A LAW OF LARGE NUMBERS FOR ARITHMETIC FUNCTIONS

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ABSTRACT. We prove the weighted strong law of large numbers for every integrable i.i.d. sequence where the weights are given by a positive strongly additive function satisfying the Lindeberg condition. This result solves one of the open problems raised in the paper by Berkes and Weber (2007).

1. Main result

Let \( f \) be a strongly additive arithmetic function, i.e.,
\[
\begin{align*}
f(mn) &= f(m) + f(n) \text{ if } \gcd(m, n) = 1, \\
f(p^n) &= f(p) \quad \text{for all primes } p \text{ and positive integers } n.
\end{align*}
\]
Erdős-Kac \([2]\) proved that if \( f(p) = O(1) \) and \( B_p \to \infty \) where \( p \) varies along primes, then the sequence \( \{f(n)\} \) obeys the central limit theorem, i.e.,
\[
\lim_{N \to \infty} \frac{1}{N} \# \{n \leq N \mid f(n) \leq A_N + xB_N^{1/2} \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du,
\]
where
\[
A_n = \sum_{p \leq n} \frac{f(p)}{p}, \quad B_n = \sum_{p \leq n} \frac{f^2(p)}{p}.
\]

Here and in the sequel, we follow the usual convention and denote the summation along the primes by \( \sum_p \). Kubilius \([4]\) and Shapiro \([5]\) relaxed the condition \( f(p) = O(1) \) to the Lindeberg condition below:
\[
\lim_{N \to \infty} \frac{1}{B_N} \sum_{\{p \leq N : f(p) \geq \varepsilon B_N^{1/2}\}} \frac{f^2(p)}{p} = 0 \quad \text{for all } \varepsilon > 0.
\]

The purpose of this paper is to prove the following theorem and show that the irregularity of a positive strongly additive function \( f \) does not have an effect on the weighted law of large numbers.
Theorem 1.1. Suppose that a positive strongly additive function $f$ satisfies the Lindeberg condition (1.1). Then for any sequence $\{X_n\}$ of independent and identically distributed integrable random variables, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) X_n / \sum_{n=1}^{N} f(n) = EX_1 \text{ a.s.}$$

Berkes-Weber [1] proved the same conclusion by assuming

$$f(p) = o\left(\frac{B_p^{1/2}}{p}\right) \text{ and } B_p \to \infty \text{ as } p \to \infty,$$

which is stronger than the Lindeberg condition (1.1). They also proved it by assuming the Lindeberg condition and the following smoothness condition:

$$\sup_{n \leq p, p' \leq n^2} \frac{f(p)}{f(p')} = O(1),$$

and posed the question whether the Lindeberg condition alone is sufficient to have the same conclusion. Our result, which is proved by simple calculations without using a randomization technique as is used in [1], gives an affirmative answer to this question.

2. Proof

We use the following asymptotics, which are proved in [1] under the Lindeberg condition (1.1):

$$\sum_{n=1}^{N} f(n) \sim NA_N, \quad (2.1)$$

$$\sum_{n=1}^{N} f^2(n) \sim NA_N^2, \quad (2.2)$$

By (2.2), we can take a constant $C > 0$ such that

$$\sum_{n=1}^{N} f^2(n) \leq CNA_N^2, \quad (N \geq 1). \quad (2.3)$$

To prove our theorem, we appeal to the characterization by Jamison-Orey-Pruitt [3].

Lemma 2.1. Let $\{w_k\}$ be a sequence of positive numbers and put $W_N = \sum_{n=1}^{N} w_n$. Then

$$\lim_{N \to \infty} \frac{1}{W_N} \sum_{n=1}^{N} w_n X_n = EX_1 \text{ a.s.}$$

holds for any sequence $\{X_n\}$ of independent and identically distributed integrable random variables if and only if

$$\lim_{t \to \infty} \frac{1}{t} \#\{n : W_n \leq tw_n\} < \infty.$$

We apply this characterization by putting $w_n = f(n)$. Because of (2.1), it is sufficient to prove

$$\#\{n : nA_n \leq mf(n)\} \leq (1 + C)^2 m. \quad (2.4)$$
To begin with, we have
\begin{equation}
\# \{ n : nA_n \leq mf(n) \} \leq m + m^2 \sum_{n>m} \frac{f^2(n)}{n^2 A_n^2}.
\end{equation}

To bound the second term, we first prove
\begin{equation}
\sum_{m<n \leq M} \frac{f^2(n)}{n^2 A_n^2} \leq \frac{C}{m} + C \sum_{m<p \leq M} \frac{f(p)}{p^2 A_p} (m < M).
\end{equation}

By using the partial summation method, we have
\[
\sum_{m<n \leq M} \frac{f^2(n)}{n^2 A_n^2} = \sum_{m<n<M} \left( \sum_{k=m+1}^{n} f^2(k) \right) \left( \frac{1}{n^2 A_n^2} - \frac{1}{(n+1)^2 A_{n+1}^2} \right) + \left( \sum_{k=m+1}^{M} f^2(k) \right) \frac{1}{A_M^2}.
\]

Thanks to (2.3), we have \( \sum_{k=m+1}^{n} f^2(k) \leq CnA_n^2/2 \) and hence
\[
\sum_{m<n \leq M} \frac{f^2(n)}{n^2 A_n^2} \leq C \sum_{m<n} \left( \frac{1}{n(n+1)} + \frac{A_{n+1}^2 - A_n^2}{2n A_{n+1}^2} \right) + C \frac{1}{2M} = \frac{C}{m+1} - \frac{C}{M} + C \sum_{m<n} \frac{A_{n+1}^2 - A_n^2}{2n A_{n+1}^2} + C \frac{1}{2M}.
\]

Since \( A_{n+1}^2 - A_n^2 \) vanishes if \( n \) is not prime and
\[
\frac{A_{p+1}^2 - A_p^2}{2p A_{p+1}^2} = \frac{(A_{p+1} + A_p)(A_{p+1} - A_p)}{2p A_{p+1}^2} \leq \frac{2A_{p+1}}{2p A_{p+1}^2} \leq \frac{f(p)}{p^2 A_p}
\]
for prime \( p \), we have (2.6).

By applying (2.6), we have
\[
\sum_{m<p \leq M} \frac{f(p)}{p^2 A_p} \leq \sum_{m<n \leq M} \frac{f(n)}{n^2 A_n^2} \leq \left( \sum_{m<n \leq M} \frac{1}{n^2} \right)^{1/2} \left( \sum_{m<n \leq M} \frac{f^2(n)}{n^2 A_n^2} \right)^{1/2} \leq \frac{1}{\sqrt{m}} \left( \frac{C}{m} + C \sum_{m<p} \frac{f(p)}{p^2 A_p} \right)^{1/2}.
\]

Therefore
\[
m^2 \left( \sum_{m<p \leq M} \frac{f(p)}{p^2 A_p} \right)^2 \leq C + Cm \sum_{m<p \leq M} \frac{f(p)}{p^2 A_p},
\]
and thereby
\begin{equation}
\sum_{m<p \leq M} \frac{f(p)}{p^2 A_p} \leq \frac{Cm + \sqrt{C^2 m^2 + 4Cm^2}}{2m^2} \leq \frac{C+1}{m}.
\end{equation}
By letting $M \to \infty$, we see that (2.6) and (2.7) are valid even in the case $M = \infty$. Combining these with (2.5), we have (2.4).

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