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A NOTE ON THE EFFECTIVE NON-VANISHING CONJECTURE

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Abstract. We give a reduction of the irregular case for the effective non-vanishing conjecture by virtue of the Fourier-Mukai transform. As a consequence, we reprove that the effective non-vanishing conjecture holds on algebraic surfaces.

In this paper we consider the following so-called effective non-vanishing conjecture, which has been put forward by Ambro and Kawamata [Am99, Ka00].

Conjecture 1 (EN_n)**.** *Let X be a proper normal variety of dimension n, B an effective $\mathbb{R}$-divisor on X such that the pair (X, B) is Kawamata log terminal, and D a Cartier divisor on X. Assume that D is nef and that $D - (K_X + B)$ is nef and big. Then $H^0(X, D) \neq 0$.*

This conjecture is closely related to the minimal model program and plays an important role in the classification theory of Fano varieties. For a detailed introduction to this conjecture, we refer the reader to [Xie06].

By the Kawamata-Viehweg vanishing theorem, we have $H^i(X, D) = 0$ for any positive integer i. Thus $H^0(X, D) \neq 0$ is equivalent to $\chi(X, D) \neq 0$. Under the same assumptions as in Conjecture 1, the Kawamata-Shokurov non-vanishing theorem says that $H^0(X, mD) \neq 0$ for all $m \gg 0$. Thus the effective non-vanishing conjecture is an improvement of the non-vanishing theorem in some sense.

Note that EN_1 is trivial by the Riemann-Roch theorem and that EN_2 was settled by Kawamata [Ka00, Theorem 3.1] by virtue of the logarithmic semipositivity theorem. For $n \geq 3$, only a few results are known. For instance, EN_n holds trivially for toric varieties [Mu02], $EN_3(X, 0)$ holds for all canonical projective minimal threefolds X [Ka00, Proposition 4.1], and $EN_3(X, 0)$ also holds for almost all canonical projective threefolds X with $-K_X$ nef [Xie05, Corollary 4.5].

In this paper, we shall prove that, in the irregular case, the effective non-vanishing conjecture can be reduced to lower-dimensional cases by means of the Fourier-Mukai transform. As consequences, EN_2 is reproved after Kawamata, and EN_n holds for all varieties of maximal Albanese dimension.

Throughout this paper, we work over the complex number field $\mathbb{C}$. For the definition of the Kawamata log terminal (KLT, for short) and the other notions, we refer the reader to [KMM87, KM98].

For irregular varieties, the study of the Albanese map provides enough information to understand their birational structure. Therefore, through the Albanese

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map, we can utilize the Fourier-Mukai transform to give a reduction of the effective non-vanishing conjecture for irregular varieties. This idea was first used in [CH02]. First of all, we need the following lemma which follows easily from [Mu81, Theorem 2.2].

Lemma 2. *Let A be an abelian variety, and let $\mathcal{F}$ be a coherent sheaf on A. Assume that $H^i(A, \mathcal{F} \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(A)$ and all i. Then $\mathcal{F} = 0$.*

Proof. Let $\hat{A}$ be the dual abelian variety of A. The assumption implies that the Fourier-Mukai transform $\Phi(\mathcal{F})$ of $\mathcal{F}$ is the zero sheaf on $\hat{A}$. Since the Fourier-Mukai transform $\Phi : D(A) \to D(\hat{A})$ induces an equivalence of derived categories [Mu81, Theorem 2.2], we have $\mathcal{F} = 0$. □

Theorem 3. *If EN_k holds for any $k < n$, then $EN_n(X, B)$ holds for any X with irregularity $q(X) := \dim H^1(X, \mathcal{O}_X) > 0$.*

Proof. By Kodaira's lemma, we may assume that $H = D - (K_X + B)$ is ample and B is a $\mathbb{Q}$-divisor. Let $\pi : \widetilde{X} \to X$ be a resolution of X, and $\widetilde{\alpha} : \widetilde{X} \to A = \mathrm{Alb}(\widetilde{X})$ the Albanese morphism of $\widetilde{X}$. Since (X, B) is KLT, X has only rational singularities by [KM98, Theorem 5.22], hence $q(\widetilde{X}) = q(X) > 0$. Since there are no rational curves on A, we have a non-trivial proper morphism $\alpha : X \to A$.

Let $P \in \mathrm{Pic}^0(A)$, $P' = \alpha^* P$ and $\mathcal{F} = \alpha_* \mathcal{O}_X(D)$. By the Kawamata-Viehweg vanishing theorem, we have $H^i(X, D + P') = 0$ for any $i > 0$. By the relative Kawamata-Viehweg vanishing theorem [KMM87, Theorem 1-2-5], we have $R^i \alpha_* \mathcal{O}_X(D + P') = 0$ for any $i > 0$. It follows from the Leray spectral sequence that $H^i(A, \mathcal{F} \otimes P) = H^i(X, D + P') = 0$ for any $i > 0$. If $H^0(A, \mathcal{F}) = 0$, then $h^0(A, \mathcal{F} \otimes P) = \chi(A, \mathcal{F} \otimes P) = \chi(A, \mathcal{F}) = 0$; i.e. $H^0(A, \mathcal{F} \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(A)$. By Lemma 2, we have $\mathcal{F} = 0$.

Next we prove that $\mathcal{F} \neq 0$, which implies $H^0(X, D) = H^0(A, \mathcal{F}) \neq 0$. Let $a(X) = \dim \alpha(X) > 0$. If $a(X) = n$, then $\alpha : X \to \alpha(X)$ is generically finite, and it is easy to see that $\mathcal{F} \neq 0$. Assume that $a(X) < n$. Let $f : X \to Y$ be the Stein factorization of α, F a general fiber of f and $\mathcal{G} = f_* \mathcal{O}_X(D)$. Then F is a normal proper variety of dimension less than n. Note that $D|_F$ is nef Cartier, $(F, B|_F)$ is KLT and $D|_F - (K_F + B|_F) = H|_F$ is ample. By assumption, we have $\mathrm{rank}\, \mathcal{G} = h^0(F, D|_F) \neq 0$; hence $\mathcal{G} \neq 0$ as well as $\mathcal{F} \neq 0$. □

Corollary 4. *EN_2 holds, and EN_3 holds for any X with $q(X) > 0$.*

Proof. For EN_2, by the Riemann-Roch theorem, one has only to deal with the case where X is a ruled surface over a smooth projective curve C with $q(X) = g(C) \geq 2$. Since EN_1 holds, EN_2 also holds by Theorem 3. The second conclusion is obvious. □

Corollary 5. *$EN_n(X, B)$ holds for any X of maximal Albanese dimension.*

Proof. By assumption, X is of maximal Albanese dimension; i.e. the Albanese morphism $\alpha : X \to A$ satisfies $\dim \alpha(X) = \dim X = n$. So we can repeat the same argument as in Theorem 3 to complete the proof by noting that α is generically finite. □

Remark 6. Note that Corollary 5 has already appeared in [PP03] and [PP05, Theorem 5.8]. Note also that the assumption that D is nef in Conjecture 1 is not needed in the proof of Corollary 5; however [PP05, Lemma 5.1] proved that if

$D-(K_X+B)$ is nef and big, then D must be nef on the variety X of maximal Albanese dimension. Furthermore, when the assumption that $D-(K_X+B)$ is nef and big in Conjecture 1 is replaced with the weaker assumption that $D-(K_X+B)$ is either nef or of non-negative Iitaka dimension, $H^0(X,D)\neq 0$ also holds for any X of maximal Albanese dimension [PP06, Theorem 6.1]. Finally, we should mention that Theorem 3 and [PP05, Theorem 5.8] used a similar idea in proof.

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