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THE WREATH PRODUCT OF $\mathbb{Z}$ WITH $\mathbb{Z}$ HAS HILBERT COMPRESSION EXPONENT $\frac{2}{3}$

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ABSTRACT. Let G be a finitely generated group, equipped with the word metric d associated with some finite set of generators. The Hilbert compression exponent of G is the supremum over all $\alpha \geq 0$ such that there exists a Lipschitz mapping $f : G \to L_2$ and a constant $c > 0$ such that for all $x, y \in G$ we have $\|f(x) - f(y)\|_2 \geq cd(x, y)^\alpha$. It was previously known that the Hilbert compression exponent of the wreath product $\mathbb{Z} \wr \mathbb{Z}$ is between $\frac{2}{3}$ and $\frac{3}{4}$. Here we show that $\frac{2}{3}$ is the correct value. Our proof is based on an application of K. Ball's notion of Markov type.

1. Introduction

Let G be a finitely generated group. Fix a finite set of generators $S \subseteq G$, which we will always assume to be symmetric (i.e. $S^{-1} = S$). Let d be the left-invariant word metric induced by S on G. The **Hilbert compression exponent** of G, which we denote by $\alpha^*(G)$, is the supremum over all $\alpha \geq 0$ such that there exists a 1-Lipschitz mapping $f : G \to L_2$ and a constant $c > 0$ such that for all $x, y \in G$ we have

$$\|f(x) - f(y)\|_2 \geq cd(x, y)^\alpha.$$

Note that $\alpha^*(G)$ does not depend on the choice of the finite set of generators S, and is thus an algebraic invariant of the group G. This notion was introduced by Guentner and Kaminker in [7] as a natural quantitative measure of Hilbert space embeddabililty in situations where bi-Lipschitz embeddings do not exist (where bi-Lipschitz embeddings do exist the natural measure would be the *Euclidean distortion*). More generally, the **compression function** of a 1-Lipschitz mapping $f : G \to L_2$ is defined as

$$\rho(t) := \inf_{d(x,y) \geq t} \|f(x) - f(y)\|_2.$$

The mapping f is called a **coarse embedding** if $\lim_{t\to\infty} \rho(t) = \infty$. Coarse embeddings of discrete groups have been studied extensively in recent years. The Hilbert compression exponents of various groups were investigated in [7, 2, 5, 16, 1]—we

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refer to these papers and the references therein for group-theoretical motivation and applications.

Consider the wreath product $\mathbb{Z} \wr \mathbb{Z}$, i.e. the group of all pairs (f, x), where $x \in \mathbb{Z}$ and $f : \mathbb{Z} \to \mathbb{Z}$ has finite support, equipped with the group law

$$(f, x)(g, y) := q(z \mapsto f(z) + g(z - x), x + y).$$

In this paper we prove that $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) = \frac{2}{3}$. The problem of computing $\alpha^*(\mathbb{Z} \wr \mathbb{Z})$ was raised explicitly in [2, 16, 1]. In [2] Arzhantseva, Guba and Sapir showed that $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \in \left[\frac{1}{2}, \frac{3}{4}\right]$. In [16] Tessera claimed to improve the lower bound on $\alpha^*(\mathbb{Z} \wr \mathbb{Z})$ to $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{2}{3}$, and conjectured that $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) = \frac{2}{3}$. Unfortunately, Tessera's proof is flawed, as explained in Remark 1.4 of [12]; his method only yields the bound $\alpha^*(Z \wr Z) \geq \frac{1}{3}$. However, the inequality $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{2}{3}$ is correct, as shown by Naor and Peres in [12] using a different method. Here we obtain the matching upper bound $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \leq \frac{2}{3}$. For the sake of completeness, in Remark 2.2 below we also present the embeddings of Naor and Peres [12] which establish the lower bound $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{2}{3}$.

Our proof of the upper bound $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \leq \frac{2}{3}$ is a simple application of K. Ball's notion of **Markov type**, a metric invariant that has found several applications in metric geometry in the past two decades; see [3, 11, 9, 4, 13, 10]. Recall that a Markov chain $\{Z_t\}_{t=0}^{\infty}$ with transition probabilities $a_{ij} := \Pr(Z_{t+1} = j \mid Z_t = i)$ on the state space $\{1, \ldots, n\}$ is *stationary* if $\pi_i := \Pr(Z_t = i)$ does not depend on t and is *reversible* if $\pi_i\, a_{ij} = \pi_j\, a_{ji}$ for every $i, j \in \{1, \ldots, n\}$. Given a metric space (X, d_X) and $p \in [1, \infty)$, we say that X has Markov type p if there exists a constant $K > 0$ such that for every stationary reversible Markov chain $\{Z_t\}_{t=0}^{\infty}$ on $\{1, \ldots, n\}$, every mapping $f : \{1, \ldots, n\} \to X$ and every time $t \in \mathbb{N}$,

$$\mathbb{E}\big[d_X(f(Z_t), f(Z_0))^p\big] \leq K^p\, t\, \mathbb{E}\big[d_X(f(Z_1), f(Z_0))^p\big]. \tag{1.1}$$

The least such K is called the Markov type p constant of X and is denoted $M_p(X)$.

The fact that L_2 has Markov type 2 with constant 1, first noted by K. Ball [3], follows from a simple spectral argument (see also inequality (8) in [13]). Since for $p \in [1, 2]$ the metric space $\left(L_p, \|x - y\|_2^{p/2}\right)$ embeds isometrically into L_2 (see [17]), it follows that L_p has Markov type p with constant 1. For $p > 2$ it was shown in [13] that L_p has Markov type 2 with constants $O\left(\sqrt{p}\right)$. We refer to [13] for a computation of the Markov type of various additional classes of metric spaces.

The notion of Markov type has been successfully applied to various embedding problems of *finite* metric spaces. In this paper we observe that one can use this invariant in the context of infinite amenable groups as well. In a certain sense, our argument simply amounts to using the Markov type asymptotically along neighborhoods of Følner sequences.

For the rest of the paper, let G be an amenable group with a fixed finite symmetric set of generators S and the corresponding left-invariant word metric d. Let e denote the identity element of G, and let $\{W_t\}_{t=0}^{\infty}$ be the canonical simple random walk on the Cayley graph of G determined by S, starting at e. Our main result is:

Proposition 1.1. *Assume that there exist $c, \delta, \beta > 0$ such that for all $t \in \mathbb{N}$,*

$$\Pr\left(d(W_t, e) \geq ct^{\beta}\right) \geq \delta. \tag{1.2}$$

Let (X, d_X) be a metric space with Markov type p, and assume that $f : G \to X$ satisfies

$$\rho(d(x,y)) \le d_X(f(x), f(y)) \le d(x,y) \tag{1.3}$$

for all $x, y \in G$, where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing. Then for all $t \in \mathbb{N}$,

$$\rho\left(ct^\beta\right) \le \frac{M_p(X)}{\delta^{1/p}} t^{1/p}.$$

In particular,

$$\alpha^*(G) \le \frac{1}{2\beta}.$$

As an immediate corollary we deduce that $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \le \frac{2}{3}$. Indeed, $\mathbb{Z} \wr \mathbb{Z}$ is amenable (see for example [8, 14]), and it was shown by Revelle in [15] that $\mathbb{Z} \wr \mathbb{Z}$ has a set of generators (namely the canonical generators $S = \{(1,0), (-1,0), (0,1), (0,-1)\}$) which satisfies the assumption of Proposition 1.1 with $\beta = \frac{3}{4}$ (see also [6] for the corresponding bound on the expectation of $d(W_t, e)$).

2. Proof of Proposition 1.1

Let $\{F_n\}_{n=0}^\infty$ be a Følner sequence for G; i.e. for every $\varepsilon > 0$ and any finite $K \subseteq G$, we have $|F_n \triangle (F_n K)| \le \varepsilon |F_n|$ for large enough n. Fix an integer $t > 0$ and denote

$$A_n := \bigcup_{x \in F_n} B(x,t) \supseteq F_n,$$

where $B(x,t)$ is the ball of radius t centered at x in the word metric determined by S.

For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\varepsilon |F_n| \ge |F_n \triangle (F_n B(e,t))| = |A_n \setminus F_n|. \tag{2.1}$$

Let $\{Z_t\}_{t=0}^\infty$ be the delayed standard random walk restricted to A_n. In other words, Z_0 is uniformly distributed on A_n, and for all $j \ge 0$ and $x \in A_n$,

$$\Pr\left(Z_{j+1} = x \middle| Z_j = x\right) = 1 - \frac{|(xS) \cap A_n|}{|S|},$$

and if $s \in S$ is such that $xs \in A_n$, then

$$\Pr\left(Z_{j+1} = xs \middle| Z_j = x\right) = \frac{1}{|S|}.$$

It is straightforward to check that $\{Z_t\}_{t=0}^\infty$ is a stationary reversible Markov chain. Hence, using the Markov type p property of X, and the fact that f is 1-Lipschitz, we see that

$$\begin{aligned} \mathbb{E}\big[d_X(f(Z_t), f(Z_0))^p\big] &\overset{(1.1)}{\le} M_p(X)^p t \mathbb{E}\big[d_X(f(Z_1), f(Z_0))^p\big] \\ &\overset{(1.3)}{\le} M_p(X)^p t \mathbb{E}\big[d(Z_1, Z_0)^p\big] \le M_p(X)^p t. \end{aligned} \tag{2.2}$$

Note that

$$\mathbb{E}\big[d_X(f(Z_t), f(Z_0))^p\big] \overset{(1.3)}{\ge} \mathbb{E}\big[\rho(d(Z_t, Z_0))^p\big] \ge \frac{1}{|A_n|} \sum_{x \in F_n} \mathbb{E}\left[\rho\left(d(Z_t, Z_0)\right)^p \middle| Z_0 = x\right], \tag{2.3}$$

since the omitted summands corresponding to $x \notin F_n$ are non-negative. If $x \in F_n$, then $B(x,t) \subseteq A_n$; this implies that conditioned on the event $\{Z_0 = x\}$, the random variable $d(Z_t, Z_0)$ has the same distribution as the random variable $d(W_t, e)$. The assumption (1.2) yields that

$$\mathbb{E}\big[\rho\left(d(W_t,e)\right)^p\big] \ge \rho\left(ct^\beta\right)^p \cdot \Pr\left(d(W_t,e) \ge ct^\beta\right) \ge \rho\left(ct^\beta\right)^p \cdot \delta\,. \tag{2.4}$$

In conjunction with (2.3), this gives that

$$\begin{aligned}\mathbb{E}\big[d_X(f(Z_t), f(Z_0))^p\big] &\ge \frac{|F_n|}{|A_n|} \cdot \mathbb{E}\big[\rho\left(d(W_t,e)\right)^p\big] \\ &\overset{(2.4)}{\ge} \frac{|F_n|}{|A_n|} \cdot \rho\left(ct^\beta\right)^p \cdot \delta \overset{(2.1)}{\ge} \frac{\delta}{1+\varepsilon} \cdot \rho\left(ct^\beta\right)^p .\end{aligned} \tag{2.5}$$

Combining (2.2) and (2.5) and letting $\varepsilon \to 0$ conclude the proof of Proposition 1.1. □

Remark 2.1. Given two groups G and H, the wreath product $G \wr H$ is the group of all pairs (f,x) where $f : H \to G$ has finite support (i.e. $f(z)$ is the identity of G for all but finitely many $z \in H$) and $x \in H$, equipped with the product $(f,x)(g,y) := \big(z \mapsto f(z)g(x^{-1}z), xy\big)$. Consider the iterated wreath products $\mathbb{Z}_{(k)}$, where $\mathbb{Z}_{(1)} = \mathbb{Z}$ and $\mathbb{Z}_{(k+1)} := \mathbb{Z}_{(k)} \wr \mathbb{Z}$. In [15] it is shown that $\mathbb{Z}_{(k)}$ has a finite symmetric set of generators which satisfies the assumption of Proposition 1.1 with $\beta = 1-2^{-k}$. Thus $\alpha^*(\mathbb{Z}_{(k)}) \le \frac{1}{2-2^{1-k}}$. In fact, as shown in [12], $\alpha^*(\mathbb{Z}_{(k)}) = \frac{1}{2-2^{1-k}}$.

Remark 2.2. In [12] the lower bound $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \ge \frac{2}{3}$ is a particular case of a more general result. For the readers' convenience we present the resulting embeddings in the case of the group $\mathbb{Z} \wr \mathbb{Z}$.

In what follows $\lesssim$ and $\gtrsim$ denote the corresponding inequality up to a universal constant. Fix $\alpha \in (0, 1/2)$ and let

$$\Big\{v_g : \ g : A \to \mathbb{Z} \text{ finitely supported, } A \in \{\mathbb{Z} \cap [n,\infty)\}_{n\in\mathbb{Z}} \cup \{\mathbb{Z} \cap (-\infty, n]\}_{n\in\mathbb{Z}}\Big\}$$

be disjointly supported unit vectors in $L_2(\mathbb{R})$. For $(f,k) \in \mathbb{Z} \wr \mathbb{Z}$ define a function $\phi_\alpha(f,k) : \mathbb{R} \to \mathbb{R}$ by

$$\phi_\alpha(f,k) := \sum_{n>k} (n-k)^\alpha \cdot v_{f\restriction_{[n,\infty)}} + \sum_{n<k} (k-n)^\alpha \cdot v_{f\restriction_{(-\infty,n]}}.$$

Observe that $\phi_\alpha(f,k) - \phi_\alpha(0,0) \in L_2(\mathbb{R})$. Indeed, if f is supported on $[-m,m]$, then

$$\begin{aligned}\|\phi_\alpha(f,k) - \phi_\alpha(0,0)\|_2^2 &\lesssim m\left(m^{2\alpha} + |k|^{2\alpha}\right) + \sum_{n\in\mathbb{Z}} \left(|n|^\alpha - |n-k|^\alpha\right)^2 \\ &\lesssim m\left(m^{2\alpha} + |k|^{2\alpha}\right) + \sum_{j=1}^\infty \frac{k^2}{j^{2(1-\alpha)}} < \infty.\end{aligned}$$

We can therefore define $F_\alpha : \mathbb{Z} \wr \mathbb{Z} \to \mathbb{R} \oplus \ell_2(\mathbb{Z}) \oplus L_2(\mathbb{R})$ by

$$F_\alpha(f,k) := k \oplus f \oplus \big(\phi_\alpha(f,k) - \phi_\alpha(0,0)\big).$$

We claim that for every $(f,k) \in \mathbb{Z} \wr \mathbb{Z}$ we have

$$d_{\mathbb{Z}\wr\mathbb{Z}}\big((f,k),(0,0)\big)^{\frac{2\alpha+1}{2\alpha+2}} \lesssim \|F_\alpha(f,k)\|_2 \lesssim \frac{1}{\sqrt{1-2\alpha}} \cdot d_{\mathbb{Z}\wr\mathbb{Z}}\big((f,k),(0,0)\big). \tag{2.6}$$

Since the metric $\|F_\alpha(f_1,k_1)-F_\alpha(f_2,k_2)\|_2$ is $\mathbb{Z}\wr\mathbb{Z}$-invariant and $F_\alpha(0,0)=0$, the inequalities in (2.6) imply that $\mathbb{Z}\wr\mathbb{Z}$ has a Hilbert compression exponent at least $\frac{2\alpha+1}{2\alpha+2}$. Letting $\alpha\uparrow\frac12$ shows that $\alpha^*(\mathbb{Z}\wr\mathbb{Z})\ge\frac23$.

It suffices to check the upper bound in (2.6) (i.e. the Lipschitz condition for F_α) when (f,k) is one of the generators of $\mathbb{Z}\wr\mathbb{Z}$, i.e. $(f,k)=(0,1)$ or $(f,k)=(\delta_0,0)$. Observe that $\|F_\alpha(\delta_0,0)\|_2=1$ and

$$\|F_\alpha(0,1)\|_2^2\lesssim\sum_{n=1}^{\infty}\left(n^\alpha-(n-1)^\alpha\right)^2\lesssim\frac{1}{1-2\alpha},$$

implying the upper bound in (2.6). To prove the lower bound in (2.6) assume that $m\in\mathbb{N}$ is the minimal integer such that f is supported on $[k-m,k+m]$. Then,

$$\|F_\alpha(f,k)\|_2^2\gtrsim k^2+\sum_{j=k-m}^{k+m}f(j)^2+\sum_{\ell=1}^{m}\ell^{2\alpha}\gtrsim k^2+\frac1m\left(\sum_{j\in\mathbb{Z}}|f(j)|\right)^2+m^{2\alpha+1}$$
$$\gtrsim\left(k+m+\sum_{j\in\mathbb{Z}}|f(j)|\right)^{\frac{4\alpha+2}{2\alpha+2}}\gtrsim d_{\mathbb{Z}\wr\mathbb{Z}}\big((f,k),(0,0)\big)^{\frac{4\alpha+2}{2\alpha+2}},$$

where the penultimate inequality follows by considering the cases $\|f\|_1\ge m^{\alpha+1}$ and $\|f\|_1\le m^{\alpha+1}$ separately.

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