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BOUNDING MATRIX COEFFICIENTS FOR SMOOTH VECTORS OF TEMPERED REPRESENTATIONS

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ABSTRACT. Let G be a Lie group. Let (π, V) be a unitary representation of G which is weakly contained in the regular representation. For smooth vectors u, v in V, we give an upper bound for the matrix coefficient $\langle \pi(g)u, v\rangle$, in terms of Harish-Chandra's Ξ-function.

1. Introduction

Let G be a locally compact Hausdorff topological group, with a compact subgroup K and a closed subgroup P such that $G = KP$.

Denote by Δ_G the modular function of G, which is defined by

$$\int_G f(gx)\, dg = \Delta_G^{-1}(x) \int_G f(g)\, dg, \quad \text{for all } f \in \mathrm{L}^1(G; dg),\, x \in G,$$

where "dg" is a left invariant Haar measure on G. Denote by Δ_P the modular function of P, and write

$$\delta(p) = \frac{\Delta_G(p)}{\Delta_P(p)}, \quad p \in P.$$

Extend δ to a left K-invariant function on G, which is still denoted by δ, by the formula

$$\delta(kp) = \delta(p), \quad k \in K,\, p \in P.$$

Under this general setting, Harish-Chandra's basic spherical function Ξ is still defined by

$$\Xi(g) = \Xi_{K,P}(g) = \int_K \delta^{-\frac{1}{2}}(gk)\, dk, \quad g \in G,$$

where "dk" is the normalized Haar measure on K.

Let (π, V) be a unitary representation of G. For any $v \in V$, write

$$||v|| = \sqrt{\langle v, v\rangle}$$

and

$$d_v = \text{the dimension of the space spanned by } \pi(K)v.$$

The following is a formal generalization of a fundamental result of Cowling, Haagerup and Howe [1, Theorem 2]. Although in [1], the theorem is only stated for

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G semisimple algebraic, and $G = KP$ an Iwasawa decomposition, the same proof works in our general setting. So we omit the proof.

Theorem 1.1. *If π is weakly contained in the regular representation, and u, v are K-finite vectors in V, then*

$$|\langle \pi(g)u, v\rangle| \leq (d_u d_v)^{\frac{1}{2}}\, ||u||\, ||v||\, \Xi(g) \tag{1.1}$$

for all $g \in G$.

For the notion of weak containment, see [1, Page 98]. The left regular representation is canonically isomorphic to the right regular one. So we do not distinguish them and call any of them the regular representation. A unitary representation which is weakly contained in the regular representation is also called tempered. Notice that Ξ is the diagonal matrix coefficient for a K-fixed vector in $\mathrm{Ind}_P^G 1_P$. Here we use normalized induction, and 1_P denotes the trivial representation of P. When P is amenable, i.e., when 1_P is tempered, the unitary representation $\mathrm{Ind}_P^G 1_P$ is also tempered. Recall that P is amenable in the most interesting case, i.e., when G is reductive algebraic over a local field and P is a minimal parabolic subgroup of it.

In applications, in particular for branching problems, one usually needs the bound (1.1) to be valid for smooth vectors. When G is totally disconnected, K is usually chosen to be open. Then smooth vectors are the same as K-finite ones. But when G is a Lie group, in most cases, there are more smooth vectors than K-finite ones.

From now on assume that G is a Lie group, with complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let "U" stand for the universal enveloping algebra. Still denote by π the induced action of $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$ on the smooth vectors V^∞ of V. Recall that V^∞ is a Fréchet space under the seminorms

$$|v|_X = ||\pi(X)v||, \quad v \in V^\infty,$$

where X runs through all vectors in $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$.

Denote by $\mathfrak{k}$ the Lie algebra of K, with complexification $\mathfrak{k}_{\mathbb{C}}$, and denote by $\mathrm{U}(\mathfrak{k}_{\mathbb{C}})^K$ the centralizer of K in $\mathrm{U}(\mathfrak{k}_{\mathbb{C}})$.

Theorem 1.2. *There is an element $X \in \mathrm{U}(\mathfrak{k}_{\mathbb{C}})^K$, which depends on K only, such that if π is weakly contained in the regular representation, then*

$$|\langle \pi(g)u, v\rangle| \leq |u|_X\, |v|_X\, \Xi(g) \tag{1.2}$$

for all $u, v \in V^\infty$ and $g \in G$, and

$$|\langle \pi(g)u, v\rangle| \leq d_v^{\frac{1}{2}}\, |u|_X\, ||v||\, \Xi(g) \tag{1.3}$$

for all $v \in V$ that are K-finite, $u \in V^\infty$, and $g \in G$.

In the case that G is real reductive, K is a maximal compact subgroup, and P is a minimal parabolic subgroup, Wallach has an estimate that ([3, Proposition 5.1.2])

$$|\lambda(\pi_K(g)u)| \leq \sigma(u)\,(1 + \log||g||)^d\, \Xi(g), \quad u \in V_K^\infty,\ g \in G, \tag{1.4}$$

where (π_K, V_K^∞) is the Casselman-Wallach smooth globalization of a tempered Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module, $||g||$ is a certain "norm function" on G, d is a nonnegative number which depends on V_K^∞, λ is a K-finite continuous linear functional on V_K^∞, and σ is a continuous seminorm on V_K^∞ which depends on λ. Recall that a Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module is tempered if and only if all of its irreducible subquotients globalize to tempered unitary representations. Our estimate (1.3) improves (1.4) in the unitary case.

The following is a direct consequence of (1.2), which is a base for applying matrix coefficient integrals to branching problems.

Corollary 1.3. *Let H be a closed unimodular subgroup of G, with a fixed Haar measure dh. If π is weakly contained in the regular representation, and $\Xi|_H \in \mathrm{L}^1(H)$, then*

$$\begin{array}{rcl} V^\infty \times V^\infty & \to & \mathbb{C}, \\ u, v & \mapsto & \int_H \langle \pi(h)u, v\rangle \, dh \end{array}$$

is a well-defined continuous $(H \times H)$-invariant Hermitian form on V^∞.

The condition that $\Xi|_H \in \mathrm{L}^1(H)$ is usually easy to check. For example, this is done for the pair $(G, H) = (\mathrm{SO}(n+1) \times \mathrm{SO}(n), \mathrm{SO}(n))$ of algebraic groups over a local field [2, Proposition 1.1].

2. Proof of Theorem 1.2

Fix a K-invariant positive definite quadratic form Q on $\mathfrak{k}$. Let X_1, X_2,…, X_n be a basis for $\mathfrak{k}$ orthonormal with respect to Q. Put

$$\Omega = \Omega_Q = 1 - \sum_{i=1}^n X_i^2.$$

Then it is in $\mathrm{U}(\mathfrak{k}_{\mathbb{C}})^K$ and is independent of the orthonormal basis.

Denote by $\widehat{K}$ the set of equivalence classes of irreducible finite dimensional unitary representations of K. For every $\tau \in \widehat{K}$, fix an irreducible unitary representation (π_τ, V_τ) of K, of class τ. By Schur's Lemma, $\pi_\tau(\Omega)$ acts on V_τ via a scalar $c(\tau)$. It is known and also easy to see that $c(\tau)$ is real and ≥ 1. Set $d(\tau) = \dim V_\tau$.

Lemma 2.1. *If m is large enough, then*

$$\sum_{\tau \in \widehat{K}} d(\tau)\, c(\tau)^{-m} < +\infty. \tag{2.1}$$

Proof. Lemma 4.4.2.3 of [4] says that

$$\sum_{\tau \in \widehat{K}} d(\tau)^2\, c(\tau)^{-m} < +\infty \tag{2.2}$$

if m is large enough. This implies the lemma. □

Now let m_0 be the smallest nonnegative integer such that (2.1) holds. Set

$$X = (\sum_{\tau \in \widehat{K}} d(\tau)\, c(\tau)^{-m_0})\Omega^{m_0}, \tag{2.3}$$

which is an element of $\mathrm{U}(\mathfrak{k}_{\mathbb{C}})^K$.

Recall that (π, V) is a unitary representation of G, and V^∞ is the space of smooth vectors. We have a Hilbert space decomposition

$$V = \widehat{\bigoplus}_{\tau \in \widehat{K}} V(\tau),$$

where $V(\tau)$ is the τ-isotypic component of V, which is automatically closed in V. Denote by $P(\tau)$ the orthogonal projection of V onto $V(\tau)$. It maps V^∞ into itself.

By [4, Theorem 4.4.2.1], which is due to Harish-Chandra, we have

Lemma 2.2. *If $v \in V^\infty$, then the Fourier series*

$$\sum_{\tau \in \widehat{K}} P(\tau) v$$

converges absolutely to v in V^∞.

The following is a variation of Lemma 4.4.2.2 of [4].

Lemma 2.3. *For any integer $m \geq 0$, and $v \in V^\infty$, one has that*

$$||P(\tau)v|| \leq c(\tau)^{-m} \, ||\pi(\Omega^m)v||.$$

Proof. Let $v \in V^\infty$. Since $P(\tau)v \in V(\tau) \cap V^\infty$, we have that

$$\pi(\Omega^m)(P(\tau)v) = c(\tau)^m P(\tau)v. \tag{2.4}$$

Notice that $P(\tau)$ and $\pi(\Omega^m)$ are two commuting elements in $\mathrm{End}(V^\infty)$. Therefore

$$\begin{aligned} & ||P(\tau)v|| \\ = & ||c(\tau)^{-m} \, \pi(\Omega^m) \, (P(\tau)v)|| \qquad \text{by (2.4)} \\ = & c(\tau)^{-m} \, ||P(\tau) \, (\pi(\Omega^m)v)|| \\ \leq & c(\tau)^{-m} \, ||\pi(\Omega^m)v|| \qquad \text{since } P(\tau) \text{ is an orthogonal projection.} \end{aligned}$$ □

Lemma 2.4. *Let X be as in (2.3). If $v \in V^\infty$, then*

$$\sum_{\tau \in \widehat{K}} d(\tau) \, ||P(\tau)v)|| \leq |v|_X.$$

Proof. We have

$$\begin{aligned} & \sum_{\tau \in \widehat{K}} d(\tau) \, ||P(\tau)v)|| \\ \leq & \sum_{\tau \in \widehat{K}} d(\tau) \, c(\tau)^{-m_0} \, ||\pi(\Omega^{m_0})v|| \qquad \text{by Lemma 2.3} \\ = & ||\pi(X)v|| \qquad \text{by (2.3)} \\ = & |v|_X. \end{aligned}$$ □

Now we are ready to prove Theorem 1.2. Assume that π is tempered. Let $g \in G$, $u \in V^\infty$ and $v \in V$. Then

$$\begin{aligned} & |\langle \pi(g)u, v\rangle| \\ = & |\langle \pi(g)(\sum_{\tau \in \widehat{K}} P(\tau)u), v\rangle| \qquad \text{by Lemma 2.2} \\ \leq & \sum_{\tau \in \widehat{K}} |\langle \pi(g)(P(\tau)u), v\rangle|. \end{aligned} \tag{2.5}$$

As a part of [1, Theorem 2], it is known and also easy to see that

$$d_{P(\tau)u} \leq d(\tau)^2. \tag{2.6}$$

Let X be as in (2.3). If v is K-finite, then

$$\begin{aligned}&|\langle \pi(g)u, v\rangle| \\ &\le \sum_{\tau\in\widehat{K}} d(\tau)\, d_v^{\frac{1}{2}}\, ||P(\tau)u)||\; ||v||\; \Xi(g) \qquad \text{by (2.5), (2.6) and Theorem 1.1} \\ &\le d_v^{\frac{1}{2}}\, |u|_X\, ||v||\; \Xi(g) \qquad \text{by Lemma 2.4.}\end{aligned}$$

This proves (1.3) of the theorem.

Now assume that $v \in V^{\infty}$. Then

$$\begin{aligned}&|\langle \pi(g)u, v\rangle| \\ &\le \sum_{\tau\in\widehat{K}} |\langle \pi(g)(P(\tau)u), \sum_{\tau'\in\widehat{K}} P(\tau')v\rangle| \qquad \text{by (2.5) and Lemma 2.2} \\ &\le \sum_{\tau\in\widehat{K}}\sum_{\tau'\in\widehat{K}} |\langle \pi(g)(P(\tau)u), P(\tau')v\rangle| \\ &\le \sum_{\tau\in\widehat{K}}\sum_{\tau'\in\widehat{K}} d(\tau)d(\tau')\, ||P(\tau)u||\; ||P(\tau')v||\, \Xi(g) \qquad \text{by (2.6) and Theorem 1.1} \\ &\le |u|_X\, |v|_X\; \Xi(g) \qquad \text{by Lemma 2.4.}\end{aligned}$$

This proves (1.2) of the theorem.

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