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ON A WEYL INEQUALITY OF OPERATORS IN BANACH SPACES

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ABSTRACT. Let $s = (s_n)$ be an injective and surjective s-number sequence in the sense of Pietsch. We show for a Riesz-operator $T : X \to X$ acting on a (complex) Banach space the following Weyl inequality between geometric means of eigenvalues and s-numbers: For any $0 < \delta \leq 1$ and all $n \in \mathbb{N}$,

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{\frac{1}{n}} \leq c_0 \left(1 + \frac{1}{\delta}\ln\left(\frac{1}{\delta}\right)\right) \left(\prod_{i=1}^{\left[\frac{n}{1+\delta}\right]} s_i(T)\right)^{\frac{1}{\left[\frac{n}{1+\delta}\right]}},$$

where $c_0 \geq 1$ is an absolute constant. The proof rests on an elementary mixing multiplicativity of an arbitrary s-number sequence and a striking result of G. Pisier. The inequality is a contribution to the problem of estimating eigenvalues by s-numbers first started in a strong sense by H. König (1986, 2001).

1. s-NUMBERS

We recall some basic definitions and notions from Banach space theory and from s-numbers of operators. If X is a Banach space, we denote by X' its dual Banach space and by B_X and $\mathring{B}_X$ the closed and open unit balls of X, respectively. In what follows, X, Y, Z, etc., always denote Banach spaces, $\mathcal{L}(X,Y)$ is the set of (bounded linear) operators from X into Y equipped with the operator norm and $\mathcal{L}$ stands for the class of all operators between arbitrary Banach spaces.

We use the notion of an s-number sequence given in [P80b] or [P87]. A rule $s = (s_n) : \mathcal{L} \to [0, \infty]$ assigning to every operator $T \in \mathcal{L}$ a non-negative scalar sequence $(s_n(T))_{n\in\mathbb{N}}$ is called an *s-number sequence* if the following conditions are satisfied:

(S1) *Monotonicity:*
$\|T\| = s_1(T) \geq s_2(T) \geq \ldots \geq 0$ for $T \in \mathcal{L}(X,Y)$.

(S2) *Additivity:*
$s_{n+m-1}(S+T) \leq s_m(S) + s_n(T)$ for $S, T \in \mathcal{L}(X,Y)$.

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(S3) *Ideal-property:*
$s_n(STR) \leq \|S\| s_n(T)\|R\|$ for $R \in \mathcal{L}(X_0, X)$, $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y_1, Y_0)$.

(S4) *Rank-property:*
If rank $(T) < n$, then $s_n(T) = 0$.

(S5) *Norming property:*
$s_n(I : l_2^n \to l_2^n) = 1$ for $n \in \mathbb{N}$,

where I denotes the identity operator on the n-dimensional Hilbert space l_2^n.
Furthermore, we additionally need the following notions:

(J) An s-number sequence $s = (s_n)$ is called *injective* if, given any metric injection $J \in \mathcal{L}(Y, \tilde{Y})$, i.e. $\|Jy\| = \|y\|$ for $y \in Y$, $s_n(T) = s_n(JT)$ for all $T \in \mathcal{L}(X,Y)$ and all Banach spaces X.

(S) An s-number sequence $s = (s_n)$ is called *surjective* if, given any metric surjection $Q \in \mathcal{L}(\tilde{X}, X)$, i.e. $Q(\overset{\circ}{B}_{\tilde{X}}) = \overset{\circ}{B}_X$, $s_n(T) = s_n(TQ)$ for all $T \in \mathcal{L}(X,Y)$ and all Banach spaces Y.

(JS) An s-number sequence is called *injective and surjective* if it satisfies (J) and (S).

We note that on the class of Hilbert spaces there is only one s-number sequence satisfying the properties (S1) - (S5) that coincides with the singular numbers [P80b]. Basic examples for our purposes are the *approximation numbers* given by

$$a_n(T) := \inf\{\|T - L\| : \; L \in \mathcal{L}(X,Y), \operatorname{rank}(L) < n\},$$

the *Gelfand numbers* given by

$$c_n(T) := \inf\{\|TJ_M\| : M \subset X, \operatorname{codim}\,(M) < n\},$$

where $J_M : M \to X$ is the natural embedding from a subspace M of X into X, and the *Kolmogorov number* given by

$$d_n(T) := \inf\{\|Q_N T\| : N \subset Y, \; \dim N < n\},$$

where $Q_n : Y \to Y/N$ defines the canonical quotient map from Y onto the quotient space Y/N.

Moreover, we need also the following characterization of Gelfand and Kolmogorov numbers,

$$c_n(T) = a_n(J_\infty T) \text{ and } d_n(T) = a_n(TQ_1),$$

where $J_\infty : Y \to l_\infty(B_{Y'})$ is the *metric injection* defined by $J_\infty y := (\langle y, a\rangle)_{a \in B_{Y'}}$ and with values in the space $l_\infty(B_{Y'})$ of bounded sequences and where $Q_1 : l_1(B_X) \to X$ is the *metric surjection* from the space of summable sequences $l_1(B_X)$ onto X, defined by $Q_1((\xi_x)) := \sum_{x \in B_X} \xi_x x$.

Now we show an elementary but very useful multiplicativity property of an arbitrary s-number sequence that we call mixing multiplicativity.

(MI) *Mixing multiplicativity:* Let $s = (s_n)$ be an arbitrary s-number sequence. Then for $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(Y,Z)$,

(i) $s_{n+m-1}(TS) \leq s_n(T) a_m(S)$ and $s_{n+m-1}(TS) \leq a_n(T) s_m(S)$.
Moreover, if $s = (s_n)$ is injective, then

(ii) $s_{n+m-1}(TS) \leq c_n(T) s_m(S)$
and if $s = (s_n)$ is surjective, then

(iii) $s_{n+m-1}(TS) \leq s_n(T) d_m(S)$.

Proof. We show the first inequality of (i); the second one can be similarly proved. To this end let $L \in \mathcal{L}(X,Y)$ be an operator with $\mathrm{rank}(L) < m$. Then the additivity (S2), the rank property (S4), and the ideal property (S3) of an s-number sequence immediately yield

$$\begin{aligned} s_{n+m-1}(TS) &= s_{n+m-1}(T(S-L)+TL) \\ &\le s_n(T(S-L)) + s_m(TL) = s_n(T(S-L)) \le s_n(T)\|S-L\|, \end{aligned}$$

implying by definition of the approximation numbers the desired inequality $s_{n+m-1}(TS) \le s_n(T)a_m(S)$.

The inequalities (ii) and (iii) follow from (i) and the characterization of Gelfand and Kolmogorov numbers, respectively. For example, (ii) follows from $s_{n+m-1}(TS) = s_{n+m-1}(J_\infty TS) \le a_n(J_\infty T)s_m(S) = c_n(T)s_m(S)$. □

2. A Weyl inequality

For a Riesz operator $T \in \mathcal{L}(X)$ acting on a complex Banach space (cf. [K86], [P87], [CS90]) we assign an eigenvalue sequence $(\lambda_n(T))$ as follows: The eigenvalues are arranged in an order of non-increasing absolute values and each eigenvalue is counted according to its algebraic multiplicity,

$$|\lambda_1(T)| \ge |\lambda_2(T)| \ge \cdots \ge 0 .$$

If T possesses less than n eigenvalues λ with $\lambda \neq 0$, we put $\lambda_n(T) = \lambda_{n+1}(T) = \cdots = 0$. In this section we establish for an arbitrary injective and surjective s-number sequence a Weyl inequality for Riesz operators. The following Weyl inequality complements Weyl inequalities for s-numbers given in [K86], [K01], [KRT80], [P80a], [P87], [H05] and [CHi07]. In the sequel, $[x]$ denotes the integer part of x for $1 \le x < \infty$ and if $0 < x \le 1$ we put $[x] := 1$.

Theorem. *Let $s = (s_n)$ be an injective and surjective s-number sequence. Then for any $0 < \delta \le 1$ there exists a constant $c(\delta) \ge 1$ such that for all (complex) Banach spaces X, all Riesz operators $T \in \mathcal{L}(X)$ and all $n \in \mathbb{N}$ the inequality*

$$\left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{\frac{1}{n}} \le c(\delta) \left(\prod_{i=1}^{\left[\frac{n}{1+\delta}\right]} s_i(T) \right)^{\frac{1}{\left[\frac{n}{1+\delta}\right]}}$$

holds. For the constant $c(\delta)$ we obtain the estimate $c(\delta) \le c_0(1 + \frac{1}{\delta}\ln\frac{1}{\delta})$, where $c_0 \ge 1$ is an absolute constant.

Proof. In order to prove the inequality we need a striking result of Pisier [Pi89a], [Pi89b] concerning the existence of isomorphisms between arbitrary n-dimensional Banach spaces and n-dimensional Hilbert spaces and the mixing multiplicativity of an injective and surjective s-number sequence.

Pisier's isomorphism states the following:

For each $\alpha > \frac{1}{2}$, there is a constant $b(\alpha)$ such that for any n and any n-dimensional (real or complex) Banach space E, there is an isomorphism $u : l_2^n \to E$ such that

$$d_k(u) \le b(\alpha)\left(\frac{n}{k}\right)^\alpha \text{ and } c_k(u^{-1}) \le b(\alpha)\left(\frac{n}{k}\right)^\alpha$$

for $1 \le k \le n$, where $b(\alpha) \le b_0(\alpha - \frac{1}{2})^{-\frac{1}{2}}$ and $b_0 > 0$ is an absolute constant.

Furthermore, for $0 < \delta \le 1$ and a non-increasing sequence of positive numbers $(s_k)_{k\in\mathbb{N}}$ we use the estimate

$$\left(\prod_{k=1}^{n} s_k\right)^{\frac{1}{n}} \le \left(\prod_{k=1}^{\left[\frac{n}{1+\delta}\right]} s_{[\delta k]+k-1}\right)^{\frac{1}{\left[\frac{n}{1+\delta}\right]}} , \ n \in \mathbb{N} .$$

Moreover, the proof uses ideas presented in [CHe91] and [DJ93]. If $\lambda_n(T) = 0$, then there is nothing to prove. So we assume $\lambda_n(T) \neq 0$. By [K86], [P87] or [CS90] we can find an n-dimensional Banach space X_n of X invariant under T such that the restriction T_n of T to X_n has precisely $\lambda_1(T), \ldots, \lambda_n(T)$ as its eigenvalues. By Pisier we get for $\frac{1}{2} < \alpha \le 1$ an isomorphism $u : l_2^n \to X_n$ such that

$$d_k(u) \le b(\alpha) \left(\frac{n}{k}\right)^{\alpha} \text{ and } c_k(u^{-1}) \le b(\alpha) \left(\frac{n}{k}\right)^{\alpha}$$

for $1 \le k \le n$. Applying Weyl's inequality to the Hilbert space operator $u^{-1}T_n u$ and inserting the principle of related operators [P87] we arrive at

$$\left(\prod_{k=1}^{n} |\lambda_k(T)|\right)^{\frac{1}{n}} \le \left(\prod_{k=1}^{n} s_k(u^{-1}T_n u)\right)^{\frac{1}{n}} .$$

For estimating the right-hand side of the inequality we put $m := \left[\frac{n}{1+\delta}\right]$ for $0 < \delta \le 1$. Then we have

$$\left(\prod_{k=1}^{n} s_k(u^{-1}T_n u)\right)^{\frac{1}{n}} \le \left(\prod_{k=1}^{m} s_{[\delta k]+k-1}(u^{-1}T_n u)\right)^{\frac{1}{m}} .$$

The mixing multiplicativity (MI) (ii) and (iii) of an injective and surjective s-number sequence yields for the single s-numbers the estimate

$$\begin{aligned} s_{[\delta k]+k-1}(u^{-1}T_n u) &\le c_{\left[\frac{\delta k}{2}\right]}(u^{-1}) s_k(T_n) d_{\left[\frac{\delta k}{2}\right]}(u) \\ &\le b^2(\alpha) \left(\frac{n}{\left[\frac{\delta k}{2}\right]}\right)^{2\alpha} s_k(T_n) \\ &\le b^2(\alpha) \left(\frac{n}{\left[\frac{\delta k}{2}\right]}\right)^{2\alpha} s_k(T) \text{ for } 1 \le k \le m . \end{aligned}$$

Hence,

$$\left(\prod_{k=1}^{m} s_{[\delta k]+k-1}(u^{-1}T_n u)\right)^{\frac{1}{m}} \le b^2(\alpha) \left(\prod_{k=1}^{m} \frac{n}{\left[\frac{\delta k}{2}\right]}\right)^{\frac{2\alpha}{m}} \left(\prod_{k=1}^{m} s_k(T)\right)^{\frac{1}{m}} .$$

From $\left[\frac{\delta k}{2}\right] \ge \frac{\delta k}{4}$, $m = \left[\frac{n}{1+\delta}\right] \ge \frac{n}{2(1+\delta)}$ and $e^m \ge \frac{m^m}{m!}$ we obtain for the first term on the right-hand side of the inequality the estimate

$$\begin{aligned} \left(\prod_{k=1}^{m} \frac{n}{\left[\frac{\delta k}{2}\right]}\right)^{\frac{2\alpha}{m}} &\le \left(\tfrac{4}{\delta}\right)^{2\alpha} e^{2\alpha} \left(\tfrac{n}{m}\right)^{2\alpha} \\ &\le \left(\tfrac{4}{\delta}\right)^{2\alpha} e^{2\alpha} (2(1+\delta))^{2\alpha} \le 16^2 e^2 \left(\tfrac{1}{\delta}\right)^{2\alpha} . \end{aligned}$$

Combining the previous inequalities we arrive at

$$\left(\prod_{k=1}^{n}|\lambda_k(T)|\right)^{\frac{1}{n}} \leq 16^2e^2b^2(\alpha)\left(\frac{1}{\delta}\right)^{2\alpha}\left(\prod_{k=1}^{m}s_k(T)\right)^{\frac{1}{m}} .$$

Finally, it remains to estimate the constant appearing on the right-hand side of the inequality by choosing an appropriate α, $\frac{1}{2}<\alpha\leq 1$. To this end, first let $\frac{1}{e}\leq\delta\leq 1$. If we choose $\alpha=1$, then for the constant $c(\delta)$ we obtain $c(\delta)\leq 2\cdot 16^2e^4b_0^2$. If $0<\delta\leq\frac{1}{e}$, then we choose $\alpha=\frac{1}{2}+\frac{1}{2\ln\left(\frac{1}{\delta}\right)}$, which guarantees $\frac{1}{2}<\alpha\leq 1$ and for $c(\delta)$ we check that

$$c(\delta)\leq 2\cdot 16^2e^3b_0^2\frac{1}{\delta}\ln\left(\frac{1}{\delta}\right) .$$

Summarizing the previous estimates we obtain

$$c(\delta)\leq 2\cdot 16^2e^3b_0^2\max\{e,\frac{1}{\delta}\ln\left(\frac{1}{\delta}\right)\}\leq 2\cdot 16^2e^4b_0^2\left(1+\frac{1}{\delta}\ln\left(\frac{1}{\delta}\right)\right) \text{ for } 0<\delta\leq 1 .$$

□

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