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MULTIPLE POINTS IN $\mathbf{P}^2$ AND DEGENERATIONS TO ELLIPTIC CURVES

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Abstract. We consider the problem of bounding the dimension of the linear system of curves in $\mathbf{P}^2$ of degree d with prescribed multiplicities $m_1, \ldots, m_n$ at n general points (Harbourne (1986), Hirschowitz (1985)). We propose a new method, based on the work of Ciliberto and Miranda (2000, 2003), by specializing the general points to an elliptic curve in $\mathbf{P}^2$.

1. Introduction

Let $P_1, \ldots, P_n$ be a set of n general points in $\mathbf{P}^2$. For any n-tuple of positive integers $\mathbf{m} = (m_1, \ldots, m_n)$ consider the "fat-point" scheme

$$\Gamma^{(\mathbf{m})} = \bigcup P_i^{(m_i)}.$$

Determining the dimension of the linear system $|\mathcal{I}_{\Gamma^{(\mathbf{m})}}(d)|$ of d-ics in $\mathbf{P}^2$ passing through each point P_i with multiplicity m_i is an open problem of algebraic geometry. In the present work we propose a new technique that allows us to give an upper bound on this dimension in some cases.

To set up the notation, let $\mathbf{P}'$ be the blowup of $\mathbf{P}^2$ at $P_1, \ldots, P_n$. Then, $Pic(\mathbf{P}') = \mathbf{Z}H \oplus \mathbf{Z}E_1 \oplus \cdots \oplus \mathbf{Z}E_n$, where H is the pull-back of a line in $\mathbf{P}^2$ and E_i is the exceptional divisor at P_i. Define the line bundle

$$\mathcal{L}_{\mathbf{m}} \cong \mathcal{O}_{\mathbf{P}'}(dH - \sum_{i=1}^{n} m_i E_i),$$

so that $|\mathcal{L}_{\mathbf{m}}| \cong |\mathcal{I}_{\Gamma^{(\mathbf{m})}}(d)|$. In the future, we will omit the subscript $_{\mathbf{m}}$ and will simply write $\mathcal{L}$. By Riemann-Roch, the expected dimension v of $|\mathcal{L}|$ is

$$v = \chi(\mathcal{L}) - 1 = \frac{d(d+3)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}.$$

We say that the linear system $|\mathcal{L}|$ is *special* if both cohomology groups $H^0(\mathcal{L})$ and $H^1(\mathcal{L})$ are nontrivial. We say that $|\mathcal{L}|$ is *homogeneous* if all multiplicities m_i are equal to some fixed m. We have the following:

Conjecture (Harbourne-Hirschowitz [12]). *The linear system $|\mathcal{L}|$ is special if and only if $Bs(|\mathcal{L}|)$ contains a (-1)-curve D with multiplicity at least two.*

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In the homogeneous case, the conjecture would imply that there are no special linear systems with $n \geq 9$ (see [4]).

Recently, Ciliberto, Cioffi, Miranda and Orrechia verified the conjecture for all homogeneous linear systems $|\mathcal{L}|$ with $m \leq 20$ (see [3], [4], [5]). The basic idea is to specialize some of the general points to a line and study the degeneration of the linear system $|\mathcal{L}|$. Using different methods, M. Dumnicki and W. Jarnicki verified the conjecture for $m \leq 42$ (see [7], [8]).

Motivated by Ciliberto-Miranda's approach ([3]), we propose to specialize the general points in $\mathbf{P}^2$ to an elliptic curve instead of a line. In Section 2, we describe a degeneration of $\mathbf{P}^2$ into a union of two surfaces, namely a rational surface and an elliptic ruled surface. The basic construction, known as *the deformation to the normal cone* (see [10]), is similar to the one used in [3].

In Section 3 we prove our main result (Theorem 3.1), which gives a bound on the dimension of $|\mathcal{L}|$ by the dimension of a (hopefully) simpler linear system in $\mathbf{P}^2$.

Finally, in Section 4, we give some applications of our result.

Remark 1.1. The content of Sections 2 and 3 generalizes to any smooth surface containing an elliptic curve, not just $\mathbf{P}^2$. We hope to find new interesting applications in the future.

Remark 1.2. Specialization of multiple points to elliptic curves was also considered by Caporaso and Harris in unpublished notes [2], where they used semi-stable reduction instead of deformation to the normal cone.

Notation and Conventions. We work over an algebraically closed field of characteristic 0. Recall some notation/terminology from [13]. Let C be a nonsingular elliptic curve. A *ruled surface* S over C is a nonsingular surface together with a $\mathbf{P}^1$-fibration $\pi : S \to C$. A *minimal section* C_0 of S is a section with minimal self-intersection. By a theorem of Atiyah ([1]), S is uniquely determined (up to a translation of C) by its *invariant* $e = -C_0^2$. For two divisors Y and Y' on S, $Y \sim Y'$ denotes rational equivalence and $Y \equiv Y'$ denotes numerical equivalence. Recall that $Pic(S) = \mathbf{Z}C_0 \oplus Pic(C)$ and $Num(S) = \mathbf{Z}C_0 \oplus \mathbf{Z}f$, where f is the class of a fiber. Thus, every divisor Y on S is rationally equivalent to some divisor $\mu C_0 + \mathfrak{b}f$, where $\mathfrak{b}$ is a divisor on C and $\mathfrak{b}f := \pi^*(\mathfrak{b})$.

2. Basic construction

Denote by Δ the affine line over the base field. The following lemma is motivated by the main construction in [3] for degenerating $\mathbf{P}^2$.

Lemma 2.1. *Fix positive integers $n \geq k \geq 10$. There exists a flat family of surfaces $X \to \Delta$ such that:*

(i) the general fiber X_t is isomorphic to the blowup of $\mathbf{P}^2$ at n general points;

(ii) the special fiber X_0 is the union of two components $S \cup \mathbf{P}'$ intersecting transversally along an elliptic curve C. Here, S is an indecomposable ruled surface over C; the component $\mathbf{P}'$ is isomorphic to the blowup of $\mathbf{P}^2$ at $n-k$ general points in $\mathbf{P}^2$ and k general points on C.

Proof. We construct X as a sequence of blowups as follows.

Step 1. Let $Y = \mathbf{P}^2 \times \Delta \to \Delta$ be the trivial family of planes. Let $\mathbf{P}_0$ be the fiber of the projection map $Y \to \Delta$ at $t = 0$. Fix a nonsingular elliptic curve $C \subset \mathbf{P}_0$. Notice that

$$N_{C/Y} \cong \mathcal{O}_C \oplus \mathcal{O}_C(3).$$

Step 2. We choose n families of general points $P_1, \dots, P_n$ on the plane such that the first k points specialize to general points on C as $t \to 0$. More precisely, let $P_1, \dots, P_n$ be n sections of the projection $Y \to \Delta$, such that:

(i) for t general, $P_i|_{X_t}$ is a general point in $\mathbf{P}^2$;

(ii) for $i \le k$, $P_i|_{X_0}$ is a general point on C;

(iii) for $i > k$, $P_i|_{X_0}$ is a general point in $\mathbf{P}_0$ and

(iv) for $i \le n$, the curve P_i intersects $\mathbf{P}_0$ transversally.

Let $\pi : \widetilde{Y} \to Y$ be the blowup of Y along $P_1, \dots, P_n$. Let $\widetilde{C}$ be the strict transform of C in $\widetilde{Y}$.

For $i \le k$, denote $p_i = P_i \cap C$ and let $\widetilde{p}_i$ be the corresponding point on $\widetilde{C}$. We have the following exact sequence on $\widetilde{C}$:

$$(*) \qquad 0 \to N_{\widetilde{C}/\widetilde{Y}} \to \pi^* N_{C/Y} \to \bigoplus_{i=1}^{k} \mathcal{F}_i \to 0,$$

where each $\mathcal{F}_i$ is a rank 1 skyscraper sheaf supported on $\widetilde{p}_i$ and naturally isomorphic to $N_{C \cup P_i/Y}|_{p_i}$.

Step 3. Finally, let $X \to \widetilde{Y}$ be the blowup of $\widetilde{C}$ and let S be the corresponding exceptional divisor. In particular, $S = \mathbf{P}(N_{\widetilde{C}/\widetilde{Y}})$ is a ruled surface over the elliptic curve C. From the exact sequence $(*)$, it follows that S is obtained from the ruled surface $\mathbf{P}(\mathcal{O}_C \oplus \mathcal{O}_C(3))$ by applying *elementary transforms*[1] at k general points. Since $k \ge 10$, it follows that S is indecomposable. The resulting threefold X has the required properties. □

3. Main result

Fix positive integers d, n, k and $m_1, \dots, m_n$ where $n \ge k \ge 10$. Let $X \to \Delta$ be the family of surfaces constructed in the previous section. For any t, we denote by X_t the fiber of X at t. For any i, denote by E_i the exceptional divisor of the map $X \to \mathbf{P}^2 \times \Delta$ corresponding to the i-th point. For t general, denote by $E_i^{(t)}$ the restriction $E_i|_{X_t}$. Thus, $E_i^{(t)}$ is an exceptional divisor of the blowup $X_t \to \mathbf{P}^2$. For $t = 0$ and $i \le k$, the restriction $E_i|_{X_0}$ has two components, which we denote by $E_i^{(0)}$ and D_i, where $E_i^{(0)}$ is an exceptional divisor of the blowup $\mathbf{P}' \to \mathbf{P}^2$ and D_i is some fiber of the ruled surface S. If $i > k$, then $E_i|_{X_0}$ has only one component, which we again denote by $E_i^{(0)}$.

Consider the line bundle $\mathcal{L} = \mathcal{O}_X(dH - \sum_{i=1}^n m_i E_i)$ on the threefold X. We have

$$\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}(dH - \sum_{i=1}^{n} m_i E_i^{(t)}),$$

for t general. At the special fiber, we have:

$$\mathcal{L}|_{\mathbf{P}'} \cong \mathcal{O}_{\mathbf{P}'}(dH - \sum_{i=1}^{n} E_i^{(0)})$$

and

$$\mathcal{L}|_S \cong \mathcal{O}_S(dH - \sum_{i=1}^{k} m_i D_i) \cong \mathcal{O}_S(\mathfrak{b} f),$$

[1] See [13], Example V.5.7.1.

for some suitable divisor $\mathfrak{b}$ on C (by construction, $\mathfrak{b}$ is general).

For any integer μ, consider the twist

$$\mathcal{L}(\mu) = \mathcal{L} \otimes \mathcal{O}_X(-\mu S).$$

Since $\mathcal{O}_X(S + \mathbf{P}') \cong \mathcal{O}_X(X_t) \cong \mathcal{O}_X$, we conclude that $\mathcal{O}_X(-S) \cong \mathcal{O}_X(\mathbf{P}')$. Therefore,

$$\mathcal{L}(\mu)|_{\mathbf{P}'} \cong \mathcal{O}_{\mathbf{P}'}((d - 3\mu)H - \sum_{i=1}^{k}(m_i - \mu)E_i^{(0)} - \sum_{i=k+1}^{n} m_i E_i^{(0)})$$

and

$$\mathcal{L}(\mu)|_S \cong \mathcal{O}_S(\mu C + \mathfrak{b} f).$$

Notice that $\mathcal{L}(\mu)|_{X_t} \cong \mathcal{L}|_{X_t}$ for $t \in \Delta$ general and any $\mu \in \mathbf{Z}$. Thus, we should think of $\mathcal{L}(\mu)|_{X_0}$ as a limit of the linear system $\mathcal{L}|_{X_t}$ as $t \to 0$. In particular, any choice μ leads to a possible limit (compare with the theory of limit linear series on curves, introduced by Eisenbud and Harris in [9]).

We are now in a position to formulate the main result in this section.

Theorem 3.1. *Let μ be a positive integer such that $\chi(\mathcal{L}(\mu)|_{\mathbf{P}'}) \geq \chi(\mathcal{L}|_{X_t})$ for a general $t \in \Delta$. Then $h^0(\mathcal{L}(\mu)|_{\mathbf{P}'}) \geq h^0(\mathcal{L}|_{X_t})$.*

The number μ should be interpreted as follows: let $\mathcal{U}$ be a curve in $\mathbf{P}^2$ passing through n general points $P_1, \dots, P_n$ with multiplicities $m_1, \dots, m_n$. As we specialize the first k of the points to an elliptic curve C (in a general fashion), at least μ copies of C must split off from $\mathcal{U}$.

The following lemma plays an essential role in the proof of the theorem.

Lemma 3.2. *Let S be an indecomposable ruled surface over an elliptic curve and let C be a section of S. Let $D \sim \mu C + \mathfrak{b} f$ be an effective divisor on S, where $\mu > 0$ and $\mathfrak{b} \in Pic(C)$ is general. Then, D is ample and $\chi(\mathcal{O}_S(D)) > 0$.*

Proof. Let C_0 be a minimal section of S and let $e = -C_0^2$. We may write $D \sim \mu C_0 + \mathfrak{b}' f$, where $\mathfrak{b}' \in Pic(C)$ is general. The canonical divisor of S is $K_S \equiv -2C_0 - ef$ and the arithmetic genus of S is $p_a = -1$ (see [13], Ch. V.2). By Riemann-Roch,

$$\chi(\mathcal{O}_S(D)) = \frac{1}{2} D \cdot (D - K_S) + p_a + 1 = (\mu + 1)(b' - \frac{1}{2}\mu e),$$

where $b' = \deg \mathfrak{b}'$. Therefore, to show that $\chi(\mathcal{O}_S(D)) > 0$, it suffices to show that

$$b' - \frac{1}{2}\mu e > 0.$$

Since S is indecomposable, $e = 0$ or -1 ([13], Thm. V.2.15). Suppose that $e = 0$. Since $C_0 \cdot D = b'$ and C_0 is nef, we have $b' \geq 0$. In fact, $b' > 0$, because $\mathfrak{b}'$ is general (it suffices to assume that the line bundle $\mathcal{O}_{C_0}(D)$ is not a multiple twist of $\mathcal{O}_{C_0}(C_0)$).

Suppose that $e = -1$. Then, it is well-known that S contains a nonsingular elliptic curve $Y \equiv 2C_0 - f$ (see [6], p.24). Since $Y^2 = 0$, it follows that Y is nef. Therefore, $Y \cdot D = 2b' + \mu \geq 0$. In fact, $2b' + \mu > 0$, because $\mathfrak{b}'$ is general (it suffices to assume that the line bundle $\mathcal{O}_Y(D)$ is not a multiple twist of $\mathcal{O}_Y(Y)$).

The fact that D is ample follows from the description of the ample cone of S (see [13], Prop. V.2.20 and 2.21).

This completes the proof of the lemma. □

Proof of Theorem 3.1. It will be notationally more convenient to replace μ with $\mu+1$ in the statement of the theorem. In other words, given that $\chi(\mathcal{L}(\mu+1)|_{\mathbf{P}'}) \geq \chi(\mathcal{L}|_{X_t})$ we want to show that $h^0(\mathcal{L}(\mu+1)|_{\mathbf{P}'}) \geq h^0(\mathcal{L}|_{X_t})$.

Consider the Mayer-Vietoris exact sequence on X_0:

$$0 \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{\mathbf{P}'} \oplus \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

We tensor the above sequence with $\mathcal{L}(\mu)$ and take cohomology:

$$0 \longrightarrow H^0(\mathcal{L}(\mu)|_{X_0}) \longrightarrow H^0(\mathcal{L}(\mu)|_{\mathbf{P}'}) \oplus H^0(\mathcal{L}(\mu)|_S) \xrightarrow{f \oplus g} H^0(\mathcal{L}(\mu)|_C).$$

We have:

$$\chi(\mathcal{L}(\mu)|_{X_0}) + \chi(\mathcal{L}(\mu)|_C) = \chi(\mathcal{L}(\mu)|_{\mathbf{P}'}) + \chi(\mathcal{L}(\mu)|_S).$$

Consider the following exact sequence on $\mathbf{P}'$:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}'}(-C) \longrightarrow \mathcal{O}_{\mathbf{P}'} \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

We tensor the above sequence with $\mathcal{L}(\mu)$ and take cohomology:

$$0 \longrightarrow H^0(\mathcal{L}(\mu+1)|_{\mathbf{P}'}) \longrightarrow H^0(\mathcal{L}(\mu)|_{\mathbf{P}'}) \xrightarrow{f} H^0(\mathcal{L}(\mu)|_C).$$

We have:

$$\chi(\mathcal{L}(\mu)|_{\mathbf{P}'}) = \chi(\mathcal{L}(\mu+1)|_{\mathbf{P}'}) + \chi(\mathcal{L}(\mu)|_C).$$

Adding the last two equalities gives:

$$(**) \qquad \chi(\mathcal{L}(\mu)|_{X_0}) = \chi(\mathcal{L}(\mu+1)|_{\mathbf{P}'}) + \chi(\mathcal{L}(\mu)|_S).$$

Since the Euler characteristic is constant in flat families, we have

$$\chi(\mathcal{L}(\mu)|_{X_0}) = \chi(\mathcal{L}|_{X_t}(\mu)) = \chi(\mathcal{L}|_{X_t}),$$

for t general. Now, the assumption $\chi(\mathcal{L}(\mu+1)|_{\mathbf{P}'}) \geq \chi(\mathcal{L}|_{X_t})$, together with $(**)$, implies $\chi(\mathcal{L}(\mu)|_S) \leq 0$. So, by the previous lemma,

$$H^0(\mathcal{L}(\mu)|_S) = 0.$$

Now, from the last two exact sequences in cohomology,

$$H^0(\mathcal{L}(\mu)|_{X_0}) = \ker f = H^0(\mathcal{L}(\mu+1)|_{\mathbf{P}'}).$$

Finally, by semicontinuity,

$$h^0(\mathcal{L}|_{X_t}) = h^0(\mathcal{L}(\mu)|_{X_t}) \leq h^0(\mathcal{L}(\mu)|_{X_0}) = h^0(\mathcal{L}(\mu+1)|_{\mathbf{P}'}).$$

This completes the proof. □

4. Applications

In this final section, we will use Theorem 3.1 to show that certain homogeneous linear systems in $\mathbf{P}^2$ are nonspecial. Also, we will give an example that exhibits a limitation of our theorem.

Given data (d, n, m), consider curves in $\mathbf{P}^2$ of degree d passing through $n \geq 10$ general points with multiplicity m. For simplicity, we will specialize all n points at once to a smooth cubic curve $C \subset \mathbf{P}^2$.

So, let $X \to \Delta$ and $\mathcal{L}$ be as before, with $k = n$. For any integer μ, we have:

$$\begin{aligned}\chi(\mathcal{L}(\mu)|_{\mathbf{P}'}) &- \chi(\mathcal{L}|_{X_t})\\ &= \frac{(d-3\mu)(d-3\mu+3)}{2} - n\frac{(m-\mu)(m-\mu+1)}{2} - \frac{d(d+3)}{2} + n\frac{m(m+1)}{2}\\ &= \frac{1}{2}\mu(n-9-6d+2mn-\mu(n-9)).\end{aligned}$$

In particular, $\chi(\mathcal{L}(\mu)|_{\mathbf{P}'}) \geq \chi(\mathcal{L}|_{X_t})$ if

$$0 \leq \mu \leq 1 + \frac{2mn-6d}{n-9}.$$

(Notice, that the right-hand side is just $1+2(\mathcal{L}|_{X_t}\cdot K_{X_t})/(-K^2_{X_t})$ for $t \in \Delta$ general.)

Clearly, in order to get the most information from Theorem 3.1, we should choose the greatest integral value of μ, subject to the inequality above. The best scenario is achieved when the upper bound on μ is already an integer:

Corollary 4.1. *Let (d, n, m) be as above, and assume that $\mu = 1 + \frac{2mn-6d}{n-9}$ is a positive integer. If $\mathcal{L}(\mu)|_{\mathbf{P}'}$ is nonspecial, then so is $\mathcal{L}|_{X_t}$, for t general.*

Proof. We have $\chi(\mathcal{L}(\mu)|_{\mathbf{P}'}) = \chi(\mathcal{L}|_{X_t})$ and $h^0(\mathcal{L}(\mu)|_{\mathbf{P}'}) \geq h^0(\mathcal{L}|_{X_t})$. Assuming that $h^0(\mathcal{L}(\mu)|_{\mathbf{P}'}) > 0$, we have

$$\chi(\mathcal{L}(\mu)|_{\mathbf{P}'}) = h^0(\mathcal{L}(\mu)|_{\mathbf{P}'}) \geq h^0(\mathcal{L}|_{X_t}) \geq \chi(\mathcal{L}|_{X_t}).$$

So, there is equality everywhere. It follows that $h^1(\mathcal{L}|_{X_t}) = 0$. □

We proceed with some examples.

Example 4.2. Consider the linear system corresponding to the data $(d, n, m) = (13, 10, 4)$, with expected dimension $v = \chi(\mathcal{L}|_{X_t}) - 1 = 4$. We take $\mu = 3$. We have $\mathcal{L}(3)|_{\mathbf{P}'} \cong \mathcal{O}_{\mathbf{P}'}(4H - \sum_{i=1}^{10} D_i)$. This is a nonspecial linear system, because any 10 points on an elliptic curve impose independent conditions on quartics in $\mathbf{P}^2$. It follows that the original linear system is also nonspecial.

Example 4.3. Let $(d, n, m) = (38, 10, 12)$, expected dimension $v = -1$. We take $\mu = 13$. We have $\mathcal{L}(13)|_{\mathbf{P}'} \cong \mathcal{O}_{\mathbf{P}'}(-H + \sum_{i=1}^{10} D_i)$. This is a nonspecial linear system, and so is the original one.[2]

Example 4.4. Let $(d, n, m) = (57, 10, 18)$, expected dimension $v = 0$. We take $\mu = 19$. We have $\mathcal{L}(19)|_{\mathbf{P}'} \cong \mathcal{O}_{\mathbf{P}'}(\sum_{i=1}^{10} D_i)$. This is a nonspecial linear system, and so is the original one.

Example 4.5. Let $(d, n, m) = (174, 10, 55)$, expected dimension $v = -1$. In this example, our approach does not work. Indeed, to use Corollary 4.1, we must take $\mu = 57$. But now, $\mathcal{L}(57)|_{\mathbf{P}'} \cong \mathcal{O}_{\mathbf{P}'}(3H + \sum_{i=1}^{10} 2D_i)$, which *is* special! (with $h^0 = h^1 = 10$). So, the best we can say is that $h^0(\mathcal{L}|_{X_t}) \leq 10$.

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[2] This example is proved in the thesis of Gimigliano [11] by using Horace's method (introduced in [14]). The original method of Ciliberto and Miranda does not handle this example (see [4], pp. 4048–4049).

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