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EXPONENTIAL ERGODICITY OF NON-LIPSCHITZ STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. Using the coupling method and Girsanov's theorem, we study the strong Feller property and irreducibility for the transition probabilities of stochastic differential equations with non-Lipschitz and monotone coefficients. Then, the exponential ergodicity and the spectral gap for the corresponding transition semigroups are obtained under fewer assumptions.

1. Introduction and main result

Consider the following stochastic differential equation (SDE):

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x_0 \in \mathbb{R}^d, \tag{1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are continuous functions and $(W_t)_{t\geqslant 0}$ is a d-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

When b and σ are locally Lipschitz continuous and monotonic, the asymptotic behavior of diffusion process X_t as $t \to \infty$ is well studied (see for example [2]). Recently, since the work of Malliavin in [8] and LeJan-Raimond in [7], there are increasing interests for the studies of non-Lipschitz SDE, in particular, when the coefficients have $r(1 \vee \log r^{-1})^{1/2}$-type continuous modulus. For this type of SDE, the flow property and large deviation estimate have been obtained in [1, 15, 5, 10, 11]. In this work, we shall concentrate on the study of the exponential ergodicity of solutions to this type of non-Lipschitz SDE.

Let us first recall some notions about the ergodicity (for example, see [2, Chapter 2]). Let $\{X_t(x_0), t \geqslant 0, x_0 \in \mathbb{E}\}$ be a family of Markov processes on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathbb{E}$ stands for the state space and is usually a Polish space. That defines a family of transition probabilities on Borel measurable space $(\mathbb{E}, \mathscr{B}(\mathbb{E}))$:

$$P_t(x_0, E) := \mathbf{P}(X_t(x_0) \in E), \quad E \in \mathscr{B}(\mathbb{E}).$$

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An invariant measure μ is defined as a stationary distribution of P_t, i.e.:

$$\int_{\mathbb{E}} P_t(x_0, E)\mu(\mathrm{d}x_0) = \mu(E), \quad \forall t > 0,\ E \in \mathscr{B}(\mathbb{E}).$$

The existence of such an invariant measure has been considered extensively for various systems (cf. [2, 3], etc.). But, the uniqueness (or ergodicity) of invariant measures is a more difficult problem. A standard method called the "overlap method" for the uniqueness of invariant measures of P_t is the following Doob-Khasminskii theorem (cf. [2]):

If P_t is strong Feller and irreducible, then P_t admits at most one invariant measure.

Recall that P_t is strong Feller if for each $t > 0$ and $E \in \mathscr{B}(\mathbb{E})$

$$\mathbb{E} \ni x_0 \mapsto P_t(x_0, E) \in [0, 1] \text{ is continuous;}$$

P_t is irreducible if for each $t > 0$ and $x_0 \in \mathbb{E}$

$$P_t(x_0, E) > 0 \text{ for any non-empty open set } E \subset \mathbb{E}.$$

For a dynamical system determined by an SDE with smooth coefficients, the strong Feller property is a direct consequence of the hypoellipticity of a diffusion operator associated with the SDE, which asserts by Hörmander's theorem that the transition probability has a smooth density with respect to the Lebesgue measure. Another efficient tool for proving the strong Feller property is the Bismut formula (cf. [2]). However, these two arguments seem to be hardly used to deal with the non-Lipschitz SDEs.

In order to prove the strong Feller property for non-Lipschitz SDEs, we shall use the coupling method combined with the Girsanov transformation. This method has been used in [13] to prove a Harnack type inequality for stochastic porous medium equations. An obvious advantage of this method lies in the succinctness of the proof. Here, the key point for us is the suitable choice of a coupling function. Moreover, for the irreducibility we shall use the ideas of the approximative controllability and Girsanov's transformation. The novelty of this paper is the use of Girsanov's theorem in the proof of the strong Feller property and irreducibility.

Before stating our main result, we introduce the following notations: Let $\langle \cdot, \cdot \rangle$ denote the inner product in $\mathbb{R}^d$, $|\cdot|$ the length of a vector in $\mathbb{R}^d$, and $\|\cdot\|_2$ the Hilbert-Schmit norm from $\mathbb{R}^d$ to $\mathbb{R}^d$. Let $B_b(\mathbb{R}^d)$ be the Banach space of all bounded measurable functions on $\mathbb{R}^d$; the norm in $B_b(\mathbb{R}^d)$ is denoted by $\|\cdot\|_0$.

We make the following assumptions on the continuous coefficients b and σ:

(H1) (Monotonicity) There exists a $\lambda_0 \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d$

$$\begin{aligned} &2\langle x - y, b(x) - b(y)\rangle + \|\sigma(x) - \sigma(y)\|_2^2 \\ \leqslant\ &\lambda_0 |x - y|^2 (1 \vee \log |x - y|^{-1}). \end{aligned}$$

(H2) (Growth of σ) There exists a $\lambda_1 > 0$ such that for all $x \in \mathbb{R}^d$

$$\|\sigma(x)\|_2 \leqslant \lambda_1 (1 + |x|).$$

(H3) (Non-degeneracy of σ) For some $\lambda_2 > 0$

$$\sup_{x \in \mathbb{R}^d} \|\sigma^{-1}(x)\|_2 \leqslant \lambda_2.$$

(H4) (One side growth of b) There exist a $p > 2$ and constants $\lambda_3, \lambda_4 \geqslant 0$ such that for all $x \in \mathbb{R}^d$

$$2\langle x, b(x)\rangle + \|\sigma(x)\|_2^2 \leqslant -\lambda_3 |x|^p + \lambda_4.$$

It is well known that under **(H1)** and **(H2)**, equation (1) has a unique continuous strong solution (cf. [12]), which is denoted by $X_t(x_0)$. The transition semigroup associated with $X_t(x_0)$ is defined by

$$P_t\varphi(x_0) := \mathbf{E}\varphi(X_t(x_0)), \quad t > 0, \quad \varphi \in B_b(\mathbb{R}^d).$$

The transition probability is given by

$$P_t(x_0, E) := (P_t 1_E)(x_0) = \mathbf{P}(X_t(x_0) \in E), \quad E \in \mathscr{B}(\mathbb{R}^d).$$

We are now in a position to state our main result in the present paper.

Theorem 1.1. *Assume* **(H1)-(H3)**. *Then the semigroup P_t is strong Feller and irreducibility. If in addition,* **(H4)** *holds, then there exists a unique invariant probability measure μ of P_t having full support in $\mathbb{R}^d$ such that*

(i) If $\lambda_3 = 0$ in **(H4)**, *then for all $t > 0$ and $x_0 \in \mathbb{R}^d$, μ is equivalent to $P_t(x_0, \cdot)$, and*

$$\lim_{t\to\infty} \|P_t(x_0, \cdot) - \mu\|_{\mathrm{Var}} = 0,$$

where $\|\cdot\|_{\mathrm{Var}}$ denotes the total variation of a signed measure.

(ii) If $\lambda_3 > 0$ in **(H4)**, *then for some $\alpha, C > 0$ independent of x_0 and t,*

$$\|P_t(x_0, \cdot) - \mu\|_{\mathrm{Var}} \leqslant C \cdot e^{-\alpha t}.$$

Moreover, for any $q > 1$ and each $\varphi \in L^q(\mathbb{R}^d, \mu)$

$$\|P_t\varphi - \mu(\varphi)\|_{L^q(\mathbb{R}^d,\mu)} \leqslant C_q \cdot e^{-\alpha t/q} \|\varphi\|_{L^q(\mathbb{R}^d,\mu)}, \quad \forall t > 0,$$

where α is the same as above and $\mu(\varphi) := \int_{\mathbb{R}^d} \varphi(x)\mu(\mathrm{d}x)$. In particular, let L_q be the generator of P_t in $L^q(\mathbb{R}^d, \mu)$; then L_q has a spectral gap (greater than α/q) in $L^q(\mathbb{R}^d, \mu)$.

This theorem will be proved in the next section.

2. Proof of the main result

We need the following generalization of the Gronwall-Belmman type inequality (cf. [15]).

Lemma 2.1 (Bihari's inequality)**.** *Let $\rho_\eta : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a concave function given by*

$$\rho_\eta(x) := \begin{cases} x \log x^{-1}, & x \leqslant \eta, \\ \eta \log \eta^{-1} + (\log \eta^{-1} - 1)(x - \eta), & x > \eta, \end{cases}$$

where $\eta > 0$. If $g(s), q(s)$ are two strictly positive functions on $\mathbb{R}^+$ such that

$$g(t) \leqslant g(0) + \int_0^t q(s)\rho_\eta(g(s))ds, \ t \geqslant 0, \tag{2}$$

then

$$g(t) \leqslant (g(0))^{\exp\{-\int_0^t q(s)ds\}}. \tag{3}$$

We shall separately prove the strong Feller property and irreducibility in the next two subsections, and assume that **(H1)-(H3)** hold.

2.1. **Strong Feller properties.** In the following, for the sake of simplicity, for $z \neq 0 \in \mathbb{R}^d$ we write

$$\bar{z} := z/|z|.$$

Let us now consider the following coupling SDEs:

$$\left\{\begin{array}{lll} \mathrm{d}X_t &= b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t), & X_0 = x_0, \\ \mathrm{d}Y_t &= b(Y_t)\mathrm{d}t + a(X_t - Y_t)\cdot 1_{\{t<\tau\}}\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t, & Y_0 = y_0, \end{array}\right. \tag{4}$$

where a is the coupling function defined by

$$a(z) := |x_0 - y_0|^\alpha \cdot 1_{\{z\neq 0\}} \cdot \bar{z}, \quad \alpha \in (0,1),$$

and τ is the coupling time given by

$$\tau := \inf\{t > 0 : |X_t - Y_t| = 0\}.$$

The second equation in (4) can be solved as follows: Define

$$a_\varepsilon(z) := |x_0 - y_0|^\alpha \cdot f_\varepsilon(|z|) \cdot \bar{z},$$

where $f_\varepsilon(r) : \mathbb{R}_+ \mapsto [0,1]$ is smooth and satisfies

$$f_\varepsilon(r) = 1 \quad \text{for } r > \varepsilon, \quad \text{and} \quad f_\varepsilon(r) = 0 \quad \text{for } r \in [0, \varepsilon/2].$$

It is easy to see that there is a $C_\varepsilon > 0$ such that for any $z, z' \in \mathbb{R}^d$

$$|a_\varepsilon(z) - a_\varepsilon(z')| \leqslant C_\varepsilon \cdot |z - z'|.$$

So, there is a unique solution Y^ε_t to the following SDE:

$$\mathrm{d}Y^\varepsilon_t = [b(Y^\varepsilon_t) + a_\varepsilon(X_t - Y^\varepsilon_t)]\mathrm{d}t + \sigma(Y^\varepsilon_t)\mathrm{d}W_t, \quad Y^\varepsilon_0 = y_0.$$

Define

$$\tau_\varepsilon := \inf\{t > 0 : |X_t - Y^\varepsilon_t| \leqslant \varepsilon\}.$$

For $\varepsilon' < \varepsilon$, by the uniqueness we clearly have $\tau_{\varepsilon'} \geqslant \tau_\varepsilon$ and

$$Y^\varepsilon_t = Y^{\varepsilon'}_t \quad \text{on } \{t < \tau_\varepsilon\}.$$

Thus, $\tau = \lim_{\varepsilon\downarrow 0} \tau_\varepsilon$ is just the coupling time and Y_t is well defined on $[0,\tau]$, and we also define

$$Y_t := X_t, \quad \text{for all } t \geqslant \tau.$$

Then, it is clear that Y_t solves the second equation in (4).

Set

$$Z_t := X_t - Y_t.$$

Using Itô's formula to the function $r \mapsto \sqrt{|r|^2 + \varepsilon}$, then letting $\varepsilon \downarrow 0$, we obtain by **(H1)**

$$\begin{aligned} &|Z_{t\wedge\tau}| - |x_0 - y_0| - \int_0^{t\wedge\tau} \langle \bar{Z}_s, (\sigma(X_s) - \sigma(Y_s))\mathrm{d}W_s\rangle \\ =\ & \int_0^{t\wedge\tau} (2|Z_s|)^{-1} \cdot \Big(2\langle Z_s, b(X_s) - b(Y_s)\rangle + \|\sigma(X_s) - \sigma(Y_s)\|_2^2\Big)\mathrm{d}s \\ & - \int_0^{t\wedge\tau} \langle \bar{Z}_s, a(Z_s)\rangle \mathrm{d}s - \int_0^{t\wedge\tau} (2|Z_s|)^{-1} \cdot |[\sigma(X_s) - \sigma(Y_s)]^*(\bar{Z}_s)|^2 \mathrm{d}s \\ \leqslant\ & \frac{\lambda_0}{2} \int_0^{t\wedge\tau} |Z_s|(1 \vee \log|Z_s|^{-1})\mathrm{d}s - |x_0 - y_0|^\alpha (t\wedge\tau). \end{aligned}$$

Note that there exists an $\eta > 0$ such that

$$r(1 \vee \log r^{-1}) \leqslant \rho_\eta(r), \quad \forall r > 0.$$

Taking expectations yields that

$$\begin{aligned}\mathbf{E}|Z_{t\wedge\tau}| &\leqslant |x_0-y_0|-|x_0-y_0|^\alpha\cdot\mathbf{E}(t\wedge\tau)+\frac{\lambda_0}{2}\mathbf{E}\int_0^{t\wedge\tau}\rho_\eta(|Z_s|)\mathrm{d}s\\ &\leqslant |x_0-y_0|-|x_0-y_0|^\alpha\cdot\mathbf{E}(t\wedge\tau)+\frac{\lambda_0}{2}\int_0^{t}\rho_\eta(\mathbf{E}|Z_{s\wedge\tau}|)\mathrm{d}s,\end{aligned}$$

where the second step is due to Jensen's inequality.

By the Bihari inequality (3), we get that for any $t>0$ and $|x_0-y_0|<\eta$,

$$\mathbf{E}|Z_{t\wedge\tau}|\leqslant|x_0-y_0|^{\exp\{-\lambda_0 t/2\}}$$

and

$$\mathbf{E}(t\wedge\tau)\leqslant|x_0-y_0|^{1-\alpha}+\frac{\lambda_0 t}{2}\rho_\eta(|x_0-y_0|^{\exp\{-\lambda_0 t/2\}})\cdot|x_0-y_0|^{-\alpha}. \tag{5}$$

We now fix a $T>0$ and define

$$R_T:=\exp\left[\int_0^{T\wedge\tau}\langle\mathrm{d}W_s,H(X_s,Y_s)\rangle-\frac{1}{2}\int_0^{T\wedge\tau}|H(X_s,Y_s)|^2\mathrm{d}s\right]$$

and

$$\tilde{W}_t:=W_t+\int_0^{t\wedge\tau}H(X_s,Y_s)\mathrm{d}s,$$

where

$$H(x,y):=|x_0-y_0|^\alpha\cdot[\sigma(y)]^{-1}(\overline{x-y}).$$

By **(H3)**, we have

$$|H(x,y)|^2\leqslant\lambda_2^2\cdot|x_0-y_0|^{2\alpha}.$$

Thus,

$$\mathbf{E}R_T=1\quad\text{and}\quad\mathbf{E}R_T^2\leqslant\exp\left[3T\lambda_2^2\cdot|x_0-y_0|^{2\alpha}/2\right].$$

By Girsanov's theorem, $(\tilde{W}_t)_{t\in[0,T]}$ is still a d-dimensional Brownian motion under the new probability measure $R_T\cdot\mathbf{P}$. Note that Y_t also solves

$$\mathrm{d}Y_t=b(Y_t)\mathrm{d}t+\sigma(Y_t)\mathrm{d}\tilde{W}_t,\quad Y_0=y_0.$$

So, the law of $X_T(y_0)$ under $\mathbf{P}$ is the same as the law of $Y_T(y_0)$ under $R_T\cdot\mathbf{P}$. We thus have for any $\varphi\in B_b(\mathbb{R}^d)$

$$\begin{aligned}&|P_T\varphi(x_0)-P_T\varphi(y_0)|\\ =\ &|\mathbf{E}(\varphi(X_T(x_0))-R_T\cdot\varphi(Y_T(y_0)))|\\ \leqslant\ &\mathbf{E}\left|(1-R_T)\cdot\varphi(X_T(x_0))\cdot 1_{\{\tau\leqslant T\}}\right|\\ &+\mathbf{E}\left|(\varphi(X_T(x_0))-R_T\cdot\varphi(Y_T(y_0)))\cdot 1_{\{\tau>T\}}\right|\\ \leqslant\ &\|\varphi\|_0\cdot\mathbf{E}|1-R_T|+\|\varphi\|_0\cdot\mathbf{E}\left[(1+R_T)1_{\{\tau>T\}}\right].\end{aligned}$$

By the elementary inequality $e^r-1\leqslant re^r$ for $r\geqslant 0$, we have for any $|x_0-y_0|\leqslant\eta$,

$$\begin{aligned}&(\mathbf{E}|1-R_T|)^2\leqslant\mathbf{E}|1-R_T|^2=\mathbf{E}R_T^2-1\\ \leqslant\ &\exp\left[3T\lambda_2^2\cdot|x_0-y_0|^{2\alpha}/2\right]-1\leqslant C_{T,\lambda_2,\eta}\cdot|x_0-y_0|^{2\alpha}\end{aligned}$$

and

$$\begin{aligned}&\left(\mathbf{E}\left[(1+R_T)1_{\{\tau\geqslant T\}}\right]\right)^2\leqslant(3+\mathbf{E}R_T^2)\cdot\mathbf{P}(\tau\geqslant T)\\ \leqslant\ &C_{T,\lambda_2,\eta}\cdot\mathbf{P}((2T)\wedge\tau\geqslant T)\leqslant C_{T,\lambda_2,\eta}\cdot\mathbf{E}((2T)\wedge\tau)/T.\end{aligned}$$

Taking $\alpha = \exp\{-\lambda_0 T\}/3$, then by (5) there exists a $0 < \eta' < \eta$ such that for any $|x_0 - y_0| < \eta'$,

$$\mathbf{E}((2T) \wedge \tau) \leqslant C_{T,\lambda_0,\eta'} \cdot |x_0 - y_0|^{\exp\{-\lambda_0 T\}/2}.$$

Summarizing the above calculations, we obtain the following strong Feller property:

Theorem 2.2. *Under* **(H**1**)**-**(H**3**)***, for any* $T > 0$*, there exist an* $\eta > 0$ *and constant* $C_{T,\lambda_0,\lambda_2,\eta} > 0$ *such that for all* $x_0, y_0 \in \mathbb{R}^d$ *with* $|x_0 - y_0| \leqslant \eta$ *and* $\varphi \in B_b(\mathbb{R}^d)$*,*

$$|P_T\varphi(x_0) - P_T\varphi(y_0)| \leqslant C_{T,\lambda_0,\lambda_2,\eta} \cdot \|\varphi\|_0 \cdot |x_0 - y_0|^{\exp\{-\lambda_0 T\}/4}.$$

2.2. **Irreducibility.** For proving the irreducibility of P_t, it suffices to prove that for any $x_0 \in \mathbb{R}^d, T > 0$, and $y_0 \in \mathbb{R}^d$, $a > 0$,

$$P_T(x_0, B(y_0, a)) = \mathbf{P}(|X_T(x_0) - y_0| \leqslant a) > 0, \tag{6}$$

where $B(y_0, a) := \{z \in \mathbb{R}^d : |z - y_0| \leqslant a\}$. The T, x_0, y_0 and a will be fixed in what follows.

By **(H**1**)** and **(H**2**)**, it is a standard deduction that for some $C_{T,x_0,\lambda_0,\lambda_1} > 0$

$$\sup_{t\in[0,T]} \mathbf{E}|X_t(x_0)|^2 \leqslant C_{T,x_0,\lambda_0,\lambda_1}. \tag{7}$$

Let $t_1 \in (0, T)$, whose value will be determined below. Set for $\varepsilon > 0$

$$X^\varepsilon_{t_1} := X_{t_1} \cdot 1_{\{|X_{t_1}| \leqslant \varepsilon^{-1}\}}. \tag{8}$$

Then

$$\lim_{\varepsilon\downarrow 0} \mathbf{E}|X^\varepsilon_{t_1} - X_{t_1}|^2 = 0. \tag{9}$$

Define for $s \in [t_1, T]$

$$Y^\varepsilon_s := \frac{T - s}{T - t_1} X^\varepsilon_{t_1} + \frac{s - t_1}{T - t_1} y_0$$

and

$$h^\varepsilon_s := \frac{y_0 - X^\varepsilon_{t_1}}{T - t_1} - b(Y^\varepsilon_s). \tag{10}$$

Then

$$Y^\varepsilon_{t_1} = X^\varepsilon_{t_1}, \quad Y^\varepsilon_T = y_0$$

and

$$Y^\varepsilon_t = X^\varepsilon_{t_1} + \int_{t_1}^t b(Y^\varepsilon_s)\mathrm{d}s + \int_{t_1}^t h^\varepsilon_s \mathrm{d}s, \quad t \in [t_1, T].$$

Consider the following SDE on $[t_1, T]$:

$$X^\varepsilon_t = X_{t_1} + \int_{t_1}^t b(X^\varepsilon_s)\mathrm{d}s + \int_{t_1}^t h^\varepsilon_s \mathrm{d}s + \int_{t_1}^t \sigma(X^\varepsilon_s)\mathrm{d}W_s.$$

If we define

$$X^\varepsilon_t := X_t, \quad \forall t \in [0, t_1],$$

then for any $t \in [0, T]$

$$X^\varepsilon_t = x_0 + \int_0^t b(X^\varepsilon_s)\mathrm{d}s + \int_0^t 1_{\{s>t_1\}} h^\varepsilon_s \mathrm{d}s + \int_0^t \sigma(X^\varepsilon_s)\mathrm{d}W_s.$$

We now define

$$\tilde{W}_t^\varepsilon := W_t + \int_0^t H_s^\varepsilon \mathrm{d}s$$

and

$$R_T^\varepsilon := \exp\left[\int_0^T \langle \mathrm{d}W_s, H_s^\varepsilon\rangle - \frac{1}{2}\int_0^T |H_s^\varepsilon|^2 \mathrm{d}s\right],$$

where

$$H_s^\varepsilon := 1_{\{s>t_1\}}[\sigma(X_s^\varepsilon)]^{-1} h_s^\varepsilon.$$

Noting that by **(H3)**, (8), (10) and the continuity of b

$$|H_s^\varepsilon| \leqslant \lambda_2 |h_s^\varepsilon| \leqslant C_{\lambda_2,\varepsilon,t_1},$$

we have

$$\mathbf{E}R_T^\varepsilon = 1, \quad \mathbf{P}(R_T^\varepsilon > 0) = 1,$$

and $(\tilde{W}_t^\varepsilon)_{t\in[0,T]}$ is a d-dimensional Brownian motion under the new probability measure $R_T^\varepsilon \cdot \mathbf{P}$. Thus, $X_T^\varepsilon(x_0)$ has the same law as $X_T(x_0)$. Hence, in order to prove (6), it suffices to prove that for some $t_1 \in (0,T)$ and $\varepsilon > 0$

$$\mathbf{P}(|X_T^\varepsilon(x_0) - y_0| \leqslant a) > 0,$$

or equivalently,

$$\mathbf{P}(|X_T^\varepsilon(x_0) - y_0| > a) < 1.$$

Set

$$Z_t^\varepsilon := X_t^\varepsilon - Y_t^\varepsilon.$$

By Itô's formula, we have by **(H1)** and **(H2)**

$$\begin{aligned}
\mathbf{E}|Z_t^\varepsilon|^2 &= \mathbf{E}|X_{t_1} - X_{t_1}^\varepsilon|^2 + \int_{t_1}^t \mathbf{E}\Big(2\langle Z_s^\varepsilon, b(X_s^\varepsilon) - b(Y_s^\varepsilon)\rangle + \|\sigma(X_s^\varepsilon)\|_2^2\Big)\mathrm{d}s \\
&\leqslant \mathbf{E}|X_{t_1} - X_{t_1}^\varepsilon|^2 + C_{\kappa_0}\int_{t_1}^t \mathbf{E}(|Y_s^\varepsilon|^2 + 1)\mathrm{d}s \\
&\quad + C_{\lambda_0,\kappa_0}\int_{t_1}^t \mathbf{E}(|Z_s^\varepsilon|^2(1 \vee \log|Z_s^\varepsilon|^{-1}))\mathrm{d}s.
\end{aligned}$$

Noticing that by (8) and (7)

$$\begin{aligned}
\int_{t_1}^T \mathbf{E}|Y_s^\varepsilon|^2 \mathrm{d}s &\leqslant 2(T-t_1)\cdot(\mathbf{E}|X_{t_1}^\varepsilon|^2 + |y_0|^2) \\
&\leqslant 2(T-t_1)\cdot(C_{T,x_0,\lambda_0,\lambda_1} + |y_0|^2),
\end{aligned}$$

and there exists an $\eta > 0$ such that

$$r^2(1 \vee \log r^{-1}) \leqslant \rho_\eta(r^2), \quad \forall r > 0,$$

we have by Jensen's inequality

$$\mathbf{E}|Z_t^\varepsilon|^2 \leqslant \mathbf{E}|X_{t_1} - X_{t_1}^\varepsilon|^2 + C|T-t_1| + C_{\lambda_0,\kappa_0}\int_{t_1}^t \rho_\eta(\mathbf{E}|Z_s^\varepsilon|^2)\mathrm{d}s,$$

where C is independent of ε and t_1.

By the Bihari inequality (3) again, we obtain

$$\mathbf{E}|X_T^\varepsilon - y_0|^2 \leqslant \Big[\mathbf{E}|X_{t_1} - X_{t_1}^\varepsilon|^2 + C(T-t_1)\Big]^{\exp\{-C_{\lambda_0,\kappa_0}\cdot T\}}.$$

Hence,

$$\begin{aligned}\mathbf{P}(|X^\varepsilon_T(x_0)-y_0|>a)&\leqslant\frac{1}{a^2}\mathbf{E}|X^\varepsilon_T(x_0)-y_0|^2\\ &\leqslant\frac{1}{a^2}\cdot\Big[\mathbf{E}|X_{t_1}-X^\varepsilon_{t_1}|^2+C(T-t_1)\Big]^{\exp\{-C_{\lambda_0,\kappa_0}\cdot T\}}.\end{aligned}$$

First letting t_1 close to T, and then choosing ε to be sufficiently small, we obtain by (9)

$$\mathbf{P}(|X^\varepsilon_T(x_0)-y_0|>a)<1.$$

The proof of irreducibility is thus complete.

2.3. **Proof of Theorem 1.1.** We now assume that **(H**1**)**-**(H**4**)** hold. Using Itô's formula, we have by **(H**4**)** and Hölder's inequality

$$\begin{aligned}\frac{\mathrm{d}\mathbf{E}|X_t|^2}{\mathrm{d}t}&=\mathbf{E}\Big(2\langle X_t,b(X_t)\rangle+\|\sigma(X_t)\|_2^2\Big)\\ &\leqslant-\lambda_3\mathbf{E}|X_t|^p+\lambda_4\\ &\leqslant-\lambda_3\Big(\mathbf{E}|X_t|^2\Big)^{p/2}+\lambda_4.\end{aligned}$$

Hence, for all $t>0$

$$\frac{1}{t}\int_0^t\mathbf{E}|X_s|^2\mathrm{d}s\leqslant\lambda_4.$$

The existence of an invariant probability measure μ now follows from the classical Krylov-Bogoliubov's method (cf. [2]). The first conclusion follows from the strong Feller property and irreducibility (cf. [9, 14, 2]).

On the other hand, if $\lambda_3>0$, let $f(t)$ solve the following ODE:

$$f'(t)=\lambda_4-\lambda_3f(t)^{p/2},\quad f(0)=|x_0|^2.$$

By the comparison theorem in ODE and [2, Lemma 1.2.6], we have for some $C>0$

$$\mathbf{E}|X_t|^2\leqslant f(t)\leqslant C\left[1+t^{\frac{1}{1-p/2}}\right],$$

where the right hand side is independent of x_0.

Since P_t is strong Feller and irreducible, we also have for any $r,a>0$ and $t>0$

$$\inf_{x_0\in B(0,r)}P_t(x_0,B(0,a))>0.$$

The second conclusion then follows from [6, Theorem 2.5 (b) and Theorem 2.7].

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