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FINITE $\mathbb{Z}/2\mathbb{Z}$-CW COMPLEXES WHICH ARE NOT HOMOTOPICALLY STRATIFIED BY ORBIT TYPE

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ABSTRACT. For $k \geq 2$, we construct finite $\mathbb{Z}/2\mathbb{Z}$-CW complexes with one $\mathbb{Z}/2\mathbb{Z}$-cell in dimensions 0, 1 and $k+1$. Using a theorem of Bruce Hughes, we show that these complexes are not homotopically stratified by orbit type in the sense of Quinn.

Homotopically stratified sets were introduced by Quinn in [5] as a means of studying purely topological stratified phenomena. Quinn showed, under suitable conditions, that the orbit space of a finite group acting on a manifold, with the orbit type partition, is a homotopically stratified set ([5, Corollary 1.6]). In this paper we construct examples of $\mathbb{Z}/2\mathbb{Z}$-CW complexes having few $\mathbb{Z}/2\mathbb{Z}$-cells whose orbit spaces, with the orbit type partition, are not homotopically stratified.

A closed subspace Y of a space X is *forward tame* in X if there exists a neighborhood U of Y in X and a homotopy $H : U \times I \to X$ such that H_0 is inclusion $U \hookrightarrow X$, $H_t|_Y$ is inclusion $Y \hookrightarrow X$ for every $t \in I$, $H_1(U) = Y$ and $H((U - Y) \times [0, 1)) \subseteq X - Y$. The *homotopy link* of Y in X is $\text{holink}(X, Y) = \{\omega \in X^I \mid \omega(t) \in Y \text{ if and only if } t = 0\}$. A *stratification* of a space X consists of an indexed locally finite partition $\{X_i \mid i \in \mathcal{I}\}$ of X by locally closed subspaces. We refer to X together with its stratification as a *stratified space*. Given a space X with an action of a group G, the *orbit type* corresponding to a subgroup $H \subset G$ is the set of all points in X whose isotropy group is conjugate to H. The *orbit type partition* of X consists of the connected components of the orbit types of X. The orbits of these components give a partition of the orbit space $G\backslash X$.

A stratified space X is said to satisfy the *frontier condition* if for every $i, j \in \mathcal{I}$, $X_i \cap \text{closure}(X_j) \neq \emptyset$ implies that $X_i \subseteq \text{closure}(X_j)$. This induces a relation $<$ on $\mathcal{I}$, defined by $i < j$ if and only if $i \neq j$ and $X_i \subset \text{closure}(X_j)$. The orbit type stratification of a finite G-CW complex need not satisfy the frontier condition. For example, let $X = S^1 \vee S^1 \vee S^1$ be the wedge of three circles along a basepoint $*$. Express $X - *$ as the union of three disjoint 1-cells e_1^1, e_2^1, e_3^1 whose closures are the corresponding S^1 factors and give X the $\mathbb{Z}/2\mathbb{Z}$-CW complex structure: one $\mathbb{Z}/2\mathbb{Z}$-0-cell, the basepoint $*$ (isotropy $\mathbb{Z}/2\mathbb{Z}$), and two $\mathbb{Z}/2\mathbb{Z}$-1-cells: $e_1^1 \cup e_2^1$ on which $\mathbb{Z}/2\mathbb{Z}$ acts by interchanging e_1^1 and e_2^1 (trivial isotropy) and e_3^1 (isotropy $\mathbb{Z}/2\mathbb{Z}$). The orbit

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type stratification on X does not satisfy the frontier condition since $e_3^1 \cup *$ is not a subset of the closure of e_j^1 for $j = 1, 2$.

Definition 1 ([3, Definition 5.2])**.** A stratified space X satisfying the frontier condition is *homotopically stratified* if the following conditions are satisfied.

(1) Forward tameness: For every $k > i$, X_i is forward tame in $X_i \cup X_k$.
(2) Normal fibrations: For every $k > i$, evaluation at the initial point of a path, $\mathrm{ev}_0 : \mathrm{holink}(X_i \cup X_k, X_i) \to X_i$, is a Hurewicz fibration.

We construct examples of $\mathbb{Z}/2\mathbb{Z}$-CW complexes that satisfy the frontier condition and the forward tameness condition but are not homotopically stratified by orbit type.

Proposition 2. *Let X be a metric space and $Y \subseteq X$ a closed subspace. Let U be a neighborhood of Y in X. Suppose* $\mathrm{ev}_0 : \mathrm{holink}(X, Y) \to Y$ *is a Hurewicz fibration. Then the restriction of* ev_0 *to* $\mathrm{holink}(U, Y)$ *is also a Hurewicz fibration.*

Proof. Since $\mathrm{holink}(U, Y)$ and Y are metrizable, it suffices to verify the homotopy lifting property with respect to metric spaces [2, XX, Corollary 2.3]. Let Z be a metric space and $f : Z \to \mathrm{holink}(U, Y)$, $F : Z \times I \to Y$ be a lifting problem, i.e., $\mathrm{ev}_0 \circ f = F_0$. Since $\mathrm{ev}_0 : \mathrm{holink}(X, Y) \to Y$ is a Hurewicz fibration, there exists $\widetilde{F} : Z \times I \to \mathrm{holink}(X, Y)$ such that $\mathrm{ev}_0 \circ \widetilde{F} = F$ and $\widetilde{F}_0 = i \circ f$, where $i : \mathrm{holink}(U, Y) \hookrightarrow \mathrm{holink}(X, Y)$ is inclusion. As in the proof of [5, Lemma 2.4(1)], there is a continuous function $r : \mathrm{holink}(X, Y) \to (0, 1]$ such that for every ω in $\mathrm{holink}(X, Y)$, $\omega([0, r(\omega)]) \subseteq U$. Note that $\mathrm{holink}(U, Y)$ is open in $\mathrm{holink}(X, Y)$. Therefore, $\widetilde{F}^{-1}(\mathrm{holink}(U, Y))$ is open in $Z \times I$ and it contains $Z \times \{0\}$. Thus, $Z \times \{0\}$ and $Z \times I - \widetilde{F}^{-1}(\mathrm{holink}(U, Y))$ are disjoint closed sets in the metric space $Z \times I$, and so there exists a continuous $\phi : Z \times I \to [0, 1]$ such that $\phi|_{Z \times \{0\}} = 1$ and $\phi|_{Z \times I - \widetilde{F}^{-1}(\mathrm{holink}(U,Y))} = 0$. Let $\mu = \max(\phi, r \circ \widetilde{F})$. Then $\hat{F}(z, t)(s) := \widetilde{F}(z, t)(\mu(z, t)s)$ defines a homotopy $\hat{F} : Z \times I \to \mathrm{holink}(U, Y)$ such that $\mathrm{ev}_0 \circ \hat{F} = F$ and $\hat{F}_0 = f$. Hence, the restriction of ev_0 to $\mathrm{holink}(U, Y)$ is also a Hurewicz fibration. □

Our Proposition 2 is closely related to [4, Proposition 4.4], in which the same conclusion is reached under the additional hypothesis that Y is forward tame in X.

Lemma 3. *Let $U := (X \times (0, 1]) \cup_f Y$ be the half-open mapping cylinder of $f : X \to Y$, and let $N := (X \times [1/2, 1]) \cup_f Y$. Then* $\mathrm{ev}_0 : \mathrm{holink}(N, Y) \to Y$ *is a Hurewicz fibration if* $\mathrm{ev}_0 : \mathrm{holink}(U, Y) \to Y$ *is a Hurewicz fibration.*

Proof. Define a retraction $r : U \to N$ by

$$r(z, t) = \begin{cases} (z, 1/2) & \text{if} \quad 1/2 < t < 1, \\ (z, t) & \text{if} \quad 0 \le t \le 1/2, \end{cases}$$

and $r(y) = y$ if $y \in Y$. Since $r(U - Y) \subseteq N - Y$, r induces a retraction $r_* : \mathrm{holink}(U, Y) \to \mathrm{holink}(N, Y)$, defined by $r_*(\omega) = r \circ \omega$. The result now follows from the fact that the retract of a fibration is a fibration. □

Let $p : S^n \to S^m$ be a surjective map. Take two disjoint copies of S^n and map them both to S^m using p; call the resulting map $q : S^n \coprod S^n \to S^m$. Attach a pair of $(n+1)$-cells to S^m using q and call the resulting space $E(p)$. Define a $\mathbb{Z}/2\mathbb{Z}$-action on $E(p)$ by interchanging the interior of the two $(n+1)$-cells and leaving the S^m subspace fixed. The space $E(p)$ is a $\mathbb{Z}/2\mathbb{Z}$-CW complex with strata (by orbit type): $S^m \subseteq E(p)$ (isotropy $\mathbb{Z}/2\mathbb{Z}$) and the two $(n+1)$-cells (trivial isotropy). By [5, Theorem 1.4], $E(p)$ is homotopically stratified by orbit type if and only if $(\mathbb{Z}/2\mathbb{Z})\backslash E(p)$ is homotopically stratified by orbit type.

Theorem 4. *If $E(p)$ is homotopically stratified by orbit type, then the attaching map $p : S^n \to S^m$ is an approximate fibration.*

Proof. The orbit space, $X = (\mathbb{Z}/2\mathbb{Z})\backslash E(p)$, of $E(p)$ is the CW complex obtained by attaching an $(n+1)$-cell to S^m via the attaching map p. Since $E(p)$ is homotopically stratified by orbit type, so is X with strata $X_0 = S^m$ and X_1 equal to the open $(n+1)$-cell. Let $U = X - x_1$, where x_1 is in X_1. By Proposition 2, $\text{ev}_0 : \text{holink}(U, S^m) \to S^m$ is a Hurewicz fibration. Since U is homeomorphic to $(S^n \times (0,1]) \cup_p S^m$, the half-open mapping cylinder of $p : S^n \to S^m$, Lemma 3 implies that $\text{ev}_0 : \text{holink}(N, S^m) \to S^m$ is a Hurewicz fibration, where $N = (S^n \times [1/2, 1]) \cup_p S^m$. Therefore, $\text{ev}_0 : \text{holink}(\text{cyl}(p), S^m) \to S^m$ is a Hurewicz fibration, since $\text{cyl}(p) = (S^n \times [0,1]) \cup_p S^m$, the mapping cylinder of $p : S^n \to S^m$, is homeomorphic to N. Since $S^m \subseteq \text{cyl}(p)$ is forward tame, $\text{cyl}(p)$ is homotopically stratified with strata S^m and $S^n \times [0,1)$. By [3, Theorem 5.11], $p : S^n \to S^m$ is an approximate fibration. □

Lemma 5. *Suppose $p : S^k \to S^1$ is a smooth surjective map and $k > 1$. Then p is not an approximate fibration.*

Proof. By Sard's Theorem, p must have a regular value z in S^1. Then $F = p^{-1}(\{z\})$ is a smooth compact $(k-1)$-manifold. Let F_0 be a path component of F and x_0 in F_0 be a basepoint. Since F is a smooth submanifold of S^k, its shape homotopy groups coincide with its homotopy groups. Suppose p is an approximate fibration. Then the corresponding homotopy long exact sequence for approximate fibrations, [1, Corollary 3.5], implies that $\pi_m(F_0, x_0) \cong \pi_m(S^k, x_0)$ for $m \geq 1$. It follows that F_0 is $(k-1)$-connected and so by Hurewicz's Theorem, $\pi_k(F_0, x_0) \cong H_k(F_0)$. Since F_0 is a $(k-1)$-manifold and $k > 1$, $H_k(F_0) = 0$, contradicting $\pi_k(F_0, x_0) \cong \pi_k(S^k, x_0) \cong \mathbb{Z}$. □

Combining Theorem 4 and Lemma 5 yields:

Theorem 6. *Suppose $p : S^k \to S^1$ is a smooth surjective map and $k > 1$. Then the $\mathbb{Z}/2\mathbb{Z}$-CW complex $E(p)$ is not homotopically stratified by orbit type.* □

Note that for p as in Theorem 6 the space $E(p)$ is a topological manifold of dimension $(k+1)$ away from a codimension k singular set homeomorphic to S^1. Although $E(p)$ is not homotopically stratified by orbit type, the orbit type partition may have a refinement for which $E(p)$ is homotopically stratified. For example, if the surjective map $p : S^k \to S^1$ is subanalytic, then [3, Corollary 7.5] asserts that there are Whitney stratifications of S^k and S^1 such that p becomes a stratified approximate fibration. By [3, Theorem 5.11], $\text{cyl}(p)$ with its natural partition is homotopically stratified and it follows that $E(p)$ with the corresponding partition, refining the orbit type partition, is homotopically stratified.

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