

## FIXED SET THEOREMS OF KRASNOSELSKIĬ TYPE

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ABSTRACT. We revisit the fixed point problem for the sum of a compact operator and a continuous function, where the domain on which these maps are defined is not necessarily convex, the former map is allowed to be multi-valued, and the latter to be a semicontraction and/or a suitable nonexpansive map. In this setup, guaranteeing the existence of fixed points is impossible, but two types of invariant-like sets are found to exist.

### 1. INTRODUCTION

The Krasnoselskiĭ fixed point theorem (cf. [9] and [13]) is a famous fixed point principle which is useful for solving certain nonlinear functional equations and studying their stability (see [2], [4] and [5]). The classical statement of this principle, which generalizes the fundamental fixed point theorems of *both* Banach and Schauder, reads as follows:

**The Krasnoselskiĭ fixed point theorem.** *Let  $S$  be a nonempty closed and convex subset of a Banach space  $X$ . Suppose that  $\varphi : S \rightarrow X$  and  $\gamma : S \rightarrow X$  satisfy the following properties:*

- (i)  $\varphi$  is a contraction;
- (ii)  $\gamma$  is compact and continuous;
- (iii)  $\varphi(S) + \gamma(S) \subseteq S$ .

*Then there exists an  $x \in S$  such that  $x = \varphi(x) + \gamma(x)$ .*

The objective of this paper is to offer suitable modifications of this result when the hypothesis of convexity of  $S$  is relaxed. Obviously, the conclusion of the theorem need not hold in this case, even if  $\varphi$  is the zero operator. Our point of departure is the observation that when the lack of a fixed point of an operator is due to the nonconvexity of the domain, one may still be able to find a fixed (invariant) set of that operator [11]. Given that invariant sets play an important role in various branches of applied nonlinear analysis, say, in dynamical systems (in the garb of a global attractor of a semiflow) or in fractal geometry (in the garb of a self-similar set), it seems worthwhile to pursue the implications of this idea in the context of the Krasnoselskiĭ fixed point theorem.

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There are at least two interesting ways of reformulating Krasnoselskiĭ's said theorem in which the "fixed point" of the original result is allowed to be a "set." First, we may look for a nonempty subset  $T$  of  $S$  such that

$$T = \varphi(T) + \gamma(T).$$

Indeed, the existence of such a set is guaranteed under the conditions of the original theorem (with  $S$  being an arbitrary nonempty closed subset of  $X$ ). In fact, we find that this is the case even if  $\varphi$  is a semicontraction and  $\gamma$  a compact and continuous set-valued map. If, in addition,  $\varphi$  is compact, then  $T$  can be taken to be compact here, even when  $\varphi$  is only known to be continuous.

Looking for a fixed point of the map  $A \mapsto \varphi(A) + \gamma(A)$  on  $2^S \setminus \{\emptyset\}$  is not the only way one can modify the Krasnoselskiĭ fixed point theorem when  $S$  is nonconvex. A second, perhaps more interesting, modification is to look for a nonempty  $K \subseteq S$  such that

$$\{x - \varphi(x) : x \in K\} = \gamma(K).$$

This statement reduces as well to the conclusion of the Krasnoselskiĭ fixed point theorem when  $K$  is a singleton. We show below that such a compact set  $K$  exists when  $S$  is a nonempty closed subset of  $X$ ,  $\varphi$  is a semicontraction and  $\gamma$  a compact and continuous set-valued map. If  $S$  is known to be star-shaped, then the statement remains valid still for any compact and nonexpansive map  $\varphi$ .

## 2. SUMS OF COMPACT MAPS WITH COMPACT CORRESPONDENCES

**2.1. Definition.** Let  $X, Y$  and  $Z$  be any topological spaces. By a **correspondence** from  $X$  to  $Y$  we understand a function  $\Gamma$  that maps  $X$  into  $2^Y \setminus \{\emptyset\}$ ; in this case we write  $\Gamma : X \rightarrow Y$ . (Throughout the exposition, single-valued correspondences and functions are identified with each other.) For any subset  $S$  of  $X$ , we write  $\Gamma(S)$  for  $\bigcup\{\Gamma(x) : x \in S\}$ . We say that  $\Gamma$  is **compact** if  $\Gamma(X)$  is a relatively compact subset of  $Y$ , and **closed-valued** if  $\Gamma(x)$  is a closed subset of  $Y$  for each  $x \in X$ . The correspondence  $\Gamma$  is said to be **continuous** if it is both **upper hemicontinuous** (i.e.  $\{x \in X : \Gamma(x) \subseteq O\}$  is open for every open  $O \subseteq Y$ ) and **lower hemicontinuous** (i.e.  $\{x \in X : \Gamma(x) \cap O \neq \emptyset\}$  is open for every open  $O \subseteq Y$ ).

If  $X = Y$ , we refer to  $\Gamma$  as a **self-correspondence** on  $X$ , while any function in  $X^X$  is called a **self-map**. The **composition** of two correspondences  $\Gamma : X \rightarrow Y$  and  $\Upsilon : Y \rightarrow Z$  is the correspondence  $\Gamma \circ \Upsilon : X \rightarrow Z$  defined by  $\Gamma \circ \Upsilon(x) := \Gamma(\Upsilon(x))$ .

**2.2. Notation.** For any topological space  $X$ , we denote by  $\mathbf{c}(X)$  the class of all nonempty compact subsets of  $X$ . If  $X$  is a metric space, we view  $\mathbf{c}(X)$  also as a metric space under the Hausdorff metric. It is well-known that the metric topology of  $\mathbf{c}(X)$  induced this way coincides with the classical Vietoris topology. (See [1, Theorem 3.91].)

The following almost-fixed-set principle will be used throughout the paper. It can be proved by invoking Theorem 4.6 of [11], but we produce here a more direct argument.

**2.3. Lemma.** *Let  $X$  be a topological space, and  $S$  a nonempty closed subset of  $X$ . If  $\Phi$  is a compact and lower hemicontinuous self-correspondence on  $S$ , then there exists a minimal  $K \in \mathbf{c}(S)$  such that  $K = \text{cl } \Phi(K)$ .*

*Proof.* Note first that  $\Phi(\text{cl}A) \subseteq \text{cl}\Phi(A)$  for any  $A \subseteq S$ . Indeed, if  $y \in \Phi(x)$  for some  $A \subseteq S$  and  $x \in \text{cl}A$ , then  $x \in S$  (because  $S$  is closed), and we have  $x_\tau \rightarrow x$  for some net  $(x_\tau)$  in  $A$ . By lower hemicontinuity of  $\Phi$ , then, there exist a subnet of indices  $(\tau_\alpha)$  and elements  $y_\alpha \in \Phi(x_{\tau_\alpha})$  for each  $\alpha$  such that  $y_\alpha \rightarrow y$ . (See [1, Theorem 17.19].) Thus  $y \in \text{cl}\Phi(A)$ , as asserted.

Now define  $T := \text{cl}\Phi(S)$ . By what is just established, we have  $\Phi(T) \subseteq \text{cl}\Phi(\Phi(S)) \subseteq \text{cl}\Phi(S) = T$ , that is,  $\Psi := \Phi|_T$  is a self-correspondence on  $T$ . Let  $\mathcal{A}$  stand for the class of all  $A \in \mathbf{c}(T)$  such that  $\text{cl}\Psi(A) \subseteq A$ . Obviously, any chain (linearly ordered set) in the poset  $(\mathcal{A}, \supseteq)$  has the finite intersection property, so compactness of  $T$  ensures that the intersection of all members of any chain in  $(\mathcal{A}, \supseteq)$  is nonempty. It follows that any chain in  $(\mathcal{A}, \supseteq)$  has a lower bound in  $\mathcal{A}$ . So, by Zorn's lemma,  $(\mathcal{A}, \supseteq)$  has a minimal element, say  $K$ . By definition,  $\text{cl}\Psi(K) \subseteq K$ . Moreover, this implies  $\Psi(\text{cl}\Psi(K)) \subseteq \Psi(K) \subseteq \text{cl}\Psi(K)$ . Then  $\text{cl}\Psi(K) \in \mathcal{A}$ , and hence  $\text{cl}\Phi(K)$  cannot be a proper subset of  $K$ . Thus,  $K = \text{cl}\Psi(K) = \text{cl}\Phi(K)$ , while we have  $K \in \mathbf{c}(S)$  because  $\mathbf{c}(T) \subseteq \mathbf{c}(S)$ .  $\square$

It may be worth noting that  $K$  can be guaranteed to be a singleton in Lemma 2.3 under suitable convexity conditions. In particular, suppose  $X$  is a normed linear space, and  $S$  and  $\Phi$  are as in Lemma 2.3. As shown by Wu [14], if, in addition,  $S$  is convex and  $\Phi$  is convex- and closed-valued, then  $\Phi$  has a fixed point.

The following result obtains upon combining this fixed set principle with a standard hyperspace argument.

**2.4. Theorem.** *Let  $S$  be a nonempty subset of a normed linear space  $X$ . Suppose that  $\varphi : S \rightarrow X$  and  $\Gamma : S \rightarrow X$  satisfy the following properties:*

- (i)  $\varphi$  is compact and continuous;
- (ii)  $\Gamma$  is compact, continuous and closed-valued;
- (iii)  $\varphi(S) + \Gamma(S) \subseteq S$ .

*Then, there exists a  $T \in \mathbf{c}(S)$  such that  $T = \varphi(T) + \Gamma(T)$ .*

*Proof.* Define

$$Y := \text{cl}\varphi(S) + \text{cl}\Gamma(S).$$

By compactness of  $\varphi$  and  $\Gamma$ ,  $Y$  is a nonempty compact subset of  $S$ , so we have  $\mathbf{c}(Y) \subseteq \mathbf{c}(S)$ . Now define  $f : \mathbf{c}(Y) \rightarrow 2^S$  by  $f(K) := \varphi(K) + \Gamma(K)$ . Since  $\varphi$  is continuous and  $\Gamma$  is compact-valued and upper hemicontinuous, both  $\varphi(K)$  and  $\Gamma(K)$  are compact subsets of  $X$ , while  $\varphi(K) + \Gamma(K) \subseteq Y$ , for any  $K \in \mathbf{c}(Y)$ . It follows that  $f$  is a self-map on  $\mathbf{c}(Y)$ . Moreover, it is well-known that continuity of  $\varphi$  and continuity and compact-valuedness of  $\Gamma$  ensure that, respectively,  $A \mapsto \varphi(A)$  and  $A \mapsto \Gamma(A)$  are continuous maps from  $\mathbf{c}(S)$  into  $\mathbf{c}(X)$ . It follows that  $f$  is a continuous function. By Lemma 2.3, then, there exists a  $\mathcal{K} \in \mathbf{c}(\mathbf{c}(Y))$  such that  $\mathcal{K} = \text{cl}f(\mathcal{K})$ . Since  $f$  is continuous,  $f(\mathcal{K})$  is compact, and hence closed, so we have  $\mathcal{K} = f(\mathcal{K})$ . Finally, we define  $T := \bigcup \mathcal{K}$ . Then

$$T = \bigcup \{f(K) : K \in \mathcal{K}\} = f\left(\bigcup \mathcal{K}\right) = f(T) = \varphi(T) + \Gamma(T).$$

Since  $Y$  is compact, the topology induced by the Hausdorff metric is identical to the Vietoris (finite) topology on  $\mathbf{c}(Y)$ . It follows that  $T$  is a compact subset of  $Y$ , being a compact union of compact subsets of  $S$ . (See Michael [10, Theorem 2.5].) Thus  $T \in \mathbf{c}(S)$ , and the proof is complete.  $\square$

Compactness of  $\varphi$  is essential for ensuring the compactness of  $T$  in this result. For instance, if  $X = \mathbb{R} = S$ ,  $\Gamma = \{0\}$ , and  $\varphi = \text{id}_{\mathbb{R}} + 1$ , then  $T = \varphi(T) + \Gamma(T)$  holds for no proper subset  $T$  of  $\mathbb{R}$ .

### 3. SUMS OF SEMICONTRACTIONS WITH COMPACT CORRESPONDENCES

In this section we work with maps that satisfy a contraction-like property which is a minor modification of the notion of “nonlinear contraction” introduced by Boyd and Wong [3]. We introduce this property for self-maps in the context of normed linear spaces.

**3.1. Definition.** Let  $X$  be a normed linear space, and  $\emptyset \neq S \subseteq X$ . We say that a map  $\varphi : S \rightarrow X$  is a **semicontraction** if there exists an upper semicontinuous self-map  $\beta$  on  $\mathbb{R}_+$  such that

$$(3.1) \quad \beta(t) < t \text{ for all } t > 0 \quad \text{and} \quad \liminf_{t \rightarrow \infty} (t - \beta(t)) > 0,$$

while

$$(3.2) \quad \|\varphi(x) - \varphi(y)\| \leq \beta(\|x - y\|) \quad \text{for all } x, y \in S.$$

Semicontractions play an essential role in various generalizations of the Banach Fixed Point Theorem. It is clear that every contraction is a semicontraction, and every semicontraction is continuous. Moreover, if  $S$  is a nonempty closed subset of a Banach space  $X$  and  $\varphi : S \rightarrow X$  is a semicontraction with  $\varphi(S) \subseteq S$ , then  $\varphi$  has a unique fixed point (cf. [3]).<sup>1</sup>

The following is the main result of this section.

**3.2. Theorem.** *Let  $S$  be a nonempty closed subset of a Banach space  $X$ . Suppose that  $\varphi : S \rightarrow X$  and  $\Gamma : S \rightarrow X$  satisfy the following properties:*

- (i)  $\varphi$  is a semicontraction;
- (ii)  $\Gamma$  is compact, continuous and closed-valued;
- (iii)  $\varphi(S) + \Gamma(S) \subseteq S$ .

*Then, there exists a minimal  $K \in \mathbf{c}(S)$  such that*

$$(3.3) \quad \{x - \varphi(x) : x \in K\} = \Gamma(K).$$

*Proof.* Just as in the classical proof of the Krasnoselskiĭ fixed point theorem, the argument is based on the analysis of the map  $F := \text{id}_S - \varphi$ .

*Claim 1.*  $\text{cl}\Gamma(S) \subseteq F(S)$ .

*Proof of Claim 1.* Take any  $y \in \text{cl}\Gamma(S)$ . Hypothesis (iii) and the closedness of  $S$  guarantee that the map  $\omega \mapsto y + \varphi(\omega)$  is a self-map on  $S$ . But (i) ensures that this map is a semicontraction and has a unique fixed point, say  $x$ , in  $S$ . Then,  $y = x - \varphi(x) \in F(S)$ .

*Claim 2.*  $F^{-1}|_{\text{cl}\Gamma(S)}$  is a continuous single-valued correspondence.

*Proof of Claim 2.* For any  $y \in \text{cl}\Gamma(S)$ ,  $F^{-1}(y)$  is the set of all fixed points of the self-map  $\omega \mapsto y + \varphi(\omega)$  on  $S$ . Since this map has a unique fixed point, we may then conclude that  $F^{-1}|_{\text{cl}\Gamma(S)}$  is single-valued. To prove the second assertion, define the self-map  $\alpha$  on  $\mathbb{R}_+$  by  $\alpha(t) := t - \beta(t)$ , where  $\beta$  is an upper semicontinuous self-map

<sup>1</sup>The second requirement in (3.1) is not needed for the validity of the said fixed point theorem.

on  $\mathbb{R}_+$  that satisfies (3.1) and (3.2). Obviously,  $\alpha$  is lower semicontinuous, and we have  $\alpha(0) = 0$ ,  $\alpha((0, \infty)) \subseteq (0, \infty)$  and  $\liminf_{t \rightarrow \infty} \alpha(t) > 0$ .

Now fix any  $y \in \text{cl}\Gamma(S)$ , and take any sequence  $(y_m)$  in  $\text{cl}\Gamma(S)$  such that  $y_m \rightarrow y$ . Define  $a_m := \|F^{-1}(y) - F^{-1}(y_m)\|$  for each  $m \in \mathbb{N}$ . We wish to show that  $a_m \rightarrow 0$ . To this end, note first that

$$\begin{aligned} \|y - y_m\| &= \|F(F^{-1}(y)) - F(F^{-1}(y_m))\| \\ &= \|F^{-1}(y) - F^{-1}(y_m) + \varphi(F^{-1}(y_m)) - \varphi(F^{-1}(y))\| \\ &\geq \|F^{-1}(y) - F^{-1}(y_m)\| - \beta (\|F^{-1}(y) - F^{-1}(y_m)\|) \\ &= \alpha(a_m) \end{aligned}$$

for each  $m = 1, 2, \dots$ , so  $\alpha(a_m) \rightarrow 0$ . Since  $\liminf_{t \rightarrow \infty} \alpha(t) > 0$ , it follows that  $(a_m)$  is bounded. Moreover, by the lower semicontinuity of  $\alpha$ , for any convergent subsequence  $(a_{m_k})$  of  $(a_m)$ , we have  $0 = \liminf \alpha(a_{m_k}) \geq \alpha(\lim a_{m_k})$  so that  $\alpha(\lim a_{m_k}) = 0$ , that is,  $a_{m_k} \rightarrow 0$ . Thus  $\limsup a_m = 0$ , as we sought.

Combining Claims 1 and 2 and hypothesis (ii), we find that  $F^{-1} \circ \Gamma$  is a continuous self-correspondence on  $S$ . Moreover, since  $\text{cl}\Gamma(S)$  is compact and  $F^{-1}|_{\text{cl}\Gamma(S)}$  is continuous,  $F^{-1}(\text{cl}\Gamma(S))$  is compact. So, since  $\text{cl}F^{-1}(\Gamma(S)) \subseteq \text{cl}F^{-1}(\text{cl}\Gamma(S))$ , we conclude that  $F^{-1} \circ \Gamma$  is compact. By Lemma 2.3, then, we have  $K = \text{cl}F^{-1} \circ \Gamma(K)$  for some minimal  $K \in \mathbf{c}(S)$ . But, since  $F$  is continuous and  $\Gamma$  is closed-valued,  $F^{-1} \circ \Gamma$  is compact-valued, so it maps compact sets to compact sets. Then  $F^{-1} \circ \Gamma(K)$  is a compact subset of  $S$ , so  $K = F^{-1} \circ \Gamma(K)$ , which is equivalent to (3.3).  $\square$

The following consequence of Theorem 3.2 shows that under the conditions of this result the conclusion of Theorem 2.4 is valid, albeit compactness of the “fixed” set  $T$  is not ensured.

**3.3. Corollary.** *Under the conditions of Theorem 3.2, there exists a nonempty (maximal)  $T \in 2^S$  such that  $T = \varphi(T) + \Gamma(T)$ .*

*Proof.* Let  $K$  be as found in Theorem 3.2. It is readily checked that this implies  $K \subseteq \varphi(K) + \Gamma(K)$ ; that is, the family

$$\mathcal{A} := \{A \in \mathbf{c}(S) : A \subseteq \varphi(A) + \Gamma(A)\}$$

is nonempty. We define  $T := \bigcup \mathcal{A}$ . Obviously,  $T$  is a nonempty subset of  $S$ . Moreover,  $T \subseteq \varphi(T) + \Gamma(T)$ , as can be readily verified. Moreover, if  $y \in \varphi(T) + \Gamma(T)$ , then  $T \cup \{y\} \in \mathcal{A}$ , and hence  $T \cup \{y\} \subseteq T$ , that is,  $y \in T$ . It follows that  $T = \varphi(T) + \Gamma(T)$ . That  $T$  is the maximal such set is straightforward.  $\square$

We do not know at present if  $T$  can be taken to be a compact set in the statement of Corollary 3.3.

The literature provides many interesting variations of the Krasnoselskiĭ fixed point theorem. In most cases, these variations can be adapted to the case of Theorem 3.2 as well. We provide two illustrations.

**3.4. Remarks.** (a) In the version of the Krasnoselskiĭ fixed point theorem obtained by Burton [4],  $\varphi$  is taken as a self-map on  $X$ , and hypothesis (iii) is replaced by the following weaker requirement:  $x \in \varphi(x) + \gamma(S)$  implies  $x \in S$ . This variation applies to a good number of functional equations that cannot be dealt with in the original version of the theorem. It is thus worth noting that Theorem 3.2 can be amended in this way as well. Put precisely, in the statement of this result, we may assume  $\varphi \in X^X$  and, instead of (iii), require that  $x \in \varphi(x) + \text{cl}\Gamma(S)$  implies  $x \in S$ .

(b) Sehgal and Singh [12] have extended the Krasnoselskiĭ fixed point theorem to the case of the sum of a semicontraction and a compact map defined over a sequentially complete subset of a locally convex space. This extension holds true in the case of Theorem 3.2 as well. Put precisely, in Theorem 3.2 it is enough to assume that  $S$  is a sequentially complete subset of a locally convex Hausdorff topological linear space  $X$ , provided that for each of the seminorms  $p$  that induce the topology of  $X$ , there is an upper semicontinuous self-map  $\beta$  on  $\mathbb{R}_+$  such that (3.1) holds and  $p(\varphi(x) - \varphi(y)) \leq \beta(p(x - y))$  for all  $x, y \in S$ .  $\square$

Finally, we consider a brief application of Theorems 2.4 and 3.2 to the theory of self-similarity.

**3.5. Example (Self-similar sets).** Let  $S$  be a nonempty closed subset of a Banach space. If  $f_1, \dots, f_n$  are finitely many self-maps on  $S$ , then the list  $(S, \{f_1, \dots, f_n\})$  is called an **iterated function system (IFS)**. If each  $f_i$  here is continuous, we refer to this IFS as **continuous**, and if each  $f_i$  is a contraction, then we say that it is **contractive**. A nonempty set  $T \subseteq S$  is said to be **self-similar** with respect to the IFS  $(S, \{f_1, \dots, f_n\})$  if

$$T = f_1(T) \cup \dots \cup f_n(T).$$

A well-known theorem of fractal geometry says that there exists a unique compact self-similar set with respect to any contractive IFS (Hutchinson [8]). By choosing  $\varphi$  to be the zero operator in Theorem 2.4, we see that, in fact, there exists a compact self-similar set with respect to any continuous IFS  $(S, \{f_1, \dots, f_n\})$ , provided that each  $f_i$  is compact. Moreover, Theorem 3.2 provides information about the stability of self-similar sets with respect to compact perturbations: If  $(S, \{f_1, \dots, f_{n+1}\})$  is a contractive IFS, then there is a nonempty subset  $T$  of  $S$  such that

$$\{x - f_{n+1}(x) : x \in T\} = f_1(T) \cup \dots \cup f_n(T),$$

provided that each  $f_i$  is compact and  $f_i(S) + f_{n+1}(S) \subseteq S$ ,  $i = 1, \dots, n$ .

#### 4. SUMS OF NONEXPANSIVE MAPS WITH COMPACT CORRESPONDENCES

If  $S$  were compact in Theorem 3.2, then we could ask  $\varphi$  to satisfy a weaker contraction property. For instance, in this case we can replace hypothesis (i) of this result with  $\|\varphi(x) - \varphi(y)\| < \|x - y\|$  for all distinct  $x, y \in X$ . (The proof is identical to that of Theorem 3.2, except now, for proving Claim 1, one uses Edelstein's fixed point theorem instead of the Boyd-Wong fixed point theorem.) Easy examples show that this is not the case when  $\|\varphi(x) - \varphi(y)\| \leq \|x - y\|$  for all  $x, y \in X$  (i.e. when  $\varphi$  is *nonexpansive*). Nevertheless, the situation reads much better if  $S$  is known to be star-shaped.

**4.1. Theorem.** *Let  $S$  be a nonempty closed star-shaped subset of a Banach space  $X$ . Suppose that  $\varphi : S \rightarrow X$  and  $\Gamma : S \rightarrow X$  satisfy the following properties:*

- (i)  $\varphi$  is compact and nonexpansive;
- (ii)  $\Gamma$  is compact, continuous and closed-valued;
- (iii)  $\varphi(S) + \Gamma(S) \subseteq S$ .

*Then, there exists a minimal  $K \in \mathbf{c}(S)$  such that  $\{x - \varphi(x) : x \in K\} = \Gamma(K)$ .*

*Proof.* Let  $x_*$  be a star-point of  $S$ , and for each positive integer  $m$ , define

$$\varphi_m := \left(1 - \frac{1}{m}\right) \varphi \quad \text{and} \quad \Gamma_m := \left(1 - \frac{1}{m}\right) \Gamma + \frac{1}{m} x_*.$$

Clearly, for each  $m$ , the map  $\varphi_m : S \rightarrow X$  is a contraction and  $\Gamma_m$  is a compact, continuous, and closed-valued correspondence from  $S$  to  $X$ . Moreover, if  $x, y \in S$ , then

$$\begin{aligned} \varphi_m(S) + \Gamma_m(S) &= \left(1 - \frac{1}{m}\right) (\varphi(S) + \Gamma(S)) + \frac{1}{m}x_* \\ &\subseteq \left(1 - \frac{1}{m}\right) S + \frac{1}{m}x_* \\ &\subseteq S, \end{aligned}$$

$m = 1, 2, \dots$  Therefore, by Theorem 3.2, there is a sequence  $(K_m)$  in  $\mathbf{c}(S)$  such that  $K_m^* = \Gamma_m(K_m)$ , where  $K_m^* := \{x - \varphi_m(x) : x \in K_m\}$ ,  $m = 1, 2, \dots$

It is well-known that compactness of  $S$  implies that of  $\mathbf{c}(S)$ . Thus,  $(K_m)$  has a convergent subsequence. Relabelling if necessary, we denote this subsequence again by  $(K_m)$ , and write  $K_m \xrightarrow{\text{H}} K$ , where  $K \in \mathbf{c}(S)$  and  $\xrightarrow{\text{H}}$  denotes convergence with respect to the Hausdorff metric. Since  $\Gamma$  is continuous and compact-valued, the map  $A \mapsto \Gamma(A)$  is a continuous map from  $\mathbf{c}(S)$  into  $\mathbf{c}(X)$ , so we have  $\Gamma(K_m) \xrightarrow{\text{H}} \Gamma(K)$ . Moreover, we have  $\theta := \sup\{\|y\| : y \in \text{cl}(\Gamma(S))\} < \infty$  by compactness of  $\Gamma$ . Then,

$$\begin{aligned} d^{\text{H}}(\Gamma_m(K_m), \Gamma(K)) &= d^{\text{H}}\left(\left(1 - \frac{1}{m}\right)\Gamma(K_m) + \frac{1}{m}x_*, \Gamma(K)\right) \\ &\leq d^{\text{H}}\left(\left(1 - \frac{1}{m}\right)\Gamma(K_m) + \frac{1}{m}x_*, \Gamma(K_m)\right) + d^{\text{H}}(\Gamma(K_m), \Gamma(K)) \\ &\leq \frac{1}{m} \sup_{y \in \Gamma(K_m)} \|y\| + \frac{1}{m} \|x_*\| + d^{\text{H}}(\Gamma(K_m), \Gamma(K)) \\ &\leq \frac{1}{m}(\theta + \|x_*\|) + d^{\text{H}}(\Gamma(K_m), \Gamma(K)) \\ &\rightarrow 0, \end{aligned}$$

where  $d^{\text{H}}$  stands for the Hausdorff metric on  $\mathbf{c}(X)$ . Thus  $\Gamma_m(K_m) \xrightarrow{\text{H}} \Gamma(K)$ .

We next claim that  $K_m^* \xrightarrow{\text{H}} K^*$ , where  $K^* := \{x - \varphi(x) : x \in K\}$ ; that is, we wish to show that

$$\sup_{\omega \in K_m^*} d_{\|\cdot\|}(\omega, K^*) \rightarrow 0 \quad \text{and} \quad \sup_{\omega \in K^*} d_{\|\cdot\|}(\omega, K_m^*) \rightarrow 0,$$

where  $d_{\|\cdot\|}$  is the metric induced by the norm of  $X$ . Indeed,

$$\begin{aligned} \sup_{\omega \in K_m^*} d_{\|\cdot\|}(\omega, K^*) &= \sup_{x \in K_m} \inf_{y \in K} \|x - (1 - \frac{1}{m})\varphi(x) - y + \varphi(y)\| \\ &\leq \sup_{x \in K_m} \inf_{y \in K} (\|x - y\| + \|\varphi(x) - \varphi(y)\|) + \frac{1}{m} \sup_{z \in S} \|\varphi(z)\| \\ &\leq 2 \sup_{x \in K_m} \inf_{y \in K} \|x - y\| + \frac{1}{m} \sup_{z \in S} \|\varphi(z)\| \\ &\leq 2d^{\text{H}}(K_m, K) + \frac{1}{m} \sup_{z \in S} \|\varphi(z)\| \\ &\rightarrow 0, \end{aligned}$$

where the third inequality is due to the nonexpansiveness of  $\varphi$ ,  $d^{\text{H}}$  stands for the Hausdorff metric on  $\mathbf{c}(S)$ , and the final step uses the compactness of  $\text{cl}\varphi(S)$ . As one can similarly show that  $\sup_{\omega \in K^*} d_{\|\cdot\|}(\omega, K_m^*) \rightarrow 0$ , we conclude that  $K_m^* \xrightarrow{\text{H}} K^*$ .

Combining the findings of the previous two paragraphs and recalling that  $K_m^* = \Gamma(K_m)$  for each  $m$ , we conclude that  $\{x - \varphi(x) : x \in K\} = \Gamma(K)$ , as desired.  $\square$

**4.2. Remark.** In the special case where  $\Gamma$  is the zero operator, Theorem 4.1 reduces to the fact that every compact and nonexpansive self-map on a closed and star-shaped subset of a Banach space has a fixed point. This result generalizes a fixed

point theorem of Dotson [7] while being a special case of a fixed point theorem of Chandler and Faulkner [6].

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