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ON UNIVERSAL C^*-ALGEBRAS GENERATED BY n PROJECTIONS WITH SCALAR SUM

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ABSTRACT. We study the universal C^*-algebras generated by n projections $p_1, \dots, p_n$ subject to the relation $p_1 + \dots + p_n = \lambda 1$, $\lambda \in \mathbb{R}$. The questions of when these C^*-algebras are type I, nuclear or exact are considered. It is proved also that among these C^*-algebras there is a continuum of mutually nonisomorphic ones.

INTRODUCTION

We consider the relations

$$\sum_{i=1}^{n} p_i = \lambda 1,\ p_i = p_i^* = p_i^2,\ i = 1, \dots, n, \tag{1}$$

where $\lambda \in \mathbb{R}$, and their representations, that is, n-tuples $P_i, i = 1, \dots, n$, of projections on a Hilbert space such that

$$\sum_{i=1}^{n} P_i = \lambda 1. \tag{2}$$

Decompositions of scalar operators on a Hilbert space into a sum of a fixed number of projections were studied in a series of papers ([8], [9], [14], [6], [7]). In [7] the set Σ_n of all possible scalars λ in (2) was completely described. Namely it was proved that for $n < 4$, Σ_n is finite:

- $\Sigma_1 = \{0, 1\}$,
- $\Sigma_2 = \{0, 1, 2\}$,
- $\Sigma_3 = \{0, 1, 3/2, 2, 3\}$,

for $n = 4$, the set is countable:

- $\Sigma_4 = \{0, 1, 1 + \frac{k}{k+2} (k \in \mathbb{N}), 2, 3 - \frac{k}{k+2} (k \in \mathbb{N}), 3, 4\}$,

and for $n \geq 5$, Σ_n is the union of the "main" segment $[\alpha_n, \beta_n]$ and two sequences Λ_n^1 and Λ_n^2 converging to the end points of this segment. Here

$$\alpha_n = \frac{n - \sqrt{n^2 - 4n}}{2}, \quad \beta_n = \frac{n + \sqrt{n^2 - 4n}}{2}. \tag{3}$$

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We do not need the exact formula for the points from Λ_n^1 and Λ_n^2; it suffices to mention that they are rational.

A natural aim is to describe, for each $\lambda \in \Sigma_n$, all n-tuples of projections in a Hilbert space that fulfill (1) or at least to understand how complicated this problem is. The degree of complexity of such a description for an arbitrary relation can be formulated in terms of belonging of the universal C^*-algebra of the relation to some more or less tractable classes of C^*-algebras (type I, approximately finite-dimensional, nuclear C^*-algebras, and so on). We will denote the universal C^*-algebra of the relation (1) by $\mathcal{P}_{n,\lambda}$.

In [7] the authors ask for which λ does the C^*-algebra $\mathcal{P}_{n,\lambda}$ belong to the class of type I C^*-algebras. They proved that if $\lambda \in \Lambda_n^i, i = 1, 2$, then $\mathcal{P}_{n,\lambda}$ is finite-dimensional and if $\lambda \in (\alpha_n, \beta_n)$, then $\mathcal{P}_{n,\lambda}$ is not of type I (for any $n > 6$). For $\lambda = \alpha_n$ and $\lambda = \beta_n$ the question remained open. Below we will give the negative answer to this question. Moreover it will be shown that for these values of λ there do not exist unital $*$-homomorphisms from $\mathcal{P}_{n,\lambda}$ to any type I C^*-algebra.

We also show that for "most" values of λ the C^*-algebras $\mathcal{P}_{n,\lambda}$ are not nuclear or even exact.

We prove that for every λ, $\mathcal{P}_{n,\lambda}$ has a trace and use this fact in the problem of classification of these C^*-algebras. The result is that among these C^*-algebras there is a continuum of mutually nonisomorphic ones. The same technique allows us to answer another question posed in [7]: to show that for most λ's the C^*-algebra $\mathcal{P}_{n,\lambda}$ is not $*$-wild in the sense of [10].

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1. Representations of (1) in type I C^*-algebras

A C^*-algebra A is called a *CCR-algebra* if for any of its nonzero irreducible representations (H, π), the set $\pi(A)$ coincides with the set $\mathcal{K}(H)$ of all compact operators on H. Furthermore A is called a *type I C^*-algebra* if it has a composition series $(I_\rho)_{0\le\rho\le\alpha}$ such that all $I_{\rho+1}/I_\rho$ are CCR-algberas.

Recall that an increasing transfinite sequence of closed two-sided ideals $(I_\rho)_{0\le\rho\le\alpha}$ of a C^*-algebra A is called a composition series if it satisfies the following conditions:

1) $I_0 = 0, I_\alpha = A$;

2) if $\rho \le \alpha$ is a limit ordinal, then $I_\rho = \overline{\bigcup_{\rho'<\rho} I_{\rho'}}$.

We couldn't find a reference for the following result.

Proposition 1. *Any unital type I C^*-algebra has a finite-dimensional representation.*

Proof. Let A be a unital type I C^*-algebra and $(I_\rho)_{0\le\rho\le\alpha}$ be its composition series. Suppose that α is a limit ordinal and hence $A = \overline{\bigcup_{\rho<\alpha} I_\rho}$. Then the unit of A is a limit of a sequence of elements that belong to ideals of A. This is impossible because the set of invertible elements of A is open. Thus α is not a limit ordinal and hence the composition series has an ideal $I_{\alpha-1}$. Then $A/I_{\alpha-1}$ is a unital CCR-algebra. Hence all its irreducible representations are finite-dimensional (because the image of the unit is compact). Let ρ be any irreducible representation of $A/I_{\alpha-1}$, $q : A \to A/I_{\alpha-1}$ be the canonical epimorphism. Then the composition $\rho\circ q$ gives a finite-dimensional representation of A. □

For any C^*-algebra A, we denote by $\Sigma_n(A)$ the set of those λ for which *in this algebra* there exist n projections whose sum is $\lambda 1$.

Theorem 2. *Let A be a type I C^*-algebra. Then $\Sigma_n(A)$ is a finite set of rational numbers.*

Proof. By Proposition 1, there exists a representation π of A in a finite-dimensional space H. Denote by m its dimension. Let $\lambda \in \Sigma_n(A)$. Then there exist projections $p_1, \ldots, p_n$ in A such that $p_1 + \ldots + p_n = \lambda 1$. Calculating traces of the left-hand side and the right-hand side of the equality

$$\pi(p_1) + \ldots + \pi(p_n) = \lambda\pi(1),$$

we get

$$tr(p_1) + \ldots + tr(p_n) = \lambda m.$$

Since the trace of any projection in m-dimensional space is a natural number not greater than m, we obtain that λ belongs to the finite set of rational numbers $\{\frac{k_1+\ldots+k_n}{m} : k_i \le m, k_i \in \mathbb{N}, i = 1, \ldots, n\}$. Since it is true for any $\lambda \in \Sigma_n(A)$, we are done. □

Corollary 3. *$\mathcal{P}_{n,\alpha_n}$ and $\mathcal{P}_{n,\beta_n}$ are not type I C^*-algebras.*

Proof. Clearly $\lambda \in \Sigma_n$ implies $\lambda \in \Sigma_n(\mathcal{P}_{n,\lambda})$. Hence $\alpha_n \in \Sigma_n(\mathcal{P}_{n,\alpha_n})$, $\beta_n \in \Sigma_n(\mathcal{P}_{n,\beta_n})$. Since α_n and β_n are irrational we obtain, by Theorem 2, that $\mathcal{P}_{n,\alpha_n}$ and $\mathcal{P}_{n,\beta_n}$ are not type I C^*-algebras. □

2. Representations of (1) in C^*-algebras with a trace, and a classification of $P_{n,\lambda}$, $\lambda \in \Sigma_n$

By a *polynomial relation* (in n variables) is meant an equation of the form

$$f(x_1, ..., x_n, x_1^*, ..., x_n^*) = 0, \tag{4}$$

where f is a polynomial in $2n$ noncommuting variables, that is, an element of the free unital $*$-algebra $\mathcal{F}_n^*$ on generators $x_1, ..., x_n$.

Let us say that the relation (4) is *representable in a (unital) C^*-algebra A* if there is an n-tuple of elements $a_1, ..., a_n \in A$ that satisfies the equality

$$f(a_1, ..., a_n, a_1^*, ..., a_n^*) = 0. \tag{5}$$

If $\|a_i\| \le C$ for all i, where $C > 0$, then we say that this representation is *C-bounded*.

Now let us introduce a topology on the set of all polynomial relations (which can be identified with $\mathcal{F}_n^*$) by a system of seminorms ν_K, $K > 0$ on $\mathcal{F}_n^*$, defined by the formula

$$\nu_K(f) = \sup \|f(T_1, \ldots, T_n, T_1^*, \ldots, T_n^*)\|,$$

where the supremum is taken over the set of all n-tuples of operators with norms not greater than K.

By a *trace* on a unital C^*-algebra A we mean a positive linear functional g such that

$$g(xy) = g(yx),\ g(1_A) = 1.$$

Theorem 4. *Let $C > 0$. The set of polynomial relations in n variables, C-boundedly representable in unital C^*-algebras with a trace, is closed.*

Proof. Let $f_k \to f$ and, for any k, the relation f_k has a C-bounded representation π_k in a C^*-algebra A_k with a trace τ_k. Consider the C^*-algebra $E = \prod_{k=1}^{\infty} A_k$ of all bounded sequences with elements in A_k. Let J be the ideal in E consisting of all sequences vanishing at infinity and let $F = E/J$. Set

$$a_{i,k} = \pi_k(x_i),$$

$i = 1, ..., n$, where the x_i are free generators of $\mathcal{F}_n^*$. Let $e_i = (a_{i,1}, a_{i,2}, ...) \in E$ and b_i be their images under the canonical epimorphism from E to F. Then, setting

$$\pi(x_i) = b_i,$$

we define a representation of the relation f in F.

It remains to prove that F has a trace. Let ξ be a nontrivial ultrafilter on $\mathbb{N}$. We can think of it as a character of $l^\infty(\mathbb{N})$. Setting

$$h((u_k)_{k\in\mathbb{N}}) = \xi((\tau_k(u_k))_{k\in\mathbb{N}}),$$

we get a state on E. It is easy to see that $h(xy) = h(yx)$, for any $x, y \in E$, $h(1_E) = 1$ and hence h is a trace on E. Since any ultrafilter vanishes on c_0, h vanishes on J and hence defines a trace on F. □

Corollary 5. *All C^*-algebras $\mathcal{P}_{n,\lambda}$, where $\lambda \in \Sigma_n$, have a trace.*

Proof. For any rational $\lambda \in \Sigma_n$, the relation (1) has a finite-dimensional representation ([7]) and hence a representation in a C^*-algebra with a trace. Moreover, for any $\lambda \in \Sigma_n$, each representation of the relation (1) is 1-bounded. Since any irrational $\lambda \in \Sigma_n$ is a limit of rational numbers from Σ_n, the relation (1) belongs to the closure of the set of polynomial relations 1-representable in C^*-algebras with a trace. By Theorem 4, it is representable in a C^*-algebra with a trace.

Since any representation of the relation (1) defines a representation of the C^*-algebra $\mathcal{P}_{n,\lambda}$ we get that for any $\lambda \in \Sigma_n$, $\mathcal{P}_{n,\lambda}$ has a $*$-homomorphism π to a C^*-algebra A with a trace τ. Setting $\tau_1(a) = \tau(\pi(a))$ for any $a \in \mathcal{P}_{n,\lambda}$, we get a trace on $\mathcal{P}_{n,\lambda}$. □

Lemma 6. *Let A be a separable C^*-algebra with a trace. Then the set $\Sigma_n(A)$ is countable.*

Proof. Let $P(A)$ be the set of all projections in A, τ be a trace on A. Suppose $\lambda \in \Sigma_n(A)$. Then there exist $q_1, \ldots, q_n \in P(A)$ such that $\lambda = \sum_{i=1}^n \tau(q_i)$. Since close projections are equivalent, the function $p \mapsto \tau(p)$ is locally constant on the separable space $P(A)$ and hence the set of its values is countable. Hence $\Sigma_n(A)$ is countable. □

Remark 7. In the absence of a trace the lemma is not true even for separable simple nuclear C^*-algebras. As an example one can take $A = O_2$ ([7]).

Consider now the problem of classification of the C^*-algebras $\mathcal{P}_{n,\lambda}$, $\lambda \in \Sigma_n$. We do not know in general when the C^*-algebras $\mathcal{P}_{n,\lambda}$ and $P_{n,\mu}$ are isomorphic. It is natural to conjecture that it happens only when $\mu = \lambda$ or $\mu = n - \lambda$. For $n < 5$ it is true. For $n \geq 5$, the invariant $\Sigma_n(A)$ helps to prove that among these C^*-algebras there is continuum of pairwise nonisomorphic ones.

Theorem 8. *Let $E \subset \Sigma_n$ have the cardinality of the continuum. Then among C^*-algebras $\mathcal{P}_{n,\lambda}, \lambda \in E$, there is a continuum of pairwise nonisomorphic ones.*

Proof. Let us represent the set of all C^*-algebras $\mathcal{P}_{n,\lambda}$, $\lambda \in E$, as the union of classes $\{K_i : i \in I\}$ of pairwise isomorphic C^*-algebras. For all C^*-algebras A from one equivalence class K_i the set $\Sigma_n(A)$ is the same, so we can denote it by $\Sigma_n(K_i)$. By Corollary 5 and Lemma 6, for any $\lambda \in \Sigma_n$, the set $\Sigma_n(\mathcal{P}_{n,\lambda})$ is countable and hence $\Sigma_n(K_i)$ is countable for any $i \in I$. Since clearly $\lambda \in \Sigma_n(\mathcal{P}_{n,\lambda})$ we have $E = \bigcup_{i \in I} \Sigma_n(K_i)$. Thus $\bigcup_{i\in I} \Sigma_n(K_i)$ has the cardinality of the continuum and each $\Sigma_n(K_i)$ is countable. It follows that I has the cardinality of the continuum. □

We finish this section by applying the invariant $\Sigma_n(A)$ to one more question from [7]. A $*$-algebra is called *$*$-wild* if the problem of the classification of its $*$-representations contains as a subproblem the classification of the pairs of self-adjoint operators up to the unitary equivalence (we refer to [10] for a precise definition). It is proved in Proposition 66 of [13] that a C^*-algebra A is $*$-wild if and only if it has an ideal I such that the quotient A/I is isomorphic to $M_n \otimes C^*(\mathbb{F}_2)$, for some $n \in \mathbb{N} \cup \{\infty\}$. Here $C^*(\mathbb{F}_2)$ is the group C^*-algebra of the free group on two generators; by M_∞ we mean the algebra $\mathcal{K}(H)$ of all compact operators on a separable Hilbert space H.

It was asked in [7] if the algebras $\mathcal{P}_{n,\lambda}$ are $*$-wild for irrational λ. The answer turns out to be negative.

Theorem 9. *Among all C^*-algebras $\mathcal{P}_{n,\lambda}$ there are at most countably many $*$-wild ones.*

Proof. If $\mathcal{P}_{n,\lambda}$ is $*$-wild, then, by the above, its quotient is isomorphic to $M_m \otimes C^*(\mathbb{F}_2)$ for some $m \in \mathbb{N} \cup \{\infty\}$. In fact the case $m = \infty$ is impossible, because $\mathcal{P}_{n,\lambda}$ is unital.

Thus if the assertion of the theorem is not true, then there are integers $n, m \in \mathbb{N}$ and an uncountable set $E \subset \Sigma_n$ such that, for any $\lambda \in E$, there exists an ideal $I \subset \mathcal{P}_{n,\lambda}$ with $\mathcal{P}_{n,\lambda}/I \cong M_m \otimes C^*(\mathbb{F}_2)$. Hence $\lambda \in \Sigma_n(\mathcal{P}_{n,\lambda}) \subset \Sigma_n(\mathcal{P}_{n,\lambda}/I) = \Sigma_n(M_m \otimes C^*(\mathbb{F}_2))$, that is, $E \subset \Sigma_n(M_m \otimes C^*(\mathbb{F}_2))$. Since $M_m \otimes C^*(\mathbb{F}_2)$ is a separable C^*-algebra with a trace, we obtain a contradiction with Lemma 6. □

3. Nuclearity and exactness

A C^*-algebra A is *nuclear* if, for any C^*-algebra B, there is only one C^*-norm on the algebraic tensor product $A \odot B$. Our aim in this subsection is to prove that for large n, $\mathcal{P}_{n,\lambda}$ is nonnuclear for "most" of the points $\lambda \in (\alpha_n; \beta_n)$. Moreover we will show that for $n > 10$, $(\alpha_n; \beta_n)$ contains a subinterval I_n such that for any $\lambda \in I_n$, $\mathcal{P}_{n,\lambda}$ does not belong to the larger class of exact C^*-algebras.

A C^*-algebra A is called *exact* if, for any short exact sequence

$$0 \to J \to B \to C \to 0,$$

the sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes C \to 0$$

is also exact. By $\otimes$ we denote the minimal tensor product.

It is well known that the class of all nuclear C^*-algebras is contained in the class of all exact C^*-algebras. Recall also that both classes are closed under taking ideals and quotients and that the class of exact algebras is closed under taking closed $*$-subalgebras. All this information can be found in [5], [4].

Lemma 10. *In an infinite-dimensional Hilbert space there exist three projections P, Q, R generating a nonexact C^*-algebra.*

Proof. Let A be the universal C^*-algebra generated by 3 projections p_1, p_2, p_3 without any relations. By [11], A is $*$-wild. Hence there exists a closed ideal J of A such that $A/J \cong M_n \otimes C^*(\mathbb{F}_2)$, for some $n \in \mathbb{N} \cup \{\infty\}$.

This implies that A is nonexact. Indeed if A is exact, then any one of its quotients is exact. On the other hand $C^*(\mathbb{F}_2)$ is nonexact ([15]) and hence $M_n \otimes C^*(\mathbb{F}_2)$ is nonexact because it contains a nonexact C^*-algebra $C^*(\mathbb{F}_2)$ as a closed $*$-subalgebra.

Now let π be the universal representation of A. Set $P = \pi(p_1), Q = \pi(p_2), R = \pi(p_3)$. Then the C^*-algebra generated by them is nonexact. □

Theorem 11. *For each $n > 6$, there exists a nonempty subset $I_n \in \Sigma_n$ such that for any $\lambda \in I_n$, the C^*-algebra $\mathcal{P}_{n,\lambda}$ is not exact.*

Moreover if $n > 10$, then $I_n \supset [5; n-5]$.

Proof. Let us denote by E_n the set $\Sigma_n \cap (\Sigma_{n-6} + 3)$ of all points $\lambda \in \Sigma_n$ such that $\lambda - 3 \in \Sigma_{n-6}$. Using (3) it is easy to check that $E_n \neq \emptyset$, for $n > 6$, and that $\beta_{n-6} + 3 < \beta_n$, $\alpha_{n-6} + 3 > \alpha_n$, for any $n > 10$; whence we conclude that E_n contains the closed interval $I_n = [\alpha_{n-6} + 3; \beta_{n-6} + 3]$. Since $[2; n-2] \subset [\alpha_n; \beta_n]$ we get $I_n \supset [5, n-5]$, for any $n > 10$.

Let $\lambda \in E_n$ and let π be arbitrary representation of the C^*-algebra $\mathcal{P}_{n-6,\lambda-3}$. Define a representation $\tilde{\pi}$ of $\mathcal{P}_{n,\lambda}$ in the following way. Set

$\tilde{\pi}(p_k) = \pi(p_k)$, for $1 \leq k \leq n-6$,
$\tilde{\pi}(p_{n-5}) = P$, $\tilde{\pi}(p_{n-4}) = 1 - P$,
$\tilde{\pi}(p_{n-3}) = Q$, $\tilde{\pi}(p_{n-2}) = 1 - Q$,
$\tilde{\pi}(p_{n-1}) = R$, $\tilde{\pi}(p_n) = 1 - R$,

where P, Q, R are the projections constructed in Lemma 10. Since the C^*-algebra generated by them is a subalgebra of $\tilde{\pi}(\mathcal{P}_{n,\lambda})$, it is isomorphic to some subalgebra of the quotient $\mathcal{P}_{n,\lambda}/Ker\tilde{\pi}$. Hence $\mathcal{P}_{n,\lambda}$ is not exact by Lemma 10. □

Remark 12. Since $[\alpha_n; \beta_n] \subset [1; n-1]$, for any n, and I_n contains $[5; n-5]$, for $n > 10$, one can say that, for large n, I_n contains "almost whole" $[\alpha_n; \beta_n]$.

Now we are going to prove that the set of points λ such that $\mathcal{P}_{n,\lambda}$ is not nuclear is strictly larger than I_n.

Let us denote by f the map from the interval $(\alpha_n; \beta_n)$ onto itself given by the formula $f(\lambda) = n - 1 - 1/(\lambda - 1)$. Let $\mathcal{S}(f)$ be the group (under the composition) generated by f, that is, the group of all (positive and negative) powers of the map f.

Corollary 13. *Let $n > 6$. For every $\lambda \in (\alpha_n; \beta_n)$ whose orbit of the action of $\mathcal{S}(f)$ intersects I_n, the C^*-algebra $\mathcal{P}_{n,\lambda}$ is not nuclear.*

Proof. Let $\lambda_1 = f(\lambda_2)$. Suppose that $\mathcal{P}_{n,\lambda_1}$ is not nuclear. Then, by Connes's theorem ([1]), there is its factor-representation π which is not hyperfinite, which means that the closure in WOT of $\pi(\mathcal{P}_{n,\lambda_1})$ is not hyperfinite. By ([1]), its commutant $\pi(\mathcal{P}_{n,\lambda_1})'$ is not hyperfinite either.

It was proved in [7] that the categories of representations of the C^*-algebras $\mathcal{P}_{n,\lambda_1}$ and $\mathcal{P}_{n,\lambda_2}$ are equivalent. This implies that there exist a representation $\tilde{\pi}$ of $\mathcal{P}_{n,\lambda_2}$ and a WOT-bicontinuous isomorphism F from $\pi(\mathcal{P}_{n,\lambda_1})'$ onto $\tilde{\pi}(\mathcal{P}_{n,\lambda_2})'$. It follows that $\tilde{\pi}(\mathcal{P}_{n,\lambda_2})'$ is also a nonhyperfinite factor, and the algebra $\mathcal{P}_{n,\lambda_2}$ is not nuclear.

Thus all C^*-algebras $\mathcal{P}_{n,\lambda}$, with λ from one orbit of the action of $\mathcal{S}(f)$, are nuclear or nonnuclear simultaneously. Now it remains to apply Theorem 11. □

4. Concluding remarks

We will mention some additional results and questions about $\mathcal{P}_{n,\lambda}$.

4.1. $\Sigma_n(A)$ for UHF-algebras. A C^*-algebra is called *uniformly hyperfinite* (UHF) if it is the closure of the union of an increasing net of unital subalgebras isomorphic to full matrix algebras. For such C^*-algebras the set $\Sigma_n(A)$ can be written explicitly.

Theorem 14. *Let $A = \overline{\bigcup A_i}$ be a UHF-algebra with $A_i \cong M_{k_i}$. Then the set $\Sigma_n(A)$ consists of all numbers $\lambda \in \Sigma_n$ of the form p/q, where $q|k_j$ for some j.*

Proof. Let $p/q \in \Sigma_n$, $q|k_j$ for some j. Clearly one can assume that p/q is an irreducible fraction. By definition, $(p/q)1$ is the sum of n projections in q-dimensional space ([8]). If $q|k_j$ for some j, then there is an embedding of M_q into M_{k_j} and hence into A. Thus we have n projections in A with sum $(p/q)1$.

On the other hand it follows from [3] that the trace of an arbitrary projection in A is a rational number with denominator dividing k_j for some j. Hence the same is true for any $\lambda \in \Sigma_n$. □

As we know, if A is $B(H)$ or a type I C^*-algebra, then the set $\Sigma_n(A)$ is closed. It follows from Theorem 14 that this is not true for a general C^*-algebra (this answers a question of the authors of [7]).

4.2. Stability. Let $\delta > 0$. An n-tuple of operators $T_1, ..., T_n$ is called a δ-*representation* of a relation $f(x_1, ..., x_n, x_1^*, ..., x_n^*) = 0$ if

$$\|f(T_1, ..., T_n, T_1^*, \dots, T_n^*)\| \leq \delta.$$

A relation is called *stable* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any of its δ-representations $T_1, ..., T_n$, there exists its representation $S_1, ..., S_n$ in the same space such that $\|T_i - S_i\| < \varepsilon$.

Theorem 15. *For any $\lambda \in [\alpha_n, \beta_n]$, the relation* (1) *is not stable.*

Proof. It suffices to show that for each $\delta > 0$ there exists a Hilbert space H and a δ-representation of the relation (1) in H but there does not exist any representation of (1) in H.

Let $\lambda \in [\alpha_n, \beta_n]$ be irrational. Let $\lambda' \in \Sigma_n \cap \mathbb{Q}$ be such that $|\lambda' - \lambda| < \delta$. Then there exist projections $P_1, ..., P_n$ in a finite-dimensional Hilbert space H such that $\sum_{i=1}^n P_i = \lambda' 1$ ([7]). Clearly they define a δ-representation of (1). It remains to note that (1) does not have finite-dimensional representations.

Now let $\lambda \in [\alpha_n, \beta_n]$, $\lambda = \frac{p}{q}$, where $\frac{p}{q}$ is an irreducible fraction. There exists a rational number λ' of the form $\lambda' = \frac{r}{p^m}$ such that $|\lambda' - \lambda| \leq \delta$. Then there exist projections $P_1, \dots, P_n$ in $\mathbb{C}^{p^m}$ such that $P_1 + \dots + P_n = \lambda' 1$ ([7]). Hence $P_1, \dots, P_n$ define a δ-representation of (1). Suppose that this relation has some representation $Q_1, \dots, Q_n$ in $\mathbb{C}^{p^m}$. Then $trQ_1 + \dots + trQ_n = \lambda p^m$, whence $\frac{p^{m+1}}{q} = \lambda p^m \in \mathbb{Z}$. This contradicts the assumption that q and p are coprime. □

For $\lambda \in \Sigma_n \backslash [\alpha_n, \beta_n]$, the C^*-algebra $\mathcal{P}_{n,\lambda}$ is finite-dimensional and hence is stable by [12].

4.3. **Simplicity.** It is known ([7]) that among the numbers $\lambda \in \Lambda_n^i$, $i = 1, 2$, there is an infinite set such that the $\mathcal{P}_{n,\lambda}$ are isomorphic to full matrix algebras and therefore are simple. This is far from being true for $\lambda \in [\alpha_n, \beta_n]$.

(i) If $\lambda \in [\alpha_n, \beta_n]$ is rational, then $\mathcal{P}_{n,\lambda}$ is not simple because it is not finite-dimensional but has a finite-dimensional representation ([8]).

(ii) If $\lambda \in \Sigma_{n-1} \cap \Sigma_n$, then $\mathcal{P}_{n,\lambda}$ is not simple. Indeed we can define a $*$-homomorphism $\pi : \mathcal{P}_{n,\lambda} \to \mathcal{P}_{n-1,\lambda}$ setting $\pi(p_i) = q_i$, $i = \overline{1, n-1}$, $\pi(p_n) = 0$, where $p_1, \dots, p_n$ and $q_1, \dots, q_{n-1}$ are generators of $\mathcal{P}_{n,\lambda}$ and $\mathcal{P}_{n-1,\lambda}$ respectively, and take its kernel.

So the question on simplicity of $\mathcal{P}_{n,\lambda}$ remains open for irrational numbers from $[\alpha_n, \beta_n] \backslash [\alpha_{n-1}, \beta_{n-1}]$. Since the latter set is contained in $[1, 2] \cup [n-3, n-1]$ one can say that for "most" $\lambda \in \Sigma_n$ the algebras $\mathcal{P}_{n,\lambda}$ are not simple.

4.4. K**-theory.** D. Hadwin (private communication) proved that for any $\lambda \in [\alpha_n, \beta_n]$, the group $K_0(\mathcal{P}_{n,\lambda})$ contains $\mathbb{Z}^n$ as a direct summand.

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