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VERTEX COVER ALGEBRAS OF UNIMODULAR HYPERGRAPHS

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Abstract. It is proved that all vertex cover algebras of a hypergraph are standard graded if and only if the hypergraph is unimodular. This has interesting consequences on the symbolic powers of monomial ideals.

1. Introduction

A hypergraph Δ is a collection of subsets of a finite set of vertices V. These subsets are called the edges of Δ. For convenience, we assume throughout this paper that there is no inclusion between the edges of Δ. Such a hypergraph is also called a clutter.

A vertex cover of Δ is a subset of V which meets every edge of Δ. Suppose that $V = \{1, ..., n\}$. We may think of a vertex cover of V as a (0,1) vector $\mathbf{c} = (c_1, ..., c_n)$ that satisfies the condition $\sum_{i\in F} c_i \geq 1$ for all $F \in \Delta$.

Let $w\colon F \mapsto w_F$ be a weight function from Δ to the set of positive integers. We call (Δ, w) a weighted hypergraph. For $k \in \mathbb{N}$ we define a k-cover of (Δ, w) as a vector $\mathbf{c} \in \mathbb{N}^n$ that satisfies the condition $\sum_{i\in F} c_i \geq kw_F$ for all $F \in \Delta$.

Let $R = K[x_1, ..., x_n]$ be a polynomial ring over a field K. For every vector $\mathbf{c} \in \mathbb{N}^n$ we set $\mathbf{x}^{\mathbf{c}} = x_1^{c_1} \cdots x_n^{c_n}$. The *vertex cover algebra* $A(\Delta, w)$ is defined as the subalgebra of the one-variable polynomial ring $R[t]$ generated by all monomials $\mathbf{x}^{\mathbf{c}} t^k$, where $\mathbf{c}$ is a k-cover of (Δ, w), $k \geq 0$. This algebra was introduced in [4]. It was proved there that $A(\Delta, w)$ is a finitely generated normal Cohen-Macaulay ring.

For every edge $F \in \Delta$ let P_F denote the ideal of R generated by the variables x_i, $i \in F$. Let $I(\Delta, w) := \bigcap_{F\in\Delta} P_F^{w_F}$. It is not hard to see that $A(\Delta, w)$ is the symbolic Rees algebra of the ideal $I(\Delta, w)$; that is,

$$A(\Delta, w) = \bigoplus_{k\geq 0} I(\Delta, w)^{(k)} t^k,$$

where $I(\Delta, w)^{(k)}$ denotes the k-th symbolic power $\bigcap_{F\in\Delta} P_F^{w_F k}$ of $I(\Delta, w)$.

We may view $A(\Delta, w)$ as a graded algebra over R with $A(\Delta, w)_k = I(\Delta, w)^{(k)} t^k$ for $k \geq 0$. It is obvious that $A(\Delta, w)$ is standard graded ($A(\Delta, w)$ is generated by elements of degree one) if and only if $I(\Delta, w)^{(k)} = I(\Delta, w)^k$ for all $k \geq 0$. Therefore, it is of great interest to know when $A(\Delta, w)$ is a standard graded algebra over R.

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This problem has a satisfactory answer in the case w is the canonical weight; i.e. $w_F = 1$ for all $F \in \Delta$. In this case, we will use the notation $A(\Delta)$ and $I^*(\Delta)$ instead of $A(\Delta, w)$ and $I(\Delta, w)$. Notice that ideals of the form $I^*(\Delta)$ are exactly squarefree monomial ideals of R. It was proved in [5, Theorem 1.4] (and implicitly in [2, Proposition 3.4] and [3, Theorem 3.5]) that $A(\Delta)$ is standard graded over R if and only if the blocker of Δ is a Mengerian hypergraph (see the last section for more details).

The aim of this paper is to characterize hypergraphs Δ for which $A(\Delta, w)$ is a standard graded algebra over R for all weight functions w. If Δ is a graph, such a characterization was already obtained in [4, Theorem 5.1 and Theorem 5.4], where it is proved that $A(\Delta, w)$ is a standard graded algebra over R for all weight functions w if and only if Δ is bipartite. This result can be fully generalized as follows.

Let $\Delta = \{F_1, ..., F_m\}$. Let M denote the edge-vertex incidence matrix of Δ; that is, $M = (e_{ij})$, $i = 1, ..., m$, $j = 1, ..., n$, with $e_{ij} = 1$ if $i \in F_j$ and $e_{ij} = 0$ if $i \notin F_j$. We say that Δ is a *unimodular hypergraph* if M is totally unimodular, i.e., if each subdeterminant of M is $0, \pm 1$.

Theorem 1.1. *$A(\Delta, w)$ is a standard graded algebra for all weight functions w if and only if Δ is a unimodular hypergraph.*

It is known that if Δ has no alternating chain $v_1, F_1, v_2, F_2, ..., v_s, F_s, v_{s+1} = v_1$ of odd length $s \geq 3$, where $v_1, ..., v_s$ and $F_1, ..., F_s$ are different vertices and edges of Δ and $v_i, v_{i+1} \in F_i$ for all $i = 1, ..., s$, then Δ is unimodular (see e.g. [1, Theorem 5, p. 164]). In particular, a graph is unimodular if and only if it is bipartite. Hence the aforementioned result of [4] is a special case of Theorem 1.1.

An interesting case of Theorem 1.1 concerns the hypergraph of all n-subsets of $n+1$ vertices. In this case, $I^*(\Delta)$ is the intersection of the defining ideals of the $n+1$ points $(0, ..., 1, ..., 0)$ in $\mathbb{P}^n$. Since every set of $n+1$ points in general position in $\mathbb{P}^n$ can be transformed into this case, we obtain the following consequence.

Corollary 1.2. *Let $P_0, ..., P_n$ be the defining ideals of $n+1$ points in general position in $\mathbb{P}^n$. Then*

$$(P_0^{w_0} \cap \cdots \cap P_n^{w_n})^k = P_0^{w_0 k} \cap \cdots \cap P_n^{w_n k},$$

for all integers $w_0, ..., w_s \geq 1$ and $k \geq 0$.

2. Decomposition property versus unimodularity

We adhere to the notions of the preceding section.

Let $\Delta = \{F_1, ..., F_m\}$ be a hypergraph on the set of vertices $\{1, ..., n\}$. Let w be a weight function on Δ and $\mathbf{w} = (w_{F_1}, ..., w_{F_m})$. Note that $\mathbf{w} \in \mathbb{Z}_+^m$. Let M be the edge-vertex incidence matrix of Δ. By definition, a vector $\mathbf{c} \in \mathbb{N}^n$ is a k-cover of the weighted hypergraph (Δ, w) if and only if $M \cdot \mathbf{c} \geq k\mathbf{w}$.

It is obvious that the vertex cover algebra $A(\Delta, w)$ is standard graded over R if and only if every monomial of $A(\Delta, w)_k$ is the product of k monomials of $A(\Delta, w)_1$ for all $k \geq 1$ or, equivalently, if and only if every k-cover of (Δ, w) can be written as a sum of k 1-covers of (Δ, w) for all $k \geq 1$. This observation leads us to the following notion in hypergraph theory.

Let Q be an arbitrary polyhedron in $\mathbb{R}^n$ and $kQ = \{k\mathbf{c} |\ \mathbf{c} \in Q\}$, $k \geq 1$. We say that Q has the *integer decomposition property* if for each $k \geq 1$ and each integral vector $\mathbf{c} \in kQ$ there exist integral vectors $\mathbf{c}_1, ..., \mathbf{c}_k \in Q$ such that $\mathbf{c} = \mathbf{c}_1 + \cdots + \mathbf{c}_k$.

If $Q = \{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \geq \mathbf{w}\}$, then $kQ = \{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \geq k\mathbf{w}\}$. Therefore, we have the following lemma.

Lemma 2.1. *$A(\Delta, w)$ is standard graded over R if and only if the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \geq \mathbf{w}\}$ has the integer decomposition property.*

If $Q = \{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \leq \mathbf{w}\}$, there is the following characterization of the integer decomposition property in terms of M.

Theorem 2.2 (Baum and Trotter; see e.g. [6, Theorem 19.4])**.** *Let M be an $m \times n$ integral matrix. Then the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \leq \mathbf{w}\}$ has the integer decomposition property for all integral vectors $\mathbf{w} \in \mathbb{Z}^m$ if and only if M is totally unimodular.*

Since M is totally unimodular if and only if $-M$ is totally unimodular and since $\mathbf{w}$ may have negative components, we can replace M by $-M$ and $\mathbf{w}$ by $-\mathbf{w}$ in Theorem 2.2. Therefore, Theorem 2.2 remains true if we replace the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \leq \mathbf{w}\}$ by the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \geq \mathbf{w}\}$, which appears in Lemma 2.1.

As a consequence, if the incidence matrix of a hypergraph Δ is totally unimodular, then $A(\Delta, w)$ is standard graded for all weight functions w. This proves the sufficient part of Theorem 1.1. The necessary part of Theorem 1.1 does not follow from Theorem 2.2 because of the condition $\mathbf{w} \in \mathbb{Z}^m$.

By Lemma 2.1, Theorem 1.1 follows from the following modification of Theorem 2.2 for integral matrices with non-negative entries and for positive integral vectors $\mathbf{w} \in \mathbb{Z}_+^m$.

Theorem 2.3. *Let M be an $m \times n$ integral matrix with non-negative entries. Then the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \geq \mathbf{w}\}$ has the integer decomposition property for all integral vectors $\mathbf{w} \in \mathbb{Z}_+^m$ if and only if M is totally unimodular.*

The proof of this theorem is given at the end of the next section. Following the proof of [6, Theorem 19.4], we need to prepare some results on the relationship between the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \geq \mathbf{w}\}$ and the total unimodularity of M.

3. Unimodularity versus integrality

Recall that a matrix is called *unimodular* if each maximal minor equals 0, ± 1 and that a rational polyhedron is called *integral* if all of its vertices are integral.

By a result of Veinott and Dantzig (see e.g. [6, Theorem 19.2]) unimodular integral matrices can be characterized by the integrality of associated polyhedra. This result can be modified for matrices with non-negative entries as follows.

Theorem 3.1. *Let M be an integral $m \times n$ matrix with non-negative entries of full row rank. Then M is unimodular if and only if the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} = \mathbf{w}\}$ is integral for all integral vectors $\mathbf{w} \in \mathbb{Z}_+^m$.*

Proof. If M is unimodular, then $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} = \mathbf{w}\}$ is integral by the aforementioned result of Veinott and Dantzig. For the converse assume that $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} = \mathbf{w}\}$ is integral for each integral vector $\mathbf{w} \in \mathbb{Z}_+^m$. Let E be an arbitrary non-singular maximal square submatrix of M. We have to prove that $\det E = \pm 1$. Since $\det E$ is an integer, E^{-1} is not an integral matrix if $\det E \neq \pm 1$. Therefore, it suffices to show that $E^{-1} \cdot \mathbf{a}$ is an integral vector for all $\mathbf{a} \in \mathbb{Z}_+^m$.

First, we can find an integral vector $\mathbf{a}' \in \mathbb{Z}_+^m$ such that $E^{-1} \cdot \mathbf{a} + \mathbf{a}' \in \mathbb{Q}_+^m$. Let $\mathbf{b} := E^{-1} \cdot \mathbf{a} + \mathbf{a}'$ and $\mathbf{w} := E \cdot \mathbf{b}$. Then $\mathbf{w} = \mathbf{a} + E \cdot \mathbf{a}' \in \mathbb{Z}_+^m$ because E has non-negative entries and no row of E is zero. Let $\mathbf{b}^*$ be the vector obtained from $\mathbf{b}$ by adding $n - m$ zero-components so that $M \cdot \mathbf{b}^* = E \cdot \mathbf{b} = \mathbf{w}$. Then $\mathbf{b}^*$ lies on at least $n - m$ facets of the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} = \mathbf{w}\}$. Hence $\mathbf{b}^*$ is a vertex of the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} = \mathbf{w}\}$. By the assumption, $\mathbf{b}^*$ is integral. Therefore, $\mathbf{b}$ and hence $E^{-1} \cdot \mathbf{a} = \mathbf{b} - \mathbf{a}'$ are integral vectors. $\square$

The following corollary is again a modification of a result of Hoffman and Kruskal for integral matrices with non-negative entries (see e.g. [6, Corollary 19.2a]).

Corollary 3.2. *An integral $m \times n$ matrix M with non-negative entries is totally unimodular if and only if the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \geq \mathbf{w}\}$ is integral for all integral vectors $\mathbf{w} \in \mathbb{Z}_+^m$.*

Proof. Let I denote the unit matrix of rank m. It is well known that M is totally unimodular if and only if the composed matrix (I, M) is unimodular. On the other hand, the vertices of the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \geq \mathbf{w}\}$ are integral if and only if the vertices of the polyhedron $\{\mathbf{b} \geq 0 |\ (I, M) \cdot \mathbf{b} = \mathbf{w}\}$ in $\mathbb{R}^{m+n}$ are integral. Therefore, the assertion follows from Theorem 3.1. $\square$

We shall use Corollary 3.2 to prove Theorem 2.3. As observed in the preceding section, Theorem 1.1 follows from Theorem 2.3.

Proof of Theorem 2.3. The necessity already follows from Theorem 2.2. To prove the sufficiency we assume that the polyhedron $Q = \{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} \geq \mathbf{w}\}$ has the integer decomposition property for all integral vectors $\mathbf{w} \in \mathbb{Z}_+^m$. By Corollary 3.2, we only need to show that Q is integral. Let $\mathbf{a}$ be an arbitrary vertex of Q. Suppose that $\mathbf{a}$ is not integral. Let k be the least common multiple of the denominators occuring in $\mathbf{a}$. Put $\mathbf{c} := k\mathbf{a}$. Then $\mathbf{c}$ is an integral vector in kQ. Therefore, there are integral vectors $\mathbf{c}_1, ..., \mathbf{c}_k \in Q$ such that $\mathbf{c} = \mathbf{c}_1 + \cdots + \mathbf{c}_k$. From this it follows that $\mathbf{a} = (\mathbf{c}_1 + \cdots + \mathbf{c}_k)/k$. Since $\mathbf{a}$ is a vertex of Q, we must have $\mathbf{c}_1 = \cdots = \mathbf{c}_k$ and hence $\mathbf{a} = \mathbf{c}_1 \in \mathbb{N}^n$. $\square$

4. Remarks

For a vector $\mathbf{w}$ we will use the notation $\mathbf{w} \gg 0$ if all components of $\mathbf{w}$ are large enough. In the proof of Theorem 3.1 we may choose $\mathbf{b} \gg 0$, which implies $\mathbf{w} \gg 0$. Therefore, we obtain the following stronger statement for the converse of Theorem 3.1.

Theorem 4.1. *Let M be an integral $m \times n$ matrix with non-negative entries of full row rank. Then M is unimodular if the polyhedron $\{\mathbf{c} \geq 0 |\ M \cdot \mathbf{c} = \mathbf{w}\}$ is integral for all integral vectors $\mathbf{w} \gg 0$ of $\mathbb{Z}_+^m$.*

Similarly as above, this result yields the following improvement of Theorem 1.1, where $w \gg 0$ means $\mathbf{w} \gg 0$.

Theorem 4.2. *If $A(\Delta, w)$ is a standard graded algebra for all weight functions $w \gg 0$, then Δ is a unimodular hypergraph.*

The above theorem doesn't hold for a sequence of weight functions $w \gg 0$. In fact, if Δ is a unimodular hypergraph, then $A(\Delta)$ is standard graded over R by Theorem 1.1. But $A(\Delta)$ need not be standard graded over R if $A(\Delta, w)$ is standard graded over R for a sequence of weight functions $w \gg 0$. This follows from the following observation.

Lemma 4.3. *For every hypergraph Δ there exists a number d such that $A(\Delta, w)$ is a standard graded algebra over R for all weight functions $w_F = kd$, $F \in \Delta$, $k \geq 1$.*

Proof. It is known that $A(\Delta)$ is always a finitely generated algebra over R [4, Theorem 4.1]. Therefore, there exists a number d such that the Veronese subalgebra $A(\Delta)^{(kd)}$ is standard graded over R for all $k \geq 1$ [4, Theorem 2.1]. But $A(\Delta)^{(kd)} = A(\Delta, w)$ where w is the weight function $w_F = kd$, $F \in \Delta$. □

On the other hand, one may ask whether $A(\Delta, w)$ is a standard graded algebra over R for all weight functions w if $A(\Delta)$ is standard graded.

This question has a positive answer if Δ is a graph. In this case, if $A(\Delta)$ is standard graded over R, then Δ is bipartite [4, Theorem 5.1]; whence $A(\Delta, w)$ is standard graded for all weight functions w by [4, Theorem 5.4]. However, we can give an example of a hypergraph Δ such that $A(\Delta)$ is standard graded over R, whereas $A(\Delta, w)$ is not standard graded over R for all weight functions w. For that we shall need the following result.

Let M be the edge-vertex incidence matrix of a hypergraph Δ. Let $m \times n$ be the size of M. One calls Δ a *Mengerian hypergraph* if

$$\min\{\mathbf{a} \cdot \mathbf{c} \mid \mathbf{a} \in \mathbb{N}^n,\ M \cdot \mathbf{a} \geq \mathbf{1}\} = \max\{\mathbf{b} \cdot \mathbf{1} \mid \mathbf{b} \in \mathbb{N}^m,\ M^T \cdot \mathbf{b} \leq \mathbf{c}\},$$

for all $\mathbf{c} \in \mathbb{N}^n$, where $\mathbf{1}$ denotes the vector $(1, ..., 1) \in \mathbb{N}^m$. One calls the hypergraph of the minimal covers of Δ the *blocker* of Δ, which we denote by Δ^*. It is proved in [5, Theorem 1.4] that $A(\Delta)$ is standard graded over R if and only if Δ^* is Mengerian.

Example 4.4. Let Δ be the hypergraph on 6 vertices which has the edges

$$\{1, 2, 3\}, \{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}.$$

Then Δ^* is the simplicial complex with the edges

$$\{1, 4\}, \{2, 5\}, \{3, 6\}, \{1, 2, 3\}, \{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}.$$

It is known that Δ^* is Mengerian, whereas Δ is not (see e.g. [4, Example 1.8]). Therefore, $A(\Delta)$ is standard graded over R. On the other hand, since unimodular hypergraphs are Mengerian (see e.g. [1, Corollary 1, p. 170]), Δ is not unimodular. Therefore, $A(\Delta, w)$ is not standard graded over R for all weight functions w.

Finally, we would like to point out that one can test totally unimodular matrices (and hence unimodular hypergraphs) in polynomial time (see e.g. [6, Chapter 19]).

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