A NOTE ON THE EFFECTIVE NON-VANISHING CONJECTURE

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Abstract. We give a reduction of the irregular case for the effective non-vanishing conjecture by virtue of the Fourier-Mukai transform. As a consequence, we reprove that the effective non-vanishing conjecture holds on algebraic surfaces.

In this paper we consider the following so-called effective non-vanishing conjecture, which has been put forward by Ambro and Kawamata [Am99, Ka00].

**Conjecture 1 (EN<sub>n</sub>).** Let X be a proper normal variety of dimension n, B an effective R-divisor on X such that the pair (X, B) is Kawamata log terminal, and D a Cartier divisor on X. Assume that D is nef and that D − (K<sub>X</sub> + B) is nef and big. Then H<sup>0</sup>(X, D) ≠ 0.

This conjecture is closely related to the minimal model program and plays an important role in the classification theory of Fano varieties. For a detailed introduction to this conjecture, we refer the reader to [Xie06].

By the Kawamata-Viehweg vanishing theorem, we have H<sup>i</sup>(X, D) = 0 for any positive integer i. Thus H<sup>0</sup>(X, D) ≠ 0 is equivalent to χ(X, D) ≠ 0. Under the same assumptions as in Conjecture 1, the Kawamata-Shokurov non-vanishing theorem says that H<sup>0</sup>(X, mD) ≠ 0 for all m ≫ 0. Thus the effective non-vanishing conjecture is an improvement of the non-vanishing theorem in some sense.

Note that EN<sub>1</sub> is trivial by the Riemann-Roch theorem and that EN<sub>2</sub> was settled by Kawamata [Ka00, Theorem 3.1] by virtue of the logarithmic semipositivity theorem. For n ≥ 3, only a few results are known. For instance, EN<sub>n</sub> holds trivially for toric varieties [Mu02], EN<sub>3</sub>(X, 0) holds for all canonical projective minimal threefolds X [Ka00, Proposition 4.1], and EN<sub>3</sub>(X, 0) also holds for almost all canonical projective threefolds X with −K<sub>X</sub> nef [Xie05, Corollary 4.5].

In this paper, we shall prove that, in the irregular case, the effective non-vanishing conjecture can be reduced to lower-dimensional cases by means of the Fourier-Mukai transform. As consequences, EN<sub>2</sub> is reproved after Kawamata, and EN<sub>n</sub> holds for all varieties of maximal Albanese dimension.

Throughout this paper, we work over the complex number field C. For the definition of the Kawamata log terminal (KLT, for short) and the other notions, we refer the reader to [KM98].

For irregular varieties, the study of the Albanese map provides enough information to understand their birational structure. Therefore, through the Albanese
map, we can utilize the Fourier-Mukai transform to give a reduction of the effective non-vanishing conjecture for irregular varieties. This idea was first used in [CH02]. First of all, we need the following lemma which follows easily from [Mu81 Theorem 2.2].

**Lemma 2.** Let $A$ be an abelian variety, and let $\mathcal{F}$ be a coherent sheaf on $A$. Assume that $H^i(A, \mathcal{F} \otimes P) = 0$ for all $P \in \text{Pic}^0(A)$ and all $i$. Then $\mathcal{F} = 0$.

**Proof.** Let $\hat{A}$ be the dual abelian variety of $A$. The assumption implies that the Fourier-Mukai transform $\Phi(\mathcal{F})$ of $\mathcal{F}$ is the zero sheaf on $A$. Since the Fourier-Mukai transform $\Phi: D(A) \to D(\hat{A})$ induces an equivalence of derived categories [Mu81 Theorem 2.2], we have $\mathcal{F} = 0$.

**Theorem 3.** If $\text{EN}_k$ holds for any $k < n$, then $\text{EN}_n(X, B)$ holds for any $X$ with irregularity $q(X) := \dim H^1(X, \mathcal{O}_X) > 0$.

**Proof.** By Kodaira’s lemma, we may assume that $H = D - (K_X + B)$ is ample and $B$ is a $\mathbb{Q}$-divisor. Let $\pi: \tilde{X} \to X$ be a resolution of $X$, and $\tilde{\alpha}: \tilde{X} \to A = \text{Alb}(\tilde{X})$ the Albanese morphism of $\tilde{X}$. Since $(X, B)$ is KLT, $X$ has only rational singularities by [KM98 Theorem 5.22], hence $q(\tilde{X}) = q(X) > 0$. Since there are no rational curves on $A$, we have a non-trivial proper morphism $\alpha: X \to A$.

Let $P \in \text{Pic}^0(A)$, $P' = \alpha^*P$ and $\mathcal{F} = \alpha_*\mathcal{O}_X(D)$. By the Kawamata-Viehweg vanishing theorem, we have $H^i(X, D + P') = 0$ for any $i > 0$. By the relative Kawamata-Viehweg vanishing theorem [KMM87 Theorem 1-2-5], we have $R^i\alpha_*\mathcal{O}_X(D + P') = 0$ for any $i > 0$. It follows from the Leray spectral sequence that $H^i(A, \mathcal{F} \otimes P) = H^i(X, D + P') = 0$ for any $i > 0$. If $H^0(A, \mathcal{F}) = 0$, then $h^0(A, \mathcal{F} \otimes P) = \chi(A, \mathcal{F} \otimes P) = \chi(A, \mathcal{F}) = 0$; i.e. $H^0(A, \mathcal{F} \otimes P) = 0$ for all $P \in \text{Pic}^0(A)$. By Lemma 2 we have $\mathcal{F} = 0$.

Next we prove that $\mathcal{F} \neq 0$, which implies $H^0(X, D) = H^0(A, \mathcal{F}) \neq 0$. Let $a(X) = \dim a(X) > 0$. If $a(X) = n$, then $\alpha: X \to \alpha(X)$ is generically finite, and it is easy to see that $\mathcal{F} \neq 0$. Assume that $a(X) < n$. Let $f: X \to Y$ be the Stein factorization of $\alpha$, $F$ a general fiber of $f$ and $\mathcal{G} = f_*\mathcal{O}_X(D)$. Then $F$ is a normal proper variety of dimension less than $n$. Note that $D\mid_F$ is nef Cartier, $(F, B\mid_F)$ is KLT and $D\mid_F - (K_F + B\mid_F) = H\mid_F$ is ample. By assumption, we have $\text{rank} \mathcal{G} = h^0(F, D\mid_F) \neq 0$; hence $\mathcal{G} \neq 0$ as well as $\mathcal{F} \neq 0$.

**Corollary 4.** $\text{EN}_2$ holds, and $\text{EN}_3$ holds for any $X$ with $q(X) > 0$.

**Proof.** For $\text{EN}_2$, by the Riemann-Roch theorem, one has only to deal with the case where $X$ is a ruled surface over a smooth projective curve $C$ with $q(X) = g(C) \geq 2$. Since $\text{EN}_1$ holds, $\text{EN}_2$ also holds by Theorem 3. The second conclusion is obvious.

**Corollary 5.** $\text{EN}_n(X, B)$ holds for any $X$ of maximal Albanese dimension.

**Proof.** By assumption, $X$ is of maximal Albanese dimension; i.e. the Albanese morphism $\alpha: X \to A$ satisfies $\dim \alpha(X) = \dim X = n$. So we can repeat the same argument as in Theorem 3 to complete the proof by noting that $\alpha$ is generically finite.

**Remark 6.** Note that Corollary 5 has already appeared in [PP03] and [PP05 Theorem 5.8]. Note also that the assumption that $D$ is nef in Conjecture 1 is not needed in the proof of Corollary 5 however [PP05, Lemma 5.1] proved that if
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$D - (K_X + B)$ is nef and big, then $D$ must be nef on the variety $X$ of maximal Albanese dimension. Furthermore, when the assumption that $D - (K_X + B)$ is nef and big in Conjecture \[1\] is replaced with the weaker assumption that $D - (K_X + B)$ is either nef or of non-negative Iitaka dimension, $H^0(X, D) \neq 0$ also holds for any $X$ of maximal Albanese dimension \[PP06\] Theorem 6.1. Finally, we should mention that Theorem \[3\] and \[PP05\] Theorem 5.8 used a similar idea in proof.

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REFERENCES


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