A NOTE ON ALMOST DISJOINT FAMILIES

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Abstract. We give a short proof of the existence of arbitrarily large chromatic almost disjoint systems. With the same method we solve a problem of P. Szeptycki.

One of the questions posed by P. Erdős and A. Hajnal in [2] was, do there exist arbitrarily large chromatic almost disjoint systems of countable sets? This was eventually proved by G. Elekes and G. Hoffmann in [1]. Later the result was extended to systems of sets of larger cardinality ([3]). These proofs applied a Baire category type argument in a topological setting.

Here we give a short, direct proof of the Elekes-Hoffmann-Komjáth theorem. With the same method we settle a problem of P. Szeptycki ([4]). He asked if it is consistent that for every family $H$ of almost disjoint countable subsets of $\mathbb{R}$ there are sets $B_0, B_1, \ldots \subseteq \mathbb{R}$ such that every $H \in \mathcal{H}$ has $1 \leq |H \cap B_i| < \omega$ for some $i < \omega$.

Notation and definitions. We apply the standard axiomatic set theory notation. If $V$ is a set and $\mu$ is a cardinal, then $[V]^\mu = \{X \subseteq V : |X| = \mu\}$. If $X, Y$ are sets, then $X^Y$ denotes the set of functions from $X$ to $Y$. A set system $\mathcal{H} \subseteq [S]^\mu$ is almost disjoint if $|A \cap B| < \mu$ holds whenever $A, B \in \mathcal{H}, A \neq B$. The chromatic number $\text{Chr}(\mathcal{H})$ of some set system $\mathcal{H}$ on the underlying set $S$ is the least cardinality $\kappa$ such that there is a coloring $f : S \to \kappa$ so that no $H \in \mathcal{H}$ gets monocolored.

Theorem 1 ([1], [3]). If $\kappa \geq \mu$ are infinite cardinals, then there exists an almost disjoint system $\mathcal{H} \subseteq [S]^\mu$ for some $|S| = 2^\kappa$ with $\text{Chr}(\mathcal{H}) > \kappa$.

Proof. Set $\Phi = \{f : \alpha \to \kappa, \alpha < \kappa^+\}$. Clearly $|\Phi| = 2^\kappa$. For some of the functions $f \in \Phi$ we define $H(f) \in [\Phi]^\mu$ as follows. Let $f : \alpha \to \kappa$. If it is possible to find a set $X \subseteq \alpha$ of order type $\mu$, cofinal in $\alpha$ (and so $\text{cf}(\alpha) = \text{cf}(\mu)$) such that $f$ is constant on $X$, then set $f \in \Phi^*$ and let $H(f) = \{f|\beta : \beta \in X\}$ for one such $X$. If no such set $X$ can be found, make $f \notin \Phi^*$ and leave $H(f)$ undefined. Our system is $\mathcal{H} = \{H(f) : f \in \Phi^*\}$.

Lemma 1. $\mathcal{H}$ is almost disjoint.

Proof. Assume that $|H(f) \cap H(g)| = \mu$. Then $H(f) = \{f|\gamma : \gamma \in X\}$ and $H(g) = \{g|\delta : \delta \in Y\}$ where $f : \alpha \to \kappa$, $g : \beta \to \kappa$, $X \subseteq \alpha, Y \subseteq \beta$ are cofinal subsets of
ordinal $\mu$. By our assumption, for a cofinal subset $X' \subseteq X$ it is true that if $\gamma \in X'$, then there is a $\delta \in Y$ such that $f|\gamma = g|\delta$. This implies that $f|\gamma \subseteq g$ and as $X'$ is cofinal in $\alpha$, $f \subseteq g$. Reversing the argument we get that $g \subseteq f$, so $f = g$. \hfill \square

**Lemma 2.** $\text{Chr} (H) > \kappa$.

**Proof.** Let $F : \Phi \rightarrow \kappa$ be a coloring. By transfinite recursion on $\alpha < \kappa^+$ we define the functions $\{f_\alpha : \alpha < \kappa^+\}$ such that $f_\alpha : \alpha \rightarrow \kappa$ and $f_\beta \subseteq f_\alpha$ for $\beta < \alpha$. Set $f_0 = \emptyset$ and let $f_\alpha = \bigcup\{f_\beta : \beta < \alpha\}$ if $\alpha$ is a limit. If $f_\alpha$ is already determined, then let $f_{\alpha+1}$ be that extension of $f_\alpha$ which has $f_{\alpha+1}(\alpha) = F(f_\alpha)$. Clearly, there is a unique function $f : \kappa^+ \rightarrow \kappa$ such that $f_\alpha = f|\alpha$ for every $\alpha < \kappa^+$. As $\kappa^+$ is regular, $f^{-1}(i)$ is cofinal for some $i < \kappa$. Let $X$ be the set of the first $\mu$ elements of $f^{-1}(i)$, $\alpha = \sup(X)$. $X$ witnesses that the required condition holds for $f_\alpha$. Therefore $f_\alpha \in \Phi^*$ and a set $H(f_\alpha) \subseteq \{f_\beta : \beta < \alpha\}$ is chosen using some $X'$ which may be different from $X$: $H(f_\alpha) = \{f_\beta : \beta \in X'\}$. But then, $f_\alpha(\beta) = \beta$ for all $\beta \in X'$ and so $F(f_\beta) = f(\beta) = j$ ($\beta \in X'$). Therefore, $H(f_\alpha)$ is monocolored by $F$. \hfill \square

The lemmas conclude the proof of the theorem. \hfill \square

**Theorem 2.** There is an almost disjoint family $H \subseteq [\mathbb{R}]^\omega$ such that there are no sets $B_i \subseteq \mathbb{R}$ ($i < \omega$) such that for every $H \in H$, $1 \leq |H \cap B_i| < \omega$ holds for some $i < \omega$.

**Proof.** It suffices to construct a family on any set of cardinality the continuum. Set $\Phi = \{f : \alpha \rightarrow \omega, \alpha < \omega_1\}$. Clearly $|\Phi| = 2^{\aleph_0}$ and our system will be a family of subsets of $\Phi$.

Assume that $f : \alpha \rightarrow \omega$ is a given, where $\alpha < \omega_1$ is a limit. Partition $\omega = X(f) \cup Y(f)$ as follows. For $i < \omega$, add $i$ to $X(f)$ if $\gamma_i = \sup(\beta < \alpha : f(\beta)(i) = 1)$ is less than $\alpha$; otherwise add $i$ to $Y(f)$. We do not define $H(f)$ unless $\gamma = \sup\{\gamma_i : i \in X(f)\} < \alpha$. In this case let $\alpha_0 < \alpha_1 < \cdots$ be a sequence converging to $\alpha$ such that $\gamma < \alpha_0$ and for every $i \in \gamma(f)$ the set $\{\alpha_n : f(\alpha_n)(i) = 1\}$ is infinite. This is possible by the definition of $Y(f)$. Set $H(f) = \{f|\alpha_n : n < \omega\}$ and let $\Phi^*$ be the set of those elements of $\Phi$ for which the above condition holds. Then $H = \{H(f) : f \in \Phi^*\}$ is a system of countable subsets of $\Phi$.

**Lemma 3.** $H$ is almost disjoint.

**Proof.** Assume that $f \neq g \in \Phi^*$ and $|H(f) \cap H(g)| = \omega$. Then $H(f) = \{f|\alpha_n : n < \omega\}$ and $H(g) = \{g|\beta_n : n < \omega\}$ where $\alpha_n \rightarrow \alpha$, $\beta_n \rightarrow \beta$ and $\alpha$, $\beta$ are the domains of $f$, $g$, respectively. If $f|\alpha_n = g|\beta_m$, then necessarily $\alpha_n = \beta_m$ and $f|\alpha_n = g|\beta_m$. If this holds for infinitely many $n$ and $m$, then clearly $\alpha = \beta$ and $f = g$. \hfill \square

Assuming now that $B_i \subseteq \Phi$ for $i < \omega$, we are going to find an $H(f) \in H$ such that for no $i$ does $1 \leq |H(f) \cap B_i| < \omega$ hold.

Construct the sequence $\{f_\alpha : \alpha < \omega_1\}$ as follows: $f_0 = \emptyset$. If $\alpha$ is a limit, let $f_\alpha = \bigcup\{f_\beta : \beta < \alpha\}$. Finally, if $f_\alpha$ is given, let $x_\alpha \in \omega^\omega$ be the following function: $x_\alpha(i) = 1$ if $f_\alpha \in B_i$; otherwise $x_\alpha(i) = 0$ and let $f_{\alpha+1}$ be that unique function on $\alpha + 1$ with $f_{\alpha+1} \supseteq f_\alpha$ and $f_{\alpha+1}(\alpha) = x_\alpha$.

For $i < \omega$ define $A_i = \{\beta : f_\beta \in B_i\}$. Now $A_0, A_1, \ldots$ are various subsets of $\omega$. Put $i \in X$ if $A_i$ is countable and then set $\gamma_i = \sup(A_i) < \omega_1$. If $A_i$ is uncountable, let $i \in Y$. Set $\gamma = \sup\{\gamma_i : i \in X\}$. Let $\alpha < \omega_1$ be a limit ordinal with $\gamma < \alpha$ such that every $A_i$ with $i \in Y$ is cofinal in $\alpha$. There are ordinals that satisfy this; actually those ordinals form a closed, unbounded subset of $\omega_1$. 
Now observe that $X(f_α) = X$, $Y(f_α) = Y$, and the condition on the boundedness of $γ$ also holds, so $f_α ∈ Φ^*$; moreover $H(f_α) ∩ B_i = ∅$ for $i ∈ X$ and $|H(f_α) ∩ B_i| = ω$ for $i ∈ Y$.

References


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