PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 1, January 2009, Pages 303–305
S 0002-9939(08)09527-0
Article electronically published on July 8, 2008

A NOTE ON ALMOST DISJOINT FAMILIES

PÉTER KOMJÁTH

(Communicated by Julia Knight)

ABSTRACT. We give a short proof of the existence of arbitrarily large chromatic almost disjoint systems. With the same method we solve a problem of P. Szeptycki.

One of the questions posed by P. Erdős and A. Hajnal in [2] was, do there exist arbitrarily large chromatic almost disjoint systems of countable sets? This was eventually proved by G. Elekes and G. Hoffmann in [1]. Later the result was extended to systems of sets of larger cardinality ([3]). These proofs applied a Baire category type argument in a topological setting.

Here we give a short, direct proof of the Elekes-Hoffmann-Komjáth theorem. With the same method we settle a problem of P. Szeptycki ([4]). He asked if it is consistent that for every family $\mathcal{H}$ of almost disjoint countable subsets of $\mathbb{R}$ there are sets $B_0, B_1, \ldots \subseteq \mathbb{R}$ such that every $H \in \mathcal{H}$ has $1 \leq |H \cap B_i| < \omega$ for some $i < \omega$.

Notation and definitions. We apply the standard axiomatic set theory notation. If V is a set and μ is a cardinal, then $[V]^\mu = \{X \subseteq V : |X| = \mu\}$. If X, Y are sets, then ${}^XY$ denotes the set of functions from X to Y. A set system $\mathcal{H} \subseteq [S]^\mu$ is *almost disjoint* if $|A \cap B| < \mu$ holds whenever $A, B \in \mathcal{H}$, $A \neq B$. The *chromatic number* $\mathrm{Chr}(\mathcal{H})$ of some set system $\mathcal{H}$ on the underlying set S is the least cardinality κ such that there is a coloring $f : S \to \kappa$ so that no $H \in \mathcal{H}$ gets monocolored.

Theorem 1 ([1], [3])**.** *If $\kappa \geq \mu$ are infinite cardinals, then there exists an almost disjoint system $\mathcal{H} \subseteq [S]^\mu$ for some $|S| = 2^\kappa$ with* $\mathrm{Chr}(\mathcal{H}) > \kappa$.

Proof. Set $\Phi = \{f : \alpha \to \kappa, \alpha < \kappa^+\}$. Clearly $|\Phi| = 2^\kappa$. For some of the functions $f \in \Phi$ we define $H(f) \in [\Phi]^\mu$ as follows. Let $f : \alpha \to \kappa$. If it is possible to find a set $X \subseteq \alpha$ of order type μ, cofinal in α (and so $\mathrm{cf}(\alpha) = \mathrm{cf}(\mu)$) such that f is constant on X, then set $f \in \Phi^*$ and let $H(f) = \{f|\beta : \beta \in X\}$ for one such X. If no such set X can be found, make $f \notin \Phi^*$ and leave $H(f)$ undefined. Our system is $\mathcal{H} = \{H(f) : f \in \Phi^*\}$.

Lemma 1. *$\mathcal{H}$ is almost disjoint.*

Proof. Assume that $|H(f) \cap H(g)| = \mu$. Then $H(f) = \{f|\gamma : \gamma \in X\}$ and $H(g) = \{g|\delta : \delta \in Y\}$ where $f : \alpha \to \kappa$, $g : \beta \to \kappa$, $X \subseteq \alpha$, $Y \subseteq \beta$ are cofinal subsets of

Received by the editors November 20, 2007.
2000 *Mathematics Subject Classification.* Primary 03E05.
Key words and phrases. Almost disjoint sets.
This research was partially supported by Hungarian National Research Grant T 046991.

ordinal μ. By our assumption, for a cofinal subset $X' \subseteq X$ it is true that if $\gamma \in X'$, then there is a $\delta \in Y$ such that $f|\gamma = g|\delta$. This implies that $f|\gamma \subseteq g$ and as X' is cofinal in α, $f \subseteq g$. Reversing the argument we get that $g \subseteq f$, so $f = g$. $\square$

Lemma 2. $\mathrm{Chr}\,(\mathcal{H}) > \kappa$.

Proof. Let $F : \Phi \to \kappa$ be a coloring. By transfinite recursion on $\alpha < \kappa^+$ we define the functions $\{f_\alpha < \kappa^+\}$ such that $f_\alpha : \alpha \to \kappa$ and $f_\beta \subseteq f_\alpha$ for $\beta < \alpha$. Set $f_0 = \emptyset$ and let $f_\alpha = \bigcup\{f_\beta : \beta < \alpha\}$ if α is a limit. If f_α is already determined, then let $f_{\alpha+1}$ be that extension of f_α which has $f_{\alpha+1}(\alpha) = F(f_\alpha)$. Clearly, there is a unique function $f : \kappa^+ \to \kappa$ such that $f_\alpha = f|\alpha$ for every $\alpha < \kappa^+$. As κ^+ is regular, $f^{-1}(i)$ is cofinal for some $i < \kappa$. Let X be the set of the first μ elements of $f^{-1}(i)$, $\alpha = \sup(X)$. X witnesses that the required condition holds for f_α. Therefore $f_\alpha \in \Phi^*$ and a set $H(f_\alpha) \subseteq \{f_\beta : \beta < \alpha\}$ is chosen using some X' which may be different from X: $H(f_\alpha) = \{f_\beta : \beta \in X'\}$. But then, $f_\alpha(\beta) = j$ for all $\beta \in X'$ and so $F(f_\beta) = f(\beta) = j$ $(\beta \in X')$. Therefore, $H(f_\alpha)$ is monocolored by F. $\square$

The lemmas conclude the proof of the theorem. $\square$

Theorem 2. *There is an almost disjoint family $\mathcal{H} \subseteq [\mathbb{R}]^\omega$ such that there are no sets $B_i \subseteq \mathbb{R}$ $(i < \omega)$ such that for every $H \in \mathcal{H}$, $1 \leq |H \cap B_i| < \omega$ holds for some $i < \omega$.*

Proof. It suffices to construct a family on any set of cardinality the continuum. Set $\Phi = \{f : \alpha \to {}^\omega 2, \alpha < \omega_1\}$. Clearly $|\Phi| = 2^{\aleph_0}$ and our system will be a family of subsets of Φ.

Assume that $f : \alpha \to {}^\omega 2$ is given, where $\alpha < \omega_1$ is a limit. Partition $\omega = X(f) \cup Y(f)$ as follows. For $i < \omega$, add i to $X(f)$ if $\gamma_i = \sup\{\beta < \alpha : f(\beta)(i) = 1\}$ is less than α; otherwise add i to $Y(f)$. We do not define $H(f)$ unless $\gamma = \sup\{\gamma_i : i \in X(f)\} < \alpha$. In this case let $\alpha_0 < \alpha_1 < \cdots$ be a sequence converging to α such that $\gamma < \alpha_0$ and for every $i \in Y(f)$ the set $\{\alpha_n : f(\alpha_n)(i) = 1\}$ is infinite. This is possible by the definition of $Y(f)$. Set $H(f) = \{f|\alpha_n : n < \omega\}$ and let Φ^* be the set of those elements of Φ for which the above condition holds. Then $\mathcal{H} = \{H(f) : f \in \Phi^*\}$ is a system of countable subsets of Φ.

Lemma 3. *$\mathcal{H}$ is almost disjoint.*

Proof. Assume that $f \neq g \in \Phi^*$ and $|H(f) \cap H(g)| = \omega$. Then $H(f) = \{f|\alpha_n : n < \omega\}$ and $H(g) = \{g|\beta_n : n < \omega\}$ where $\alpha_n \to \alpha$, $\beta_n \to \beta$ and α, β are the domains of f, g, respectively. If $f|\alpha_n = g|\beta_m$, then necessarily $\alpha_n = \beta_m$ and $f|\alpha_n = g|\beta_m$. If this holds for infinitely many n and m, then clearly $\alpha = \beta$ and $f = g$. $\square$

Assuming now that $B_i \subseteq \Phi$ for $i < \omega$, we are going to find an $H(f) \in \mathcal{H}$ such that for no i does $1 \leq |H(f) \cap B_i| < \omega$ hold.

Construct the sequence $\{f_\alpha : \alpha < \omega_1\}$ as follows: $f_0 = \emptyset$. If α is a limit, let $f_\alpha = \bigcup\{f_\beta : \beta < \alpha\}$. Finally, if f_α is given, let $x_\alpha \in {}^\omega 2$ be the following function: $x_\alpha(i) = 1$ if $f_\alpha \in B_i$; otherwise $x_\alpha(i) = 0$ and let $f_{\alpha+1}$ be that unique function on $\alpha + 1$ with $f_{\alpha+1} \supseteq f_\alpha$ and $f_{\alpha+1}(\alpha) = x_\alpha$.

For $i < \omega$ define $A_i = \{\beta : f_\beta \in B_i\}$. Now $A_0, A_1, \ldots$ are various subsets of ω_1. Put $i \in X$ if A_i is countable and then set $\gamma_i = \sup(A_i) < \omega_1$. If A_i is uncountable, let $i \in Y$. Set $\gamma = \sup\{\gamma_i : i \in X\}$. Let $\alpha < \omega_1$ be a limit ordinal with $\gamma < \alpha$ such that every A_i with $i \in Y$ is cofinal in α. There are ordinals that satisfy this; actually those ordinals form a closed, unbounded subset of ω_1.

Now observe that $X(f_\alpha) = X$, $Y(f_\alpha) = Y$, and the condition on the boundedness of γ also holds, so $f_\alpha \in \Phi^*$; moreover $H(f_\alpha) \cap B_i = \emptyset$ for $i \in X$ and $|H(f_\alpha) \cap B_i| = \omega$ for $i \in Y$. □

References

1. G. Elekes, G. Hoffmann: On the chromatic number of almost disjoint families of countable sets, *Infinite and finite sets* (Colloq., Keszthely, 1973), Colloq., Math. Soc. János Bolyai, **10**, North-Holland, Amsterdam, 1975, 397–402. MR0373905 (51:10105)
2. P. Erdős, A. Hajnal: On a property of families of sets, *Acta Math. Acad. Sci. Hungar.*, **12**(1961), 87–123. MR0150047 (27:50)
3. P. Komjáth: Dense systems of almost-disjoint sets, *Finite and infinite sets* (Eger, 1981), Colloq. Math. Soc. János Bolyai, **37**, North-Holland, Amsterdam, 1984, 527–536. MR818255 (87f:04005)
4. Paul J. Szeptycki: Transversals for strongly almost disjoint families, *Proc. Amer. Math. Soc.*, **135**(2007), 2273–2282. MR2299505

Department of Computer Science, Eötvös University, P. O. Box 120, Budapest, 1518, Hungary

E-mail address: kope@cs.elte.hu