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A NOTE ON RICCI SIGNATURES

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ABSTRACT. We show that only two types of Ricci signatures cannot be realized by any left-invariant metric on 4-dimensional Lie groups.

1. INTRODUCTION

On an n-dimensional manifold, $n \geq 3$, one can ask whether there is a complete Riemannian metric whose Ricci curvature has a given signature, i.e. a given number of positive, zero and negative eigenvalues. It is well-known that such a prescribed Ricci signature problem does not always have a solution. For example, the classical Bonnet-Myers theorem says that for a complete Riemannian manifold to have positive Ricci curvature, it must be compact and its fundamental group has to be finite. Another standard example is the Cheeger-Gromoll splitting theorem, which gives topological obstructions to the existence of a complete metric of nonnegative Ricci curvature. On the other hand, J. Lohkamp ([5]) showed that any manifold of dimension ≥ 3 admits a complete metric of negative Ricci curvature. However, there seems to be little knowledge about Ricci signatures with mixed signs. Here, we just want to mention D. M. DeTurck's paper [1], which implies the local solvability of the prescribed Ricci signature problem in the absence of zeroes.

In this note, we restrict our attention to Lie groups with left-invariant metrics and study their Ricci signatures. This kind of problem has been attacked by many authors before. In his beautiful survey article [7], J. Milnor classified Ricci signatures of left-invariant metrics on 3-dimensional Lie groups and found that there are three types of Ricci signatures which cannot be realized by any left-invariant metric on 3-dimensional Lie groups. Based on his work, Milnor raised the problem of seeking possible restrictions on Ricci signatures of left-invariant metrics in higher dimensions. In [2], I. Dotti-Miatello determined Ricci signatures of left-invariant metrics on two-step solvable unimodular Lie groups.

Motivated by the works mentioned above, we consider the realization of Ricci signatures of left-invariant metrics on 4-dimensional Lie groups. First of all, we note that there are, in total, fifteen candidates for Ricci signatures of left-invariant metrics on 4-dimensional Lie groups, as indicated in Table 1.

Remark 1.1. It follows from the results of Milnor ([7], §4) that Ricci signatures S_1 to S_7 can be realized by left-invariant product metrics on product Lie groups of

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a 3-dimensional Lie group with S^1. For example, according to Milnor ([7], Corollary 4.5), Ricci signatures S_1, S_4 and S_6 can be realized by left-invariant product metrics on $SU(2) \times S^1$. However, this construction fails to give examples of left-invariant metrics with Ricci signatures S_{12}, S_{14} or S_{15}, as there are no 3-dimensional Lie groups admitting left-invariant metrics with Ricci signatures $(+,+,0)$, $(+,+,-)$ or $(+,0,-)$.

Table 1. List of Ricci signatures of left-invariant metrics

Name	Description	Name	Description	Name	Description
S_1	$(+,+,+,0)$	S_2	$(0,0,0,0)$	S_3	$(0,-,-,-)$
S_4	$(+,0,0,0)$	S_5	$(0,0,0,-)$	S_6	$(+,0,-,-)$
S_7	$(0,0,-,-)$	S_8	$(+,+,-,-)$	S_9	$(+,-,-,-)$
S_{10}	$(-,-,-,-)$	S_{11}	$(+,+,+,+)$	S_{12}	$(+,+,0,0)$
S_{13}	$(+,+,+,-)$	S_{14}	$(+,+,0,-)$	S_{15}	$(+,0,0,-)$

Remark 1.2. It follows from the results of Dotti-Miatello ([2], Proposition 4.1) that Ricci signatures S_2 and S_5 to S_9 can be realized by left-invariant metrics on 4-dimensional two-step solvable unimodular Lie groups. However, his results do not afford examples of left-invariant metrics with Ricci signatures S_{10} to S_{15}.

Remark 1.3. It follows from Jensen's classification of 4-dimensional Lie groups with left-invariant Einstein metrics ([4], p. 348) that Ricci signature S_{10} can be realized by left-invariant Einstein metrics on 4-dimensional solvable Lie groups.

From the above three remarks, we know that Ricci signatures S_1 to S_{10} can be realized by left-invariant metrics on 4-dimensional Lie groups. Now we are left with Ricci signatures S_{11} to S_{15}. An easy observation asserts that Ricci signature S_{11} cannot be realized at all; otherwise the underlying Lie group G would have positive Ricci curvature. Hence its Lie algebra $\mathfrak{g}$ must be equal to its commutator ideal $[\mathfrak{g},\mathfrak{g}]$ ([7], Lemma 2.3); i.e. $\mathfrak{g}$ would be perfect. This contradicts the fact that there is no 4-dimensional perfect Lie algebra ([8], Table 1, p. 988). Besides this observation, we have the following

Theorem 1.4. *No 4-dimensional Lie group admits a left-invariant metric with Ricci signature S_{12}.*

The complete proof of Theorem 1.4 will be given in §2. However, it is instructive to see why Theorem 1.4 is true in some special cases. Let (G,g) be a 4-dimensional Lie group with a left-invariant metric of Ricci signature S_{12}. Then (G,g) has nonnegative Ricci curvature. Hence we can apply to (G,g) the Cheeger-Gromoll splitting theorem and assume that (G,g) is a Riemannian product $(\bar{G}\times\mathbb{R}^k, \bar{g}\times g_0)$, where g_0 is the canonical flat metric of $\mathbb{R}^k$ and $(\bar{G},\bar{g})$ is a $(4-k)$-dimensional Lie group with a left-invariant metric of nonnegative Ricci curvature. Note that $0 \le k \le 2$ as the Ricci curvature of (G,g) has exactly two zero eigenvalues. If $k=2$, then $(\bar{G}^2,\bar{g})$ has positive Ricci curvature; i.e. $(\bar{G},\bar{g})$ has positive scalar curvature, which is impossible since $\bar{G}$ is solvable ([7], Theorem 3.1). If $k=1$, then $(\bar{G}^3,\bar{g})$ has Ricci signature $(+,+,0)$. This also gives a contradiction. (See Remark 1.1.)

In §3, we classify Ricci signatures of left-invariant metrics on $SU(2)\times S^1$. In particular, we show that there exist left-invariant metrics of Ricci signatures S_{13} to S_{15}.

We hope that our results can be applied to Ricci flow on 4-dimensional Lie groups. For this direction, see [3] and references therein.

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2. Proof of Theorem 1.4

Let (G, g) be a 4-dimensional Lie group with a left-invariant metric, and let $\mathfrak{g}$ be the associated Lie algebra, consisting of all smooth left-invariant vector fields on G. Assume that (G, g) has Ricci signature S_{12}. By definition, (G, g) has nonnegative Ricci curvature. Thus G is unimodular ([7], Lemma 6.4). Moreover, G cannot be solvable; otherwise g would be flat ([7], Theorem 3.1) and hence Ricci flat, which contradicts our Ricci signature assumption. Now we look at the classification of 4-dimensional unimodular Lie algebras ([6], pp 306-307). It turns out that only two of these are not solvable; their brackets are given as follows, where $\{X_i\}_1^4$ is a basis of the Lie algebra. Here we adopt the notation in [6].

(1) Class U3S1

$$\begin{array}{lll} [X_1, X_4] = 0 & [X_2, X_4] = 0 & [X_3, X_4] = 0 \\ [X_2, X_3] = X_1 & [X_3, X_1] = X_2 & [X_1, X_2] = -X_3 \end{array}$$

This Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$, whose derived algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

(2) Class U3S3

$$\begin{array}{lll} [X_1, X_4] = 0 & [X_2, X_4] = 0 & [X_3, X_4] = 0 \\ [X_2, X_3] = X_1 & [X_3, X_1] = X_2 & [X_1, X_2] = X_3 \end{array}$$

This Lie algebra is isomorphic to $\mathfrak{su}(2) \oplus \mathbb{R}$, whose derived algebra is isomorphic to $\mathfrak{su}(2)$.

In both cases, the Lie algebra $\mathfrak{g}$ is the direct sum $\mathfrak{g}' \oplus \mathbb{R}X_4$, where $\mathfrak{g}'$, spanned by $\{X_i\}_1^3$, is the derived algebra of $\mathfrak{g}$ and X_4 is a center element. Moreover, $\mathfrak{g}'$ is unimodular. So there exists an orthonormal basis $\{e_i\}_1^3$ of $\mathfrak{g}'$ such that

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3.$$

([7], pp 305-307). As pointed out by Milnor, by changing signs if necessary, we may assume that at most one of the structure constants $\{\lambda_i\}$ is negative. Therefore, for $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{su}(2)$, the signs of $\{\lambda_i\}$ are of types $(+, +, -)$ and $(+, +, +)$, respectively. In particular, $\lambda_1\lambda_2\lambda_3 \neq 0$. Now we choose e_4 to be a unit left-invariant vector field perpendicular to $\mathfrak{g}'$. Thus $\{e_i\}_1^4$ becomes an orthonormal basis of $\mathfrak{g}$. Therefore we can write X_4 as a linear combination of $\{e_i\}_1^4$, say

$$X_4 = ae_1 + be_2 + ce_3 + de_4.$$

Note that $d \neq 0$, otherwise $X_4 \in \mathfrak{g}'$, which gives a contradiction. For the sake of simplicity, we assume that $d = 1$. Then we can express e_4 as a linear combination of $\{e_1, e_2, e_3, X_4\}$, i.e.

$$e_4 = X_4 - ae_1 - be_2 - ce_3.$$

This allows us to compute Lie brackets $[e_i, e_4]$, $i = 1, 2, 3$. The results are as follows:

$$[e_1, e_4] = c\lambda_2 e_2 - b\lambda_3 e_3, \quad [e_2, e_4] = a\lambda_3 e_3 - c\lambda_1 e_1, \quad [e_3, e_4] = b\lambda_1 e_1 - a\lambda_2 e_2.$$

With these brackets in hand, it is tedious but straightforward to compute the Ricci curvature R_{ij}. The results are as follows:

$$R_{11} = \frac{1}{2}((1+b^2+c^2)\lambda_1^2 - (1+b^2)\lambda_3^2 - (1+c^2)\lambda_2^2 + 2\lambda_2\lambda_3)$$
$$R_{22} = \frac{1}{2}((1+a^2+c^2)\lambda_2^2 - (1+c^2)\lambda_1^2 - (1+a^2)\lambda_3^2 + 2\lambda_1\lambda_3)$$
$$R_{33} = \frac{1}{2}((1+a^2+b^2)\lambda_3^2 - (1+b^2)\lambda_1^2 - (1+a^2)\lambda_2^2 + 2\lambda_1\lambda_2)$$
$$R_{44} = -\frac{1}{2}(a^2(\lambda_2-\lambda_3)^2 + b^2(\lambda_3-\lambda_1)^2 + c^2(\lambda_1-\lambda_2)^2)$$
$$R_{12} = \frac{1}{2}ab(\lambda_3^2 - \lambda_1\lambda_2) \quad R_{13} = \frac{1}{2}ac(\lambda_2^2 - \lambda_1\lambda_3) \quad R_{23} = \frac{1}{2}bc(\lambda_1^2 - \lambda_2\lambda_3)$$
$$R_{14} = \frac{1}{2}a(\lambda_3-\lambda_2)^2 \quad R_{24} = \frac{1}{2}b(\lambda_3-\lambda_1)^2 \quad R_{34} = \frac{1}{2}c(\lambda_1-\lambda_2)^2$$

Note that $R_{44} \le 0$. Hence it follows from our Ricci signature assumption that $R_{44} = 0$, i.e.

$$a^2(\lambda_2-\lambda_3)^2 + b^2(\lambda_3-\lambda_1)^2 + c^2(\lambda_1-\lambda_2)^2 = 0. \tag{2.1}$$

There are three cases:

(1) $\lambda_1 = \lambda_2 = \lambda_3$
(2) $\lambda_1 = \lambda_2$, $\lambda_1 \ne \lambda_3$
(3) $\lambda_1 \ne \lambda_2$, $\lambda_1 \ne \lambda_3$, $\lambda_2 \ne \lambda_3$

In case (1), $\{e_i\}$ diagonalizes the Ricci curvature, and we have

$$R_{11} = R_{22} = R_{33} = \frac{1}{2}\lambda_1^2 > 0, \qquad R_{44} = 0.$$

In particular, the Ricci signature is S_1.

In case (2), it follows from (2.1) that $a = b = 0$; hence $\{e_i\}$ also diagonalizes the Ricci curvature. Moreover, we have

$$R_{11} = R_{22} = \lambda_1\lambda_3 - \tfrac{1}{2}\lambda_3^2, \quad R_{33} = \tfrac{1}{2}\lambda_3^2 > 0, \quad R_{44} = 0.$$

Depending on the choice of λ_i's, the Ricci signature can be either S_1 or S_4 or S_6. However, none is of type S_{12}.

In case (3), it follows from (2.1) that $a = b = c = 0$; hence $\{e_i\}$ diagonalizes the Ricci curvature too. Moreover, we have

$$R_{11} = \tfrac{1}{2}(\lambda_1+\lambda_2-\lambda_3)(\lambda_1-\lambda_2+\lambda_3) \quad R_{22} = \tfrac{1}{2}(\lambda_1+\lambda_2-\lambda_3)(\lambda_2+\lambda_3-\lambda_1)$$
$$R_{33} = \tfrac{1}{2}(\lambda_1-\lambda_2+\lambda_3)(\lambda_2+\lambda_3-\lambda_1) \quad R_{44} = 0$$

Depending on the choice of λ_i's, the Ricci signature can be either S_1 or S_4 or S_5 or S_6. Again, none is of type S_{12}.

Therefore, we can conclude that there is no left-invariant metric of Ricci signature S_{12} on Lie algebras of Classes U3S1 and U3S3. This completes the proof of Theorem 1.4.

3. Ricci signatures of left-invariant metrics on $SU(2) \times S^1$

In this section, we exhibit numerical examples of left-invariant metrics of different Ricci signatures on $SU(2) \times S^1$ and show that these examples realize all possible Ricci signatures of left-invariant metrics on $SU(2) \times S^1$. Of particular interest is that among them there are left-invariant metrics of Ricci signatures S_{13} to S_{15}. For this purpose, we introduce

Definition 3.1. We denote by $\mathfrak{g}^4(a,b,c,\lambda_1,\lambda_2,\lambda_3)$ the 4-dimensional metric Lie algebra admitting an orthonormal basis $\{e_i\}$ with multiplication table

$$\begin{array}{lll} [e_1,e_2]=\lambda_3 e_3 & [e_2,e_3]=\lambda_1 e_1 & [e_3,e_1]=\lambda_2 e_2 \\ [e_1,e_4]=c\lambda_2 e_2 - b\lambda_3 e_3 & [e_2,e_4]=a\lambda_3 e_3 - c\lambda_1 e_1 & [e_3,e_4]=b\lambda_1 e_1 - a\lambda_2 e_2 \end{array}$$

where $a,b,c,\lambda_1,\lambda_2,\lambda_3$ are parameters.

Note that the formulas for the Ricci curvature of $\mathfrak{g}^4(a,b,c,\lambda_1,\lambda_2,\lambda_3)$ have been given in §2.

Theorem 3.2. *Depending on the choice of left-invariant metrics, the Ricci signature for $SU(2)\times S^1$ can be either S_1 or S_4 or S_6 or S_8 or S_9 or S_{13} or S_{14} or S_{15}.*

Proof. According to Theorem 1.4 and the preceding argument in the Introduction, we may a priori rule out Ricci signatures S_{11} and S_{12} on $SU(2)\times S^1$. Since there is no left-invariant metric of nonpositive Ricci curvature on compact nonabelian Lie groups, we may also rule out Ricci signatures S_2, S_3, S_5, S_7 and S_{10} on $SU(2)\times S^1$. On the other hand, we have already shown in Remark 1.1 that Ricci signatures S_1, S_4 and S_6 can be realized on $SU(2)\times S^1$. Now it remains to check the following statements:

(1) $\mathfrak{g}^4(0,1,1,1,1,\frac{7}{5})$ has Ricci signature S_8.
(2) $\mathfrak{g}^4(0,\sqrt{\frac{3}{5}},2\sqrt{5},1,1,\frac{8}{5})$ has Ricci signature S_9.
(3) $\mathfrak{g}^4(0,1,1,1,1,\frac{4}{3})$ has Ricci signature S_{13}.
(4) $\mathfrak{g}^4(0,1,1,1,1,\frac{1+\sqrt{3}}{2})$ has Ricci signature S_{14}.
(5) $\mathfrak{g}^4(0,\sqrt{\frac{3}{5}},2\sqrt{5},1,1,\frac{3}{2})$ has Ricci signature S_{15}.

In the above five explicit examples, the signs of the constants $\{\lambda_i\}$ are $(+,+,+)$. Hence it follows from the proof of Theorem 1.4 that we can choose $SU(2)\times S^1$ as the underlying Lie group. This completes the proof. $\square$

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