A NOTE ON RICCI SIGNATURES

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Abstract. We show that only two types of Ricci signatures cannot be realized by any left-invariant metric on 4-dimensional Lie groups.

1. Introduction

On an $n$-dimensional manifold, $n \geq 3$, one can ask whether there is a complete Riemannian metric whose Ricci curvature has a given signature, i.e. a given number of positive, zero and negative eigenvalues. It is well-known that such a prescribed Ricci signature problem does not always have a solution. For example, the classical Bonnet-Myers theorem says that for a complete Riemannian manifold to have positive Ricci curvature, it must be compact and its fundamental group has to be finite. Another standard example is the Cheeger-Gromoll splitting theorem, which gives topological obstructions to the existence of a complete metric of nonnegative Ricci curvature. On the other hand, J. Lohkamp (§4) showed that any manifold of dimension $\geq 3$ admits a complete metric of negative Ricci curvature. However, there seems to be little knowledge about Ricci signatures with mixed signs. Here, we just want to mention D. M. DeTurck’s paper [11], which implies the local solvability of the prescribed Ricci signature problem in the absence of zeroes.

In this note, we restrict our attention to Lie groups with left-invariant metrics and study their Ricci signatures. This kind of problem has been attacked by many authors before. In his beautiful survey article [7], J. Milnor classified Ricci signatures of left-invariant metrics on 3-dimensional Lie groups and found that there are three types of Ricci signatures which cannot be realized by any left-invariant metric on 3-dimensional Lie groups. Based on his work, Milnor raised the problem of seeking possible restrictions on Ricci signatures of left-invariant metrics in higher dimensions. In [2], I. Dotti-Miatello determined Ricci signatures of left-invariant metrics on two-step solvable unimodular Lie groups.

Motivated by the works mentioned above, we consider the realization of Ricci signatures of left-invariant metrics on 4-dimensional Lie groups. First of all, we note that there are, in total, fifteen candidates for Ricci signatures of left-invariant metrics on 4-dimensional Lie groups, as indicated in Table 1.

Remark 1.1. It follows from the results of Milnor ([7], §4) that Ricci signatures $S_1$ to $S_7$ can be realized by left-invariant product metrics on product Lie groups of
a 3-dimensional Lie group with $S^1$. For example, according to Milnor ([7], Corollary 4.5), Ricci signatures $S_1$, $S_4$, and $S_6$ can be realized by left-invariant product metrics on $SU(2) \times S^1$. However, this construction fails to give examples of left-invariant metrics with Ricci signatures $S_{12}$, $S_{14}$ or $S_{15}$, as there are no 3-dimensional Lie groups admitting left-invariant metrics with Ricci signatures $(+, +, 0)$, $(+, +, -)$ or $(+, 0, -)$.

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<thead>
<tr>
<th>Name</th>
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<tbody>
<tr>
<td>$S_1$</td>
<td>$(+, +, +, 0)$</td>
<td>$S_2$</td>
<td>$(0, 0, 0, 0)$</td>
<td>$S_3$</td>
<td>$(0, -, -, -)$</td>
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<tr>
<td>$S_4$</td>
<td>$(+, 0, 0, 0)$</td>
<td>$S_5$</td>
<td>$(0, 0, 0, -)$</td>
<td>$S_6$</td>
<td>$(+, 0, 0, -)$</td>
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<td>$S_7$</td>
<td>$(0, 0, - ,-, -)$</td>
<td>$S_8$</td>
<td>$(+, +, - ,-, -)$</td>
<td>$S_9$</td>
<td>$(+, -, -, -)$</td>
</tr>
<tr>
<td>$S_{10}$</td>
<td>$(-, -, -, -)$</td>
<td>$S_{11}$</td>
<td>$(+, +, +, +)$</td>
<td>$S_{12}$</td>
<td>$(+, +, 0, 0)$</td>
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<tr>
<td>$S_{13}$</td>
<td>$(+, +, +, -)$</td>
<td>$S_{14}$</td>
<td>$(+, +, 0, -)$</td>
<td>$S_{15}$</td>
<td>$(+, 0, 0, -)$</td>
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Remark 1.2. It follows from the results of Dotti-Miatello ([2], Proposition 4.1) that Ricci signatures $S_2$ and $S_5$ to $S_9$ can be realized by left-invariant metrics on 4-dimensional two-step solvable unimodular Lie groups. However, his results do not afford examples of left-invariant metrics with Ricci signatures $S_{10}$ to $S_{15}$.

Remark 1.3. It follows from Jensen’s classification of 4-dimensional Lie groups with left-invariant Einstein metrics ([4], p. 348) that Ricci signature $S_{10}$ can be realized by left-invariant Einstein metrics on 4-dimensional solvable Lie groups.

From the above three remarks, we know that Ricci signatures $S_1$ to $S_{10}$ can be realized by left-invariant metrics on 4-dimensional Lie groups. Now we are left with Ricci signatures $S_{11}$ to $S_{15}$. An easy observation asserts that Ricci signature $S_{11}$ cannot be realized at all; otherwise the underlying Lie group $G$ would have positive Ricci curvature. Hence its Lie algebra $\mathfrak{g}$ must be equal to its commutator ideal $[\mathfrak{g}, \mathfrak{g}]$ ([7], Lemma 2.3); i.e. $\mathfrak{g}$ would be perfect. This contradicts the fact that there is no 4-dimensional perfect Lie algebra ([8], Table 1, p. 988). Besides this observation, we have the following

Theorem 1.4. No 4-dimensional Lie group admits a left-invariant metric with Ricci signature $S_{12}$.

The complete proof of Theorem 1.4 will be given in §2. However, it is instructive to see why Theorem 1.4 is true in some special cases. Let $(G, g)$ be a 4-dimensional Lie group with a left-invariant metric of Ricci signature $S_{12}$. Then $(G, g)$ has nonnegative Ricci curvature. Hence we can apply to $(G, g)$ the Cheeger-Gromoll splitting theorem and assume that $(G, g)$ is a Riemannian product $(\tilde{G} \times \mathbb{R}^k, \tilde{g} \times g_0)$, where $g_0$ is the canonical flat metric of $\mathbb{R}^k$ and $(\tilde{G}, \tilde{g})$ is a $(4 - k)$-dimensional Lie group with a left-invariant metric of nonnegative Ricci curvature. Note that $0 \leq k \leq 2$ as the Ricci curvature of $(G, g)$ has exactly two zero eigenvalues. If $k = 2$, then $(\tilde{G}^2, \tilde{g})$ has positive Ricci curvature; i.e. $(\tilde{G}, \tilde{g})$ has positive scalar curvature, which is impossible since $G$ is solvable ([7], Theorem 3.1). If $k = 1$, then $(\tilde{G}^3, \tilde{g})$ has Ricci signature $(+, +, 0)$. This also gives a contradiction. (See Remark 1.4)

In §3, we classify Ricci signatures of left-invariant metrics on $SU(2) \times S^1$. In particular, we show that there exist left-invariant metrics of Ricci signatures $S_{13}$ to $S_{15}$. 

Table 1. List of Ricci signatures of left-invariant metrics
We hope that our results can be applied to Ricci flow on 4-dimensional Lie groups. For this direction, see [3] and references therein.

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2. Proof of Theorem 1.4

Let $(G, g)$ be a 4-dimensional Lie group with a left-invariant metric, and let $g$ be the associated Lie algebra, consisting of all smooth left-invariant vector fields on $G$. Assume that $(G, g)$ has Ricci signature $S_{12}$. By definition, $(G, g)$ has nonnegative Ricci curvature. Thus $G$ is unimodular ([7], Lemma 6.4). Moreover, $G$ cannot be solvable; otherwise $g$ would be flat ([7], Theorem 3.1) and hence Ricci flat, which contradicts our Ricci signature assumption. Now we look at the classification of 4-dimensional unimodular Lie algebras ([6], pp 306-307). It turns out that only two of these are not solvable; their brackets are given as follows, where $\{X_i\}^4_1$ is a basis of the Lie algebra. Here we adopt the notation in [6].

(1) Class $U3S1$

\[
\begin{align*}
[X_1, X_4] &= 0 & [X_2, X_4] &= 0 & [X_3, X_4] &= 0 \\
[X_2, X_3] &= X_1 & [X_3, X_1] &= X_2 & [X_1, X_2] &= -X_3
\end{align*}
\]

This Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$, whose derived algebra is isomorphic to $\mathfrak{su}(2, \mathbb{R})$.

(2) Class $U3S3$

\[
\begin{align*}
[X_1, X_4] &= 0 & [X_2, X_4] &= 0 & [X_3, X_4] &= 0 \\
[X_2, X_3] &= X_1 & [X_3, X_1] &= X_2 & [X_1, X_2] &= X_3
\end{align*}
\]

This Lie algebra is isomorphic to $\mathfrak{su}(2) \oplus \mathbb{R}$, whose derived algebra is isomorphic to $\mathfrak{su}(2)$.

In both cases, the Lie algebra $g$ is the direct sum $g' \oplus \mathbb{R}X_4$, where $g'$, spanned by $\{X_i\}^4_1$, is the derived algebra of $g$ and $X_4$ is a center element. Moreover, $g'$ is unimodular. So there exists an orthonormal basis $\{e_i\}^4_1$ of $g'$ such that

\[
[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3.
\]

([7], pp 305-307). As pointed out by Milnor, by changing signs if necessary, we may assume that at most one of the structure constants $\{\lambda_i\}$ is negative. Therefore, for $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{su}(2)$, the signs of $\{\lambda_i\}$ are of types $(+, +, -)$ and $(+, +, +)$, respectively. In particular, $\lambda_1 \lambda_2 \lambda_3 \neq 0$. Now we choose $e_4$ to be a unit left-invariant vector field perpendicular to $g'$. Thus $\{e_i\}^4_1$ becomes an orthonormal basis of $g$. Therefore we can write $X_4$ as a linear combination of $\{e_i\}^4_1$, say

\[
X_4 = ae_1 + be_2 + ce_3 + de_4.
\]

Note that $d \neq 0$, otherwise $X_4 \in g'$, which gives a contradiction. For the sake of simplicity, we assume that $d = 1$. Then we can express $e_4$ as a linear combination of $\{e_1, e_2, e_3, X_4\}$, i.e.

\[
e_4 = X_4 - ae_1 - be_2 - ce_3.
\]

This allows us to compute Lie brackets $[e_i, e_4]$, $i = 1, 2, 3$. The results are as follows:

\[
[e_1, e_4] = c\lambda_2 e_2 - b\lambda_3 e_3, \quad [e_2, e_4] = a\lambda_3 e_3 - c\lambda_1 e_1, \quad [e_3, e_4] = b\lambda_1 e_1 - a\lambda_2 e_2.
\]
With these brackets in hand, it is tedious but straightforward to compute the Ricci curvature $R_{ij}$. The results are as follows:

$$R_{11} = \frac{1}{2}((1 + b^2 + c^2)\lambda_1^2 - (1 + b^2)\lambda_2^2 + (1 + c^2)\lambda_3^2 + 2\lambda_2\lambda_3)$$

$$R_{22} = \frac{1}{2}((1 + a^2 + c^2)\lambda_1^2 - (1 + c^2)\lambda_1^2 + (1 + a^2)\lambda_2^2 + 2\lambda_1\lambda_3)$$

$$R_{33} = \frac{1}{2}((1 + a^2 + b^2)\lambda_2^2 - (1 + b^2)\lambda_1^2 - (1 + a^2)\lambda_2^2 + 2\lambda_1\lambda_2)$$

$$R_{44} = -\frac{1}{2}(a^2(\lambda_2 - \lambda_3)^2 + b^2(\lambda_3 - \lambda_1)^2 + c^2(\lambda_1 - \lambda_2)^2)$$

$$R_{12} = \frac{1}{2}ab(\lambda_3^2 - \lambda_1\lambda_2) \quad R_{13} = \frac{1}{2}ac(\lambda_3^2 - \lambda_1\lambda_3) \quad R_{23} = \frac{1}{2}bc(\lambda_1^2 - \lambda_2\lambda_3)$$

$$R_{14} = \frac{1}{2}a(\lambda_3 - \lambda_2)^2 \quad R_{24} = \frac{1}{2}b(\lambda_3 - \lambda_1)^2 \quad R_{34} = \frac{1}{2}c(\lambda_1 - \lambda_2)^2$$

Note that $R_{44} \leq 0$. Hence it follows from our Ricci signature assumption that $R_{44} = 0$, i.e.

$$a^2(\lambda_2 - \lambda_3)^2 + b^2(\lambda_3 - \lambda_1)^2 + c^2(\lambda_1 - \lambda_2)^2 = 0.$$

There are three cases:

1. $\lambda_1 = \lambda_2 = \lambda_3$
2. $\lambda_1 = \lambda_2$, $\lambda_1 \neq \lambda_3$
3. $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3$

In case (1), $\{e_1\}$ diagonalizes the Ricci curvature, and we have

$$R_{11} = R_{22} = R_{33} = \frac{1}{2}\lambda_1^2 > 0, \quad R_{44} = 0.$$ 

In particular, the Ricci signature is $S_1$.

In case (2), it follows from (2.1) that $a = b = 0$; hence $\{e_1\}$ also diagonalizes the Ricci curvature. Moreover, we have

$$R_{11} = R_{22} = \lambda_1\lambda_3 - \frac{1}{2}\lambda_3^2, \quad R_{33} = \frac{1}{2}\lambda_3^2 > 0, \quad R_{44} = 0.$$ 

Depending on the choice of $\lambda_i$'s, the Ricci signature can be either $S_1$ or $S_4$ or $S_6$. However, none is of type $S_{12}$.

In case (3), it follows from (2.1) that $a = b = c = 0$; hence $\{e_1\}$ diagonalizes the Ricci curvature too. Moreover, we have

$$R_{11} = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3) \quad R_{22} = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 - \lambda_1)$$

$$R_{33} = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3)(\lambda_2 + \lambda_3 - \lambda_1) \quad R_{44} = 0$$

Depending on the choice of $\lambda_i$'s, the Ricci signature can be either $S_1$ or $S_4$ or $S_5$ or $S_6$. Again, none is of type $S_{12}$.

Therefore, we can conclude that there is no left-invariant metric of Ricci signature $S_{12}$ on Lie algebras of Classes U3S1 and U3S3. This completes the proof of Theorem 1.4.

3. Ricci signatures of left-invariant metrics on $SU(2) \times S^1$

In this section, we exhibit numerical examples of left-invariant metrics of different Ricci signatures on $SU(2) \times S^1$ and show that these examples realize all possible Ricci signatures of left-invariant metrics on $SU(2) \times S^1$. Of particular interest is
that among them there are left-invariant metrics of Ricci signatures $S_{13}$ to $S_{15}$. For this purpose, we introduce

**Definition 3.1.** We denote by $g^4(a, b, c, \lambda_1, \lambda_2, \lambda_3)$ the 4-dimensional metric Lie algebra admitting an orthonormal basis $\{e_i\}$ with multiplication table

\[
\begin{align*}
[e_1, e_2] &= \lambda_3 e_3 \\
[e_1, e_4] &= c \lambda_2 e_2 - b \lambda_3 e_3 \\
[e_2, e_3] &= \lambda_1 e_1 \\
[e_2, e_4] &= a \lambda_3 e_3 - c \lambda_1 e_1 \\
[e_3, e_1] &= \lambda_2 e_2 \\
[e_3, e_4] &= b \lambda_1 e_1 - a \lambda_2 e_2
\end{align*}
\]

where $a, b, c, \lambda_1, \lambda_2, \lambda_3$ are parameters.

Note that the formulas for the Ricci curvature of $g^4(a, b, c, \lambda_1, \lambda_2, \lambda_3)$ have been given in §2.

**Theorem 3.2.** Depending on the choice of left-invariant metrics, the Ricci signature for $SU(2) \times S^1$ can be either $S_1$ or $S_4$ or $S_6$ or $S_8$ or $S_9$ or $S_{13}$ or $S_{14}$ or $S_{15}$.

**Proof.** According to Theorem 1.4 and the preceding argument in the Introduction, we may a priori rule out Ricci signatures $S_{11}$ and $S_{12}$ on $SU(2) \times S^1$. Since there is no left-invariant metric of nonpositive Ricci curvature on compact nonabelian Lie groups, we may also rule out Ricci signatures $S_2, S_3, S_5, S_7$ and $S_{10}$ on $SU(2) \times S^1$. On the other hand, we have already shown in Remark 1.1 that Ricci signatures $S_1, S_4$ and $S_6$ can be realized on $SU(2) \times S^1$. Now it remains to check the following statements:

1. $g^4(0, 1, 1, 1, 1, \frac{7}{5})$ has Ricci signature $S_8$.
2. $g^4(0, \sqrt{\frac{2}{5}}, 2\sqrt{5}, 1, 1, \frac{8}{5})$ has Ricci signature $S_9$.
3. $g^4(0, 1, 1, 1, 1, \frac{4}{3})$ has Ricci signature $S_{13}$.
4. $g^4(0, 1, 1, 1, 1, \frac{1+\sqrt{25}}{2})$ has Ricci signature $S_{14}$.
5. $g^4(0, \sqrt{\frac{2}{5}}, 2\sqrt{5}, 1, 1, \frac{2}{3})$ has Ricci signature $S_{15}$.

In the above five explicit examples, the signs of the constants $\{\lambda_i\}$ are (+, +, +). Hence it follows from the proof of Theorem 1.4 that we can choose $SU(2) \times S^1$ as the underlying Lie group. This completes the proof.  

**References**


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