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ON THE EXPECTED NUMBER OF ZEROS OF A RANDOM HARMONIC POLYNOMIAL

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ABSTRACT. We study the distribution of complex zeros of Gaussian harmonic polynomials with independent complex coefficients. The expected number of zeros is evaluated by applying a formula of independent interest for the expected absolute value of quadratic forms of Gaussian random variables.

1. INTRODUCTION

A harmonic polynomial is a complex-valued harmonic function in $\mathbb{C}$ (the complex plane) of the form $h_{n,m}(z) := p_n(z) + q_m(\overline{z})$, where $p_n(z)$ and $q_m(z)$ are analytic polynomials of degree n and m, respectively, with $0 \leq m \leq n$. In 1992, T. Sheil-Small conjectured that the sharp upper bound for the number of zeros of $h_{n,m}(z)$ was n^2. Wilmshurst proved this in [Wi98] by using Bézout's theorem and also demonstrated by examples that the bound is sharp for $m = n$ and $m = n-1$. In addition, for $m \leq n-1$, it was conjectured in [Wi98] that the the maximal number of zeros should be $m(m-1) + 3n - 2$. The case $m = 1$ was proved in Khavinson and Swiatek [KS03] by using powerful techniques from complex dynamics. Other related works can be found in [BHS95], [BL04], [Ge03] and [KN06]. Due to the variability of the number of zeros for $h_{n,m}(z)$, unlike polynomials of fixed degree, it is natural to ask for the expected number of zeros on $\mathbb{C}$ with random coefficients.

There is a long history of studying zeros of a random polynomial whose coefficients are independent, non-degenerate random variables; see [EK95] for a survey. Exact formulae for the expected number of real zeros under independent identically distributed Gaussian coefficients are found for a random polynomial by Kac [Kac43], and for a random trigonometric polynomial by Dunnage [D66]. Much work was also done on complex roots over a fixed domain; see [SV95], [IZ97] and [PV05]. For harmonic homogeneous polynomials of degree d in $m+1$ variables, that is, the sum of powers of all $m+1$ variables in each term is always d and the Laplacians of the polynomials are equal to zero, the expected number of roots for a system of m such random harmonic homogeneous polynomial equations is $(d(d+m-1)/m)^{m/2}$; see [EK95]. One may also consider systems of homogeneous harmonic polynomials of different degrees, or one may consider underdetermined systems, and obvious generalizations of the above result still hold; see Kostlan [Ko02] for a detailed discussion.

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In this paper, we consider a challenging general case and find the expected number of zeros in an open domain $T \subset \mathbb{C}$ for a random harmonic polynomial

$$h_{n,m}(z) = \sum_{j=0}^{n} a_j z^j + \sum_{j=0}^{m} b_j \overline{z}^j$$

with $0 \le m \le n$, where a_j and b_j are centered complex Gaussian random variables, i.e. $\mathbb{E}\, a_j = \mathbb{E}\, b_j = 0$, with $\mathbb{E}\, a_j \overline{a}_k = \delta_{jk} \binom{n}{j}$ and $\mathbb{E}\, b_j \overline{b}_k = \delta_{jk} \binom{m}{j}$. Our argument also works for independent identically distributed standard complex or real Gaussian variables. The use of the Gaussian distribution is mainly due to feasibility of computation and provides a reasonable understanding of universality behavior under other distributions. The main theorem of this paper is:

Theorem 1.1. *The expected number of zeros of $h_{n,m}(z) = p_n(z) + q_m(\overline{z})$ on an open domain $T \subset \mathbb{C}$, denoted by $\mathbb{E}\, N_h(T)$, is given by:*

$$\mathbb{E}\, N_h(T) = \frac{1}{\pi} \int_T \frac{1}{|z|} \frac{r_1^2 + r_2^2 - 2r_{12}^2}{r_3^2 \sqrt{(r_1+r_2)^2 - 4r_{12}^2}}\ d\sigma(z) \tag{1.1}$$

where $\sigma(\cdot)$ denotes the Lebesgue measure on the complex plane and

$$r_{12} = mn|z|^4(1+|z|^2)^{m+n-2}\,,\ \ r_3 = (1+|z|^2)^m + (1+|z|^2)^n\,,$$
$$r_1 = r_3(1+|z|^2)^{n-2}\left(n^2|z|^4 + n|z|^2\right) - n^2|z|^4(1+|z|^2)^{2n-2}\,,$$
$$r_2 = r_3(1+|z|^2)^{m-2}\left(m^2|z|^4 + m|z|^2\right) - m^2|z|^4(1+|z|^2)^{2m-2}\,.$$

In particular, when $m = n$, the expected number of zeros over the entire $\mathbb{C}$ is

$$\begin{aligned} \mathbb{E}\, N_h(\mathbb{C}) &= \frac{\pi n^2}{4\sqrt{n-1}} + \frac{n}{2} - \frac{n^2}{2\sqrt{n-1}} \arctan\left(\frac{1}{\sqrt{n-1}}\right) \\ &\sim \frac{\pi}{4} n^{3/2} \quad \text{as} \quad n \to \infty, \end{aligned}$$

and when $m = \alpha n + o(n)$ with $0 \le \alpha < 1$, $\mathbb{E}\, N_h(\mathbb{C}) \sim n$ as $n \to \infty$.

The argument principle immediately shows that a harmonic polynomial $h_{n,m}(z)$ has at least n zeros. It is surprising to see that even when m is a fraction of n, the expected number of zeros is still n asymptotically. On the other hand, when m is n minus a constant, the expected number of zeros is of order $n^{3/2}$. Note that the finiteness of $\mathbb{E}\, N_h(\mathbb{C})$ shows that $h_{n,m}(z)$ has finite number of zeros with probability one. We can also prove $\mathbb{E}\, N_h(\mathbb{C}) < \infty$ directly by applying Bézout's theorem, which implies that $h_{n,m}(z) = p_n(z) + q_m(\overline{z})$ has at most n^2 zeros if $p_n(z)$ and $q_m(z)$ are relatively prime; see [DHL96], [Wi98] and [S02]. Following a standard argument similar to the proof of Lemma 5 in [CW03], one can show that the probability $\mathbb{P}\,(\exists\, z_0 \in \mathbb{C}\ s.t.\ p_n(z_0) = q_m(z_0) = 0)$ is zero, which is equivalent to saying that $p_n(z)$ and $q_m(z)$ are relatively prime polynomials almost surely.

Our approach follows the general framework used in [IZ97], but significantly different arguments at the technical level are needed. To be more precise, it is crucial for us to find an expectation of the form $\mathbb{E}\,|X_1^2 - X_2^2 + X_3^2 - X_4^2|$ for joint Gaussian vector (X_1, X_2, X_3, X_4). In all known early work on random functions, one only needs to deal with $\mathbb{E}\,|X_1^2 + X_2^2| = \mathbb{E}\, X_1^2 + \mathbb{E}\, X_2^2$. This is why we have a very hard time finding the variance or higher moments of the number of zeros, since we do not know how to evaluate expressions like $\mathbb{E} \prod_{j=1}^{d} |X_{1j}^2 - X_{2j}^2 + X_{3j}^2 - X_{4j}^2|$.

Note that the difference of squares comes from the combined effect of variables z and $\overline{z}$.

Our method also works in principle for the rational harmonic function, and hence provides a probabilistic interpretation of the Khavinson-Neumann theorem on the gravitationally lensed images of a light source by n masses; see [KN06]. Technical details will be given elsewhere.

The remaining sections are organized as follows: Section 2 deals with the evaluation of the absolute value of a quadratic form of Gaussian random variables. Our method is based on a representation of the absolute function. A useful corollary is given; for related topics of independent interest, see details in [LW08].

The main part of section 3 is devoted to the detailed proof of Theorem 1.1, followed by interesting asymptotic results. In the last section we discuss the alternative setting of random harmonic polynomials with independent identically distributed standard complex Gaussian coefficients. Finally, it must be mentioned that the first author's attention was drawn to the problem by an excellent lecture given by D. Khavinson.

2. Preliminary results

We start with the Rice formula, which provides a representation for the expected number of zeros of certain random fields; see [AW05] and [AW06] for details.

Lemma 2.1. *Let $f : U \to \mathbb{R}^d$ be a random field, with U an open subset of $\mathbb{R}^d$. Assume that*

(1) *f is Gaussian,*
(2) *almost surely the function $t \to h(t)$ is of class C^1,*
(3) *for each $t \in U$, $f(t)$ has a non-degenerate distribution (i.e. $Var\,(f(t)) \succ 0$),*
(4) $\mathbb{P}\,\{\exists\, t \in U \text{ s.t. } f(t) = 0, \det\,(f'(t)) = 0\} = 0$.

Then, for every Borel set T contained in U, we have

$$\mathbb{E}\,\left(N_f(T)\right) = \int_T \mathbb{E}\,\left(|\det\,(f'(t))\,|\big|f(t) = 0\right) p_0\; dt$$

where p_0 is the probability density of $f(t)$ at 0.

Note that function f in the above formula is defined on $\mathbb{R}^d$. In our application, we need to find all zeros (real and complex) of $h_{n,m}(z)$. They are real zeros of $\Re h_{n,m}(x+iy) = 0$ and $\Im h_{n,m}(x+iy) = 0$ for $(x,y) \in \mathbb{R}^2$, or equivalently, $\Re h_{n,m}(re^{i\theta}) = 0$ and $\Im h_{n,m}(re^{i\theta}) = 0$, for $(r,\theta) \in \mathbb{R}^+ \times [0, 2\pi) \subset \mathbb{R}^2$. It is easy to check the conditions in Lemma 2.1. In particular, condition (4) follows from Lemma 5 in [CW03] based on the smoothness of f.

Next we give a formula for the expectation of the absolute value of a quadratic form $\langle X, HX\rangle$ of Gaussian random variables. The formula is of independent interest and a special case is used in the proof of the main theorem.

Proposition 2.1. *For a real centered Gaussian random vector $X = (X_1, X_2, \cdots, X_n)$ with covariance matrix R and any real symmetric matrix $H = (h_{ij})_{n\times n}$,*

$$\begin{aligned}\mathbb{E}\,|\langle X, HX\rangle| &= \mathbb{E}\,|\sum_{i,j=1}^{n} h_{ij} X_i X_j\,| \\ &= \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left(1 - \frac{\det(I+2\mathtt{i}tRH)^{1/2} + \det(I-2\mathtt{i}tRH)^{1/2}}{2\det(I+4t^2R^2H^2)^{1/2}}\right) dt\end{aligned}$$

where I is the $n \times n$ identity matrix and $\mathtt{i} = \sqrt{-1}$.

Proof. We only need to consider a non-singular covariance matrix R, since for singular R the result still holds by considering $R^* = R + \delta I$ and using the standard limiting argument of taking $\delta \to 0$.

We start with the representation

$$|x| = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(xt)}{t^2}\,dt = \frac{2}{\pi}\int_0^\infty \frac{1-\mathbb{E}_\varepsilon e^{\mathtt{i}\varepsilon xt}}{t^2}\,dt$$

where $\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = 1/2$. Then we can rewrite the expectation as

$$\mathbb{E}\,|\langle X, HX\rangle| = \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left(1 - \mathbb{E}_\varepsilon \mathbb{E}_X e^{\mathtt{i}\varepsilon t\langle X,HX\rangle}\right) dt\,.$$

For a non-singular covariance matrix R, the density of X is

$$f_X(x) = (2\pi)^{-n/2}\,(\det\, R)^{-1/2}\, e^{-\frac{1}{2}\langle x, R^{-1}x\rangle}.$$

Therefore we have

$$\begin{aligned}
\mathbb{E}_X e^{\mathtt{i}\varepsilon t\langle X,HX\rangle} &= \int_{\mathbb{R}^n} (2\pi)^{-n/2}(\det\, R)^{-1/2} e^{-\frac{1}{2}\langle x,R^{-1}x\rangle} e^{\mathtt{i}\varepsilon t\langle x,Hx\rangle} dx\\
&= (\det\, R)^{-1/2}\int_{\mathbb{R}^n} (2\pi)^{-n/2} e^{-\frac{1}{2}\langle x,(R^{-1}-2\mathtt{i}t\varepsilon H)x\rangle} dx\\
&= (\det\, R)^{-1/2}\left(\det(R^{-1}-2\mathtt{i}t\varepsilon H)\right)^{-1/2}\\
&= \left(\det(I-2\mathtt{i}t\varepsilon RH)\right)^{-1/2}\,.
\end{aligned}$$

Note that

$$\det(I-2\mathtt{i}t\varepsilon RH)\cdot\det(I+2\mathtt{i}t\varepsilon RH) = \det(I+4t^2R^2H^2) \neq 0,$$

and so $\det(I-2\mathtt{i}t\varepsilon RH) \neq 0$. Hence

$$\begin{aligned}
\mathbb{E}_\varepsilon \mathbb{E}_X e^{\mathtt{i}\varepsilon t\langle X,HX\rangle} &= \frac{1}{2}\frac{1}{\det(I-2\mathtt{i}tRH)^{1/2}} + \frac{1}{2}\frac{1}{\det(I+2\mathtt{i}tRH)^{1/2}}\\
&= \frac{\det(I-2\mathtt{i}tRH)^{1/2}+\det(I+2\mathtt{i}tRH)^{1/2}}{2\det(I+4t^2R^2H^2)^{1/2}},
\end{aligned}$$

which is real since $\det(I-2\mathtt{i}tRH)^{1/2}$ and $\det(I+2\mathtt{i}tRH)^{1/2}$ are conjugate to each other. Thus we finish the proof. □

It is obvious that in the case where the matrix $H = (h_{ij})_{n\times n}$ is positive definite, we have directly

$$\mathbb{E}\,|\langle X, HX\rangle| = \mathbb{E}\,\langle X, HX\rangle = \sum_{i,j=1}^n h_{ij}\, r_{ij}$$

where $(r_{ij})_{n\times n} = R$ is the covariance matrix of X.

Next we give an important corollary which will be used in the proof of Theorem 1.1. Other interesting cases and applications can be found in [LW08].

Corollary 2.1. *Let (X_1, X_2) be a centered Gaussian with $\mathbb{E}\,X_1^2 = \sigma_1^2$, $\mathbb{E}\,X_2^2 = \sigma_2^2$, and $\mathbb{E}\,X_1X_2 = \sigma_{12}$. If (X_3, X_4) is an independent copy of (X_1, X_2), then*

$$\mathbb{E}\,|X_1^2 - X_2^2 + X_3^2 - X_4^2| = \frac{2\sigma_1^4 + 2\sigma_2^4 - 4\sigma_{12}^2}{\sqrt{(\sigma_1^2+\sigma_2^2)^2 - 4\sigma_{12}^2}}\ .$$

Proof. In this case, we have $H = diag\ (1, -1, 1, -1)$ and

$$R = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & 0 & 0 \\ \sigma_{12} & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \sigma_1^2 & \sigma_{12} \\ 0 & 0 & \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

Therefore

$$I + 2\mathtt{i}tRH = \begin{pmatrix} 1+2\mathtt{i}t\sigma_1^2 & -2\mathtt{i}t\sigma_{12} & 0 & 0 \\ 2\mathtt{i}t\sigma_{12} & 1-2\mathtt{i}t\sigma_2^2 & 0 & 0 \\ 0 & 0 & 1+2\mathtt{i}t\sigma_1^2 & -2\mathtt{i}t\sigma_{12} \\ 0 & 0 & 2\mathtt{i}t\sigma_{12} & 1-2\mathtt{i}t\sigma_2^2 \end{pmatrix}.$$

Hence the determinant is given by

$$\det(I + 2\mathtt{i}tRH) = \left(1 + 2(\sigma_1^2 - \sigma_2^2)\mathtt{i}t + (4\sigma_1^2\sigma_2^2 - 4\sigma_{12}^2)t^2\right)^2.$$

Write $p = 4\sigma_1^2\sigma_2^2 - 4\sigma_{12}^2$ and $q = \sigma_1^2 - \sigma_2^2$; then Proposition 2.1 can be simplified:

$$\begin{aligned} \mathbb{E}\,|X_1^2 + X_2^2 - X_3^2 - X_4^2| &= \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left(1 - \frac{1+pt^2}{(1+pt^2)^2 + 4q^2t^2}\right) dt \\ &= \frac{2}{\pi}\int_0^\infty \frac{p(1+pt^2) + 4q^2}{(1+pt^2)^2 + 4q^2t^2}\ dt \\ &= \frac{1}{\pi}\int_0^\infty \frac{p + 4q^2 - 2\mathtt{i}pqt}{pt^2 + 1 - 2\mathtt{i}qt} + \frac{p + 4q^2 + 2\mathtt{i}pqt}{pt^2 + 1 - 2\mathtt{i}qt}\ dt \\ &= \frac{p + 2q^2}{\sqrt{p + q^2}} = \frac{2\sigma_1^4 + 2\sigma_2^4 - 4\sigma_{12}^2}{\sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4\sigma_{12}^2}}. \end{aligned}$$

□

3. Proof of Theorem 1.1

In order to apply the representation of the number of zeros given in (2.1), we need to separate the real and imaginary parts of $h_{n,m}(z)$. Namely, we write

$$\begin{aligned} h_{n,m}(z) &= \sum_{j=0}^n a_j z^j + \sum_{j=0}^m b_j \overline{z}^j \\ &= \sum_{j=0}^n (a_{j,1} + \mathtt{i}a_{j,2}) r^j (\cos j\theta + \mathtt{i}\sin j\theta) + \sum_{j=0}^m (b_{j,1} + \mathtt{i}b_{j,2}) r^j (\cos j\theta - \mathtt{i}\sin j\theta) \\ &= Y_1(r,\theta) + \mathtt{i}Y_2(r,\theta) \end{aligned}$$

where

$$\begin{aligned} Y_1(r,\theta) &= \sum_{j=0}^n r^j (a_{j,1}\cos j\theta - a_{j,2}\sin j\theta) + \sum_{j=0}^m r^j (b_{j,1}\cos j\theta + b_{j,2}\sin j\theta), \\ Y_2(r,\theta) &= \sum_{j=0}^n r^j (a_{j,1}\sin j\theta + a_{j,2}\cos j\theta) + \sum_{j=0}^m r^j (b_{j,2}\cos j\theta - b_{j,1}\sin j\theta). \end{aligned}$$

Write $Y(r,\theta)=(Y_1(r,\theta),Y_2(r,\theta))^T$; the Jacobian determinant in (2.1) of this vector can be computed as

$$\det \nabla Y(r,\theta) = \frac{\partial Y_1(r,\theta)}{\partial r}\frac{\partial Y_2(r,\theta)}{\partial \theta} - \frac{\partial Y_1(r,\theta)}{\partial \theta}\frac{\partial Y_2(r,\theta)}{\partial r}$$
$$= \frac{1}{r}\{ \left(\sum_{j=1}^{n} jr^j(a_{j,1}\cos j\theta - a_{j,2}\sin j\theta)\right)^2 - \left(\sum_{j=1}^{m} jr^j(b_{j,1}\cos j\theta + b_{j,2}\sin j\theta)\right)^2$$
$$+ \left(\sum_{j=1}^{n} jr^j(a_{j,1}\sin j\theta + a_{j,2}\cos j\theta)\right)^2 - \left(\sum_{j=1}^{n} jr^j(b_{j,2}\cos j\theta - b_{j,1}\sin j\theta)\right)^2 \}.$$

To reduce the number of parameters, we change the expression back to a function of z. We cannot do this at the beginning because the representation of the number of zeros can only be applied to real functions. Note that $\Re z^j = r^j\cos j\,\theta$ and $\Im z^j = r^j \sin j\,\theta$, so we have

$$\det \nabla Y(r,\theta) = \frac{1}{|z|}\left(u_1^2 - u_2^2 + v_1^2 - v_2^2\right)$$

where

$$u_1 = \Re\sum_{j=1}^{n} jz^j a_j\,, \quad v_1 = \Im\sum_{j=1}^{n} jz^j a_j\,,$$
$$u_2 = \Re\sum_{j=1}^{m} jz^j a_j\,, \quad v_2 = \Im\sum_{j=1}^{m} jz^j a_j. \tag{3.1}$$

To simplify notation, we also define

$$u_3 = \Re\left(p_n(z) + q_m(\overline{z})\right)\,, \qquad v_3 = \Im\left(p_n(z)+q_m(\overline{z})\right)\,. \tag{3.2}$$

Then according to (2.1), we need to find the conditional expectation

$$\mathbb{E}\,\left(|u_1^2 - u_2^2 + v_1^2 - v_2^2| \,\Big|\, u_3 = 0, v_3 = 0\right) = \mathbb{E}\,|U_1^2 - U_2^2 + V_1^2 - V_2^2| \tag{3.3}$$

where (U_1,U_2,V_1,V_2) is the Gaussian random vector with the same distribution as (u_1,u_2,v_1,v_2) under the condition $u_3=0,\ v_3=0$. According to [T90, page 34] the covariance matrix of (U_1,U_2,V_1,V_2) is given by

$$R_c = C - BA^{-1}B^T \tag{3.4}$$

where $A_{2\times 2} = cov(u_3,v_3)$, $B_{4\times 2} = cov\left((u_1,u_2,v_1,v_2),(u_3,v_3)\right)$ and $C_{4\times 4} = cov(u_1,$ $u_2,v_1,v_2)$. From (3.1) and (3.2), we have

$$\mathbb{E}\,u_3^2 = \mathbb{E}\,v_3^2 = \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}|z|^{2j} + \frac{1}{2}\sum_{j=0}^{m}\binom{m}{j}|z|^{2j} = \frac{1}{2}(1+|z|^2)^n + \frac{1}{2}(1+|z|^2)^m\,,$$

$$\mathbb{E}\,u_1u_3 = \mathbb{E}\,v_1v_3 = \frac{1}{2}\sum_{j=1}^{n}\binom{n}{j}j|z|^{2j} = \frac{1}{2}n|z|^2(1+|z|^2)^{n-1}\,,$$

$$\mathbb{E}\,u_1^2 = \mathbb{E}\,v_1^2 = \frac{1}{2}\sum_{j=1}^{n}\binom{n}{j}j^2|z|^{2j} = \frac{1}{2}(n^2|z|^4 + n|z|^2)(1+|z|^2)^{n-2}\,.$$

Similarly we also have

$$\mathbb{E}\, u_2 u_3 = \mathbb{E}\, v_2 v_3 = \frac{1}{2} m|z|^2 (1+|z|^2)^{m-1}\,,$$
$$\mathbb{E}\, u_2^2 = \mathbb{E}\, v_2^2 = \frac{1}{2}(m^2|z|^4 + m|z|^2)(1+|z|^2)^{m-2}\,.$$

Thus the covariance matrices are

$$A_{2\times 2} = \frac{1}{2}\ diag\left((1+|z|^2)^n + (1+|z|^2)^m,\ (1+|z|^2)^n + (1+|z|^2)^m\right),$$
$$B_{4\times 2} = \frac{1}{2}\begin{pmatrix} n|z|^2(1+|z|^2)^{n-1} & 0 \\ m|z|^2(1+|z|^2)^{m-1} & 0 \\ 0 & n|z|^2(1+|z|^2)^{n-1} \\ 0 & m|z|^2(1+|z|^2)^{m-1} \end{pmatrix},$$
$$C_{4\times 4} = \frac{1}{2}\ diag\big((n^2|z|^4 + n|z|^2)(1+|z|^2)^{n-2},\ (m^2|z|^4 + m|z|^2)(1+|z|^2)^{m-2},$$
$$(n^2|z|^4 + n|z|^2)(1+|z|^2)^{n-2},\ (m^2|z|^4 + m|z|^2)(1+|z|^2)^{m-2}\big)\,.$$

Therefore following (3.4) we obtain the covariance matrix of (U_1, U_2, V_1, V_2),

$$R_c = \frac{1}{2r_3}\begin{pmatrix} r_1 & -r_{12} & 0 & 0 \\ -r_{12} & r_2 & 0 & 0 \\ 0 & 0 & r_1 & -r_{12} \\ 0 & 0 & -r_{12} & r_2 \end{pmatrix}$$

where r_1, r_2, r_{12} and r_3 are given in Theorem 1.1.

By applying Corollary 2.1, we obtain the expectation of the absolute value in (3.3):

$$\mathbb{E}\,|U_1^2 - U_2^2 + V_1^2 - V_2^2| = \frac{1}{r_3}\frac{r_1^2 + r_2^2 - 2r_{12}^2}{\sqrt{(r_1+r_2)^2 - 4r_{12}^2}}\,.$$

Note that p_0 is the probability density of $h(z)$ at 0, which means

$$p_0 = (2\pi)^{-1}\left(\det(\mathbb{E}\, Y(r,\theta)Y(r,\theta)^T)\right)^{-1/2} = (\pi r_3)^{-1}$$

where $Y(r,\theta) = (Y_1(r,\theta), Y_2(r,\theta))^T$ as defined at the beginning of this section. Combining these, we have

$$\begin{aligned} \mathbb{E}\ (N_h(T)) &= \int_T \frac{1}{|z|}\,\mathbb{E}\,|U_1^2 - U_2^2 + V_1^2 - V_2^2|\,p_0\,d\sigma(z) \\ &= \frac{1}{\pi}\int_T \frac{r_1^2 + r_2^2 - 2r_{12}^2}{|z|\,r_3^2\sqrt{(r_1+r_2)^2 - 4r_{12}^2}}\,d\sigma(z)\,. \end{aligned} \tag{3.5}$$

In the case that $m = n$, we have $r_1 = r_2$, and part of the integrand in (3.5) can be simplified:

$$\frac{r_1^2 + r_2^2 - 2r_{12}^2}{r_3^2\sqrt{(r_1+r_2)^2 - 4r_{12}^2}} = \frac{\sqrt{r_1^2 - r_{12}^2}}{r_3^2} = \frac{\sqrt{n^2|z|^4 + n^3|z|^6}}{2(1+|z|^2)^2}\,.$$

Therefore the expected number of zeros in $\mathbb{C}$ is

$$\begin{aligned}\mathbb{E}\, N_h(\mathbb{C}) &= \pi^{-1}\int_0^{2\pi}\int_0^\infty \frac{nr\sqrt{1+nr^2}}{2(1+r^2)^2}\,dr\,d\theta \\ &= n^2\int_1^\infty \left(\frac{1}{s^2+n-1}-\frac{n-1}{(s^2+n-1)^2}\right)ds \\ &= \frac{\pi n^2}{4\sqrt{n-1}}+\frac{n}{2}-\frac{n^2}{2\sqrt{n-1}}\arctan\left(\frac{1}{\sqrt{n-1}}\right) \\ &\sim \frac{\pi}{4}n^{3/2}\end{aligned}$$

by using the substitution $s=\sqrt{1+nr^2}$.

In the case that $m=\alpha n+o(n)$ with $0\le\alpha<1$, clearly r_2 and r_{12} are dominated by r_1; therefore we have

$$\lim_{n\to\infty}\frac{1}{n}\frac{r_1^2+r_2^2-2r_{12}^2}{r_3^2\sqrt{(r_1+r_2)^2-4r_{12}^2}}=\lim_{n\to\infty}\frac{1}{n}\frac{r_1}{r_3^2}=\frac{|z|^2}{(1+|z|^2)^2}\,.$$

So the asymptotic result for $\mathbb{E}\, N_h(\mathbb{C})$ is

$$\lim_{n\to\infty}\frac{1}{n}\,\mathbb{E}\, N_h(\mathbb{C})=\frac{1}{\pi}\int_{\mathbb{C}}\frac{|z|^2}{|z|(1+|z|^2)^2}\,d\sigma(z)=1\,.$$

Thus we finish the proof. $\square$

For the remaining part of this section, we briefly examine the asymptotic results for m close to n. In the case that $m=n-k$ for a fixed positive integer k, we have

$$\frac{r_1^2+r_2^2-2r_{12}^2}{r_3^2\sqrt{(r_1+r_2)^2-4r_{12}^2}}\sim\frac{n^{3/2}r^3(1+r^2)^{k/2-2}}{1+(1+r^2)^k}\quad\text{as}\quad n\to\infty.$$

Thus $\lim_{n\to\infty}\mathbb{E}\, N_h(\mathbb{C})/n^{3/2}=c_k$, where

$$c_k=\int_0^\infty\frac{2r^2(1+r^2)^{k/2-2}}{1+(1+r^2)^k}\,dr\,.$$

When $k=0$, the case degenerates into $m=n$ with $c_0=\pi/4$. For $k\ge 1$, $c_1=0.49\cdots$, $c_2=0.31\cdots$, and c_k decreases to zero as $k\to\infty$. In fact, as $k\to\infty$, $c_k\sim\sqrt{2\pi}k^{-3/2}$.

4. Independent identically distributed setting

Next we consider another general case,

$$\tilde{h}_{n,m}(z)=\sum_{j=0}^n\tilde{a}_j z^j+\sum_{j=0}^m\tilde{b}_j\overline{z}^j,$$

with $0\le m\le n$, where $\tilde{a}_j$ and $\tilde{b}_j$ are independent identically distributed complex Gaussian random variables with $\mathbb{E}\,\tilde{a}_j=\mathbb{E}\,\tilde{b}_j=0$ and $\mathbb{E}\,\tilde{a}_j\overline{\tilde{a}}_k=\delta_{jk}$. In this case, the differences when compared with the setting in the previous section are the values of $r_1,\ r_2,\ r_{12}$ and r_3. So we omit the details and only state the result.

Theorem 4.1. *The expected number of zeros of $\tilde{h}_{n,m}(z) = \tilde{p}_n(z) + \tilde{q}_m(\overline{z})$ on T, denoted by $\mathbb{E}\, N_{\tilde{h}}(T)$, is given by*

$$\mathbb{E}\, N_{\tilde{h}}(T) = \frac{1}{\pi}\int_T \frac{1}{|z|}\frac{\tilde{r}_1^2 + \tilde{r}_2^2 - 2\tilde{r}_{12}^2}{\tilde{r}_3^2\sqrt{(\tilde{r}_1+\tilde{r}_2)^2 - 4\tilde{r}_{12}^2}}\, d\sigma(z) \tag{4.1}$$

where $\sigma(\cdot)$ denotes the Lebesgue measure on the complex plane and

$$\tilde{r}_{12} = (\sum_{j=1}^{n} j|z|^{2j})(\sum_{j=1}^{m} j|z|^{2j}), \qquad \tilde{r}_3 = \sum_{j=0}^{n}|z|^{2j} + \sum_{j=0}^{m}|z|^{2j},$$

$$\tilde{r}_1 = \tilde{r}_3\sum_{j=1}^{n} j^2|z|^{2j} - (\sum_{j=1}^{n} j|z|^{2j})^2, \qquad \tilde{r}_2 = \tilde{r}_3\sum_{j=1}^{m} j^2|z|^{2j} - (\sum_{j=1}^{m} j|z|^{2j})^2.$$

Numerical analysis suggests that the asymptotic of above expectation for $T = \mathbb{C}$ is $\lim_{n\to\infty}\mathbb{E}\, N_h(\mathbb{C})/n = 1$ for fixed m, but a rigorous analytic asymptotic hasn't been found.

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