $[C_0, \ldots, C_m] \subset \tau$ opposite to $C$. By (2.8) with $C$ replaced by $C'$, we would obtain that $L \cap \tau = [C_0, \ldots, C_m, C, C']$ is a double simplex with common base $[C_0, \ldots, C_m]$. This clearly contradicts $P_i$.

It remains to show (2.8). To do this, for $0 \leq i \leq m$, we let $\tau_i = \langle C_0, \ldots, \widehat{C_i}, \ldots, C_m, C \rangle \subset \tau$ and apply $P_{m-1}$ to $\tau_i$. Then (2.6) in Lemma 2.5 gives

$$L \cap \tau_i = [C_0, \ldots, \widehat{C_i}, \ldots, C_m, C], \quad 0 \leq i \leq m.$$

In particular, for $0 \leq i < j \leq m$, the $(m-1)$-simplex

$$[C_0, \ldots, \widehat{C_i}, \ldots, \widehat{C_j}, \ldots, C_m, C]$$

is on the boundary of $L$. Let $C' \in \partial L$ be a point in the interior of this $(m-1)$-simplex and consider the plane $\sigma = \langle C_i, C_j, C' \rangle$. By $P_1$, $L \cap \sigma = [C_i, C_j, C'']$ for some $C'' \in \partial L \cap \sigma$ with $C' \in [C_i, C_j, C'']$. We claim that $C'' = C'$. This will clearly imply (2.8).

First, $C''$ cannot be in the interior of $[C_i, C_j, C'']$ since otherwise $C', C''$ and the unique intersection point $C''' = (C', C'') \cap \sigma$ would be three collinear points on $\partial L$, so that, by convexity, $[C''', C''']$ would be on the boundary of $L$.

Thus, $C'$ is on the boundary of $[C_i, C_j, C''']$, say $C' \in [C_i, C''']$. On the other hand, $C' \in [C_0, \ldots, \widehat{C_i}, \ldots, \widehat{C_j}, \ldots, C_m, C]$ and $C'' \in [C_0, \ldots, \widehat{C_i}, \ldots, \widehat{C_j}, \ldots, C_m, C]$ since $C_i, C', C''$ are collinear. As $[C_0, \ldots, \widehat{C_i}, \ldots, \widehat{C_j}, \ldots, C_m, C]$ is the side of the $m$-simplex $[C_0, \ldots, \widehat{C_i}, \ldots, \widehat{C_j}, \ldots, C_m, C]$ opposite to $C_i$, $C' = C''$ follows. The second statement of $P_m$ and hence Theorem 1.2 follow. $\square$

REFERENCES


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