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ON THE LIMIT POINTS OF $(a_n\xi)_{n=1}^{\infty}$ MOD 1 FOR SLOWLY INCREASING INTEGER SEQUENCES $(a_n)_{n=1}^{\infty}$

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ABSTRACT. In this paper, we are interested in sequences of positive integers $(a_n)_{n=1}^{\infty}$ such that the sequence of fractional parts $\{a_n\xi\}_{n=1}^{\infty}$ has only finitely many limit points for at least one real irrational number ξ. We prove that, for any sequence of positive numbers $(g_n)_{n=1}^{\infty}$ satisfying $g_n \geqslant 1$ and $\lim_{n\to\infty} g_n = \infty$ and any real quadratic algebraic number α, there is an increasing sequence of positive integers $(a_n)_{n=1}^{\infty}$ such that $a_n \leqslant n g_n$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty}\{a_n\alpha\} = 0$. The above bound on a_n is best possible in the sense that the condition $\lim_{n\to\infty} g_n = \infty$ cannot be replaced by a weaker condition. More precisely, we show that if $(a_n)_{n=1}^{\infty}$ is an increasing sequence of positive integers satisfying $\liminf_{n\to\infty} a_n/n < \infty$ and ξ is a real irrational number, then the sequence of fractional parts $\{a_n\xi\}_{n=1}^{\infty}$ has infinitely many limit points.

1. INTRODUCTION

By an old result of Weyl [16], for every increasing sequence of positive integers $(a_n)_{n=1}^{\infty}$, the set of real numbers ξ for which the sequence of fractional parts $\{a_n\xi\}_{n=1}^{\infty}$ is not uniformly distributed in $[0,1)$ is of Lebesgue measure zero. In particular, for almost all real ξ, the set $\{a_n\xi\}_{n=1}^{\infty}$ is everywhere dense in $[0,1)$. Of course, all rational numbers ξ are trivial exceptions, because the set of limit points of $\{a_n\xi\}_{n=1}^{\infty}$ is finite if $\xi \in \mathbb{Q}$. Another exception is related to the so-called PV-numbers, named after Pisot and Vijayaraghavan (see [11] and [15]). For instance, taking the PV-number $\sqrt{2}+1$ and setting $S_n = (\sqrt{2}+1)^n - (\sqrt{2}-1)^n \in \mathbb{N}$, we have $\lim_{n\to\infty}(\sqrt{2}S_n + S_n - S_{n+1}) = 0$. More precisely, $\{S_n\sqrt{2}\} \to 1$ as $n \to \infty$. So there is a geometrically growing sequence $(a_n)_{n=1}^{\infty}$ and a quadratic number ξ such that $\{a_n\xi\}_{n=1}^{\infty}$ has a unique limit point. Erdös asked whether, for every sufficiently fast growing sequence of integers $(a_n)_{n=1}^{\infty}$, there are some non-trivial exceptional $\xi \notin \mathbb{Q}$ for which $\{a_n\xi\}_{n=1}^{\infty}$ is not dense in $[0,1)$. For every lacunary sequence $(a_n)_{n=1}^{\infty}$, namely, the sequence satisfying $a_{n+1} \geqslant \tau a_n$ for some $\tau > 1$ and each $n \in \mathbb{N}$, the question of Erdös was answered in the affirmative by de Mathan [5] and Pollington [12], independently. See also Hilfssatz III in Khintchine's paper [8].

However, if $(a_n)_{n=1}^{\infty}$ is a slowly increasing sequence of positive integers, then it can be no exceptional ξ in the sense that the sequence $\{a_n\xi\}_{n=1}^{\infty}$ is everywhere

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dense in $[0, 1)$ for every real irrational number ξ. In this direction, Furstenberg [6] proved a remarkable result which implies that if an increasing sequence of positive integers $a_1 < a_2 < a_3 < \dots$ is a multiplicative semigroup which is not generated by powers of a single integer, then the sequence of fractional parts $\{a_n\xi\}_{n=1}^{\infty}$ is everywhere dense in $[0, 1)$ for each irrational real number ξ. The set A is said to be a *multiplicative semigroup* if it is closed under multiplication, namely, if $aa' \in A$ for any $a, a' \in A$. For example, the set of integers of the form $p^k q^m$, where $p < q$ are two fixed primes and k, m run over all non-negative integers, is a multiplicative semigroup. It is easy to see that a semigroup with at least two generators must satisfy the condition $\lim_{n\to\infty} a_{n+1}/a_n = 1$.

Later, a simpler proof of Furstenberg's theorem was given by Boshernitzan [4], whereas the papers of Berend [1], [2], [3], Kra [9] and Urban [14] contain various generalizations of Furstenberg's result. See also [13] for a collection of many slowly increasing sequences $(a_n)_{n=1}^{\infty}$ such that, for each $\xi \notin \mathbb{Q}$, the sequence $\{a_n\xi\}_{n=1}^{\infty}$ is everywhere dense in $[0, 1)$. Such are, for instance, the sequences $a_n = n$, $a_n = P(n)$, where $P(x) \in \mathbb{Z}[x]$ has degree $\geqslant 1$, $a_n = P(p_n)$, where p_n is the nth prime. Nevertheless, a similar question on whether, for the sequence of positive integers $(a_n)_{n=1}^{\infty}$ of the form $p^k + q^m$, where $p < q$ are two fixed primes and k, m run over all non-negative integers, the sequence $\{a_n\xi\}_{n=1}^{\infty}$ is everywhere dense in $[0, 1)$ remains open [10].

In this paper, we investigate whether, for a given increasing sequence of positive integers $(a_n)_{n=1}^{\infty}$, there is an *exceptional* real irrational number ξ in the sense that the sequence of fractional parts $\{a_n\xi\}_{n=1}^{\infty}$ has only finitely many limit points. Then no Furstenberg type theorem holds. *How slowly can such a sequence $(a_n)_{n=1}^{\infty}$ for which at least one exceptional $\xi \notin \mathbb{Q}$ exists increase?* The above examples show that for each rapidly increasing sequence, e.g., a lacunary sequence $(a_n)_{n=1}^{\infty}$, such exceptional ξ exist, but for most 'natural' slowly increasing sequences such exceptional ξ do not exist.

We shall prove that there is a sequence of positive integers $a_1 < a_2 < a_3 < \dots$ satisfying $a_n \leqslant ng_n$ for each $n \in \mathbb{N}$ such that $\{a_n\xi\}_{n=1}^{\infty}$ has only finitely many limit points for some $\xi \notin \mathbb{Q}$, if and only if $\lim_{n\to\infty} g_n = \infty$, no matter how slowly g_n tends to infinity. Moreover, it turns out that it is possible to construct an 'extreme' sequence $(a_n)_{n=1}^{\infty}$ for which the sequence $\{a_n\xi\}_{n=1}^{\infty}$, where $\xi \notin \mathbb{Q}$, has not just finitely many, but only one limit point, say, 0. In fact, our construction of an 'extreme' sequence of positive integers $a_1 < a_2 < a_3 < \dots$ of slowest possible growth involves the properties of this exceptional ξ (which will be taken as an arbitrary real quadratic algebraic number α) and the properties of some recurrence sequences related to some algebraic integer in the field $\mathbb{Q}(\alpha)$.

Theorem 1. *Let α be a real quadratic algebraic number, and let $g_1, g_2, g_3, \dots$ be a sequence of real numbers such that $g_n \geqslant 1$ for each $n \geqslant 1$ and $\lim_{n\to\infty} g_n = \infty$. Then there exists an increasing sequence of positive integers $a_1 < a_2 < a_3 < \dots$ satisfying $a_n \leqslant ng_n$ for each $n \in \mathbb{N}$ such that $\lim_{n\to\infty}\{a_n\alpha\} = 0$.*

The bound $a_n \leqslant ng_n$ for $n \in \mathbb{N}$ on the growth of $(a_n)_{n=1}^{\infty}$ in Theorem 1 is the best possible in the sense that the condition $\lim_{n\to\infty} g_n = \infty$ cannot be weakened. Indeed, suppose that there is a constant $g \geqslant 1$ and an increasing sequence of positive integers $(a_n)_{n=1}^{\infty}$ satisfying $a_n \leqslant gn$ for infinitely many $n \in \mathbb{N}$. Then $\liminf_{n\to\infty} a_n/n \leqslant g < \infty$, so the sequence $A = (a_n)_{n=1}^{\infty}$ has a positive upper density $\overline{d}(A) = \limsup_{n\to\infty} n/a_n \geqslant 1/g$ (see [7]). For such sequences $(a_n)_{n=1}^{\infty}$, we prove the following:

Theorem 2. *Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence of positive integers with positive upper density, i.e., $\liminf_{n\to\infty} a_n/n < \infty$, and let ξ be an irrational real number. Then the sequence of fractional parts $\{a_n\xi\}_{n=1}^{\infty}$ has infinitely many limit points.*

In this respect we recall the paper of Vijayaraghavan [15] once again. He proved that, for any rational non-integer number $p/q > 1$ and any real number $\xi \neq 0$, the sequence of fractional parts $\{(p/q)^n\xi\}_{n=1}^{\infty}$ has infinitely many limit points.

In the next section, we shall prove two auxiliary results necessary for the proof of Theorem 1. Section 3 contains the proof of Theorem 1. We do not know whether a similar construction of the slowly increasing sequence $(a_n)_{n=1}^{\infty}$ is possible for other real numbers α (see the end of Section 3). In Section 4, we prove Theorem 2. The proofs of both theorems are completely self contained.

2. Auxiliary results

Lemma 3. *Let α be a real quadratic algebraic number. Then there exist $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ such that the number $\beta = p\alpha + q$ is a positive quadratic reciprocal unit with minimal polynomial $x^2 - tx + 1$, where $t \geqslant 4$ is an even integer.*

Proof. Suppose that the minimal polynomial of α is

$$ax^2 + bx + c = a(x - \alpha)(x - \alpha'),$$

where $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, $c \neq 0$. Since α is a real quadratic number, the discriminant $\Delta = b^2 - 4ac$ is a positive integer which is not a perfect square. Hence the Pell equation $X^2 - \Delta Y^2 = 1$ has a solution $X, Y \in \mathbb{N}$ with $X \geqslant 2$. Set $p = 2aY$ and $q = bY + X$, so that

$$\beta = 2aY\alpha + bY + X.$$

Then $\beta' = 2aY\alpha' + bY + X$. From $\alpha + \alpha' = -b/a$ it follows that

$$\beta + \beta' = 2aY(\alpha + \alpha') + 2bY + 2X = 2aY(-b/a) + 2bY + 2X = 2X.$$

Similarly, using $\alpha\alpha' = c/a$, $\alpha + \alpha' = -b/a$ and $X^2 - (b^2 - 4ac)Y^2 = 1$, we obtain

$$\begin{aligned}\beta\beta' &= 4a^2Y^2\alpha\alpha' + 2aY(bY + X)(\alpha + \alpha') + (bY + X)^2 \\ &= 4acY^2 - 2bY(bY + X) + b^2Y^2 + 2bXY + X^2 = (4ac - b^2)Y^2 + X^2 = 1.\end{aligned}$$

This proves that β is a reciprocal real quadratic unit with minimal polynomial $x^2 - 2Xx + 1$. From $\beta = (\beta^2 + 1)/(2X)$, we conclude that β is positive. $\square$

Lemma 4. *Let $\beta > 1$ be a reciprocal quadratic unit with minimal polynomial $x^2 - tx + 1$, where $t \geqslant 4$ is an even integer. Set $T_m = \beta^m + \beta^{-m}$ and $U_m = (\beta^m - \beta^{-m})/\sqrt{(t/2)^2 - 1}$. Then $T_m, U_m \in \mathbb{N}$,*

$$T_m\beta - T_{m+1} = \beta^{-m+1}(1 - \beta^{-2})$$

and

$$U_m\beta^{-1} - U_{m-1} = \beta^{-m+1}(1 - \beta^{-2})/\sqrt{(t/2)^2 - 1}$$

for each $m \in \mathbb{N}$. Furthermore, $gcd(T_m, T_{m+1})$=$gcd(U_m, U_{m+1}) = 2$ for each $m \geqslant 1$.

Proof. Clearly, $T_0 = 2$, $T_1 = t$ and $T_{m+1} = tT_m - T_{m-1}$ for each $m \geqslant 1$. Similarly, $U_0 = 0$, $U_1 = 2$ and $U_{m+1} = tU_m - U_{m-1}$ for each $m \geqslant 1$. This proves that $T_m, U_m \in \mathbb{N}$ for each $m \in \mathbb{N}$. The numbers $T_1, T_2, \dots$ are all even, hence $\gcd(T_m, T_{m+1}) \geqslant 2$. If, however, some $d > 2$ divides T_m and T_{m+1}, then from the recurrence relation on T_{m+1}, T_m, T_{m-1} we see that d also divides T_{m-1}, and so on up to $d|T_0$, i.e., $d|2$, which is impossible. This proves that $\gcd(T_m, T_{m+1}) = 2$. The proof of $\gcd(U_m, U_{m+1}) = 2$ is the same.

From the representation $T_m = \beta^m + \beta^{-m}$, we have

$$T_m\beta - T_{m+1} = \beta(\beta^m + \beta^{-m}) - (\beta^{m+1} + \beta^{-m-1}) = \beta^{-m+1}(1 - \beta^{-2}).$$

Likewise,

$$\sqrt{(t/2)^2 - 1}(U_m\beta^{-1} - U_{m-1}) = \beta^{-1}(\beta^m - \beta^{-m}) - (\beta^{m-1} - \beta^{-m+1}) = \beta^{-m+1}(1 - \beta^{-2}).$$

This finishes the proof. □

3. Proof of Theorem 1

Suppose that α is a real quadratic algebraic number and α' is its reciprocal over $\mathbb{Q}$. There are two cases, $\alpha > \alpha'$ and $\alpha < \alpha'$. In the first case, take $\beta = p\alpha + q$ with p, q as in Lemma 3. Then $\beta > 1 > \beta' = \beta^{-1}$. In the second case, the role of α belongs to α'. So we take $\beta = p\alpha' + q$ with p, q as in Lemma 3. Then $\beta > 1 > \beta' = p\alpha + q = \beta^{-1}$. Note that, in both cases, we have $\beta > 1$, so Lemma 4 can be applied. Below, we shall construct the sequence $a_1 < a_2 < a_3 < \dots$ using T_m, $m = 1, 2, \dots$ (in the first case) and U_m, $m = 1, 2, \dots$ (in the second case).

Note that by replacing each g_n with $g_n = \inf_{j \geqslant n} g_j$, we can assume that the sequence $g_1, g_2, g_3, \dots$ is non-decreasing. By replacing each g_n with its integer part $[g_n]$, we can assume that each g_n is a positive integer. Finally, by reducing each positive gap $k = g_{n+1} - g_n$, where $k \geqslant 2$, to the gap with $k = 1$, we can assume without loss of generality that $g_{n+1} - g_n \leqslant 1$.

Take $\beta > 1$ as above (namely, $\beta = p\alpha + q$ or $\beta = p\alpha' + q$),

$$c = 8p\beta^5 \quad \text{and} \quad k_m = [c\beta^m/g_m] = [8p\beta^{m+5}/g_m].$$

Let

$$A_m = \{pkT_{m+1} + p\ell T_m \mid k = 1, \dots, k_{m+1},\ \ell = 1, \dots, k_m\},$$
$$A_m^{'} = \{pkU_{m+1} + p\ell U_m \mid k = 1, \dots, k_{m+1},\ \ell = 1, \dots, k_m\}.$$

Consider the sets $B = \bigcup_{m=1}^{\infty} A_m$ and $B' = \bigcup_{m=1}^{\infty} A_m^{'}$. Denote their distinct elements by $b_1 < b_2 < b_3 < \dots$ and $b_1^{'} < b_2^{'} < b_3^{'} < \dots$, respectively. The required sequence $A = \{a_1 < a_2 < a_3 < \dots\}$ will be obtained from B in the first case and from B' in the second case. In both cases, we just replace several first elements of B (resp. B') by smaller positive integers.

Let us first show that, in the first case,

$$\lim_{n\to\infty} \{b_n\alpha\} = 0.$$

Suppose that $b_n \in A_m$. Such $m \in \mathbb{N}$ is not necessarily unique, but $m \to \infty$ provided that $n \to \infty$, and, vice versa, $n \to \infty$ as $m \to \infty$. By the above, $b_n = pkT_{m+1} + p\ell T_m$ with some $k, \ell \in \mathbb{N}$ satisfying $1 \leqslant k, \ell \leqslant \max\{k_m, k_{m+1}\} \leqslant c\beta^{m+1}/g_m$. From $\beta = p\alpha + q$ it follows that

$$\{b_n\alpha\} = \{(kT_{m+1} + \ell T_m)p\alpha\} = \{(kT_{m+1} + \ell T_m)\beta\}.$$

Using the upper bound for k and ℓ, the formulae $c = 8p\beta^5$ and Lemma 4, we deduce that

$$\begin{aligned}\{b_n\alpha\} = \{(kT_{m+1} + \ell T_m)\beta\} &= k(T_{m+1}\beta - T_{m+2}) + \ell(T_m\beta - T_{m+1}) \\ = \beta^{-m}(1-\beta^{-2})(k+\ell\beta) &\leqslant \beta^{-m}(1-\beta^{-2})(1+\beta)c\beta^{m+1}/g_m \\ &= (\beta+\beta^2)(1-\beta^{-2})c/g_m < 16p\beta^7/g_m\end{aligned}$$

for each sufficiently large m. (Certainly, this holds for those m for which $g_m > 16p\beta^7$.) If $n \to \infty$, then $m \to \infty$ and $g_m \to \infty$. Hence $\lim_{n\to\infty}\{b_n\alpha\} = 0$, as claimed.

Similarly, in the second case, the equality $p\alpha + q = \beta' = \beta^{-1}$ combined with the representation $b_n^{'} = pkU_{m+1} + p\ell U_m$ yields $\{b_n^{'}\alpha\} = \{(kU_{m+1} + \ell U_m)\beta^{-1}\}$. Using the fact that $U_m\beta^{-1} - U_{m-1}$ is 'small' (see Lemma 4), in exactly the same manner as above we can prove that, in the second case, $\lim_{n\to\infty}\{b_n^{'}\alpha\} = 0$.

Our next goal is to show that the elements of the set $A_m = \{pkT_{m+1} + p\ell T_m \,|\, k = 1,\dots,k_{m+1}, \ell = 1,\dots,k_m\}$ are distinct for $m \geqslant m_1$. Assume that $pkT_{m+1}+p\ell T_m = pk'T_{m+1}+p\ell' T_m$, where $\ell \neq \ell'$. Then $(k-k')T_{m+1}/2 = (\ell'-\ell)T_m/2$. By Lemma 4, the integers $T_{m+1}/2$ and $T_m/2$ are coprime. It follows that $T_{m+1}/2$ divides $|\ell - \ell'|$. Therefore, $\beta^{m+1} < T_{m+1} \leqslant 2|\ell-\ell'| \leqslant 2k_m \leqslant 2c\beta^m/g_m$. Setting m_1 to be the least integer for which $g_{m_1} \geqslant 2c$, we derive that $\beta^{m+1} < \beta^m$ for $m \geqslant m_1$, a contradiction. Likewise, the elements of the set $A_m^{'} = \{pkU_{m+1} + p\ell U_m \mid k = 1,\dots,k_{m+1},\ \ell = 1,\dots,k_m\}$ are distinct for $m \geqslant m_2$.

Let us take an integer $M \geqslant \max\{m_1, m_2\}$, where M is so large that

$$m \leqslant k_m < \beta^2 k_{m-1} \quad \text{for} \quad m \geqslant M.$$

Such an M exists, because the quotient k_m/k_{m-1} is 'approximately' $\beta g_m/g_{m-1}$, which is less than or equal to $\beta(1+g_{m-1})/g_{m-1} < \beta(1+\varepsilon)$ for m large enough.

For any integer $n > k_{M-1}k_M$, there is a unique integer $m \geqslant M$ such that

$$k_{m-1}k_m < n \leqslant k_m k_{m+1}.$$

Since all $k_{m+1}k_m$ elements of A_m (resp. $A_m^{'}$) are distinct, the nth element of B (resp. B') does not exceed the nth element of A_m (resp. $A_m^{'}$). The largest element of A_m is $pk_{m+1}T_{m+1} + pk_mT_m$. Hence, using the bounds $k_{m+1} < \beta^4 k_{m-1}$, $T_{m+1} < 2\beta^{m+1}$ and $\beta^m < 2g_mk_m/c$, we obtain

$$\begin{aligned}b_n \leqslant pk_{m+1}T_{m+1} + pk_mT_m &< 2pk_{m+1}T_{m+1} < 4p\beta^4 k_{m-1}\beta^{m+1} \\ = 4p\beta^5 k_{m-1}\beta^m &< 8p\beta^5 k_{m-1}k_m g_m/c = k_{m-1}k_m g_m.\end{aligned}$$

This is less than ng_n, because $m \leqslant k_{m-1}k_m$, the sequence $g_1, g_2, \dots$ is non-decreasing, and $k_{m-1}k_m < n$. Consequently, $b_n < ng_n$ for each $n > k_{M-1}k_M$. Similarly, using $U_{m+1} < \beta^{m+1}$, we obtain

$$\begin{aligned}b_n^{'} \leqslant pk_{m+1}U_{m+1} + pk_mU_m &< 2pk_{m+1}U_{m+1} < 2p\beta^4 k_{m-1}\beta^{m+1} \\ &< 4p\beta^5 k_{m-1}k_m g_m/c < k_{m-1}k_m g_m < ng_n\end{aligned}$$

for each $n > k_{M-1}k_M$. This proves the required upper bound for b_n and $b_n^{'}$ provided that n is large enough.

Trivially, $b_n \geqslant n$ and $b_n^{'} \geqslant n$ for each positive integer n. Thus, by the above, there exists a positive integer n_0, say $n_0 = k_{M-1}k_M$, such that $n \leqslant b_n < ng_n$ and $n \leqslant b_n^{'} < ng_n$ for each $n \geqslant n_0 + 1$. In the first case, $\alpha > \alpha'$, the required increasing sequence of positive integers $A = \{a_1 < a_2 < a_3 < \dots\}$ can be obtained from $B = \bigcup_{m=1}^{\infty} A_m = \{b_1 < b_2 < b_3 < \dots\}$ by setting $a_n = n$ for $n \leqslant n_0$ and $a_n = b_n$ for $n \geqslant n_0 + 1$. In the second case, $\alpha' > \alpha$, the required increasing sequence of positive integers $A = \{a_1 < a_2 < a_3 < \dots\}$ can be obtained from $B' = \bigcup_{m=1}^{\infty} A'_m = \{b'_1 < b'_2 < b'_3 < \dots\}$ by setting $a_n = n$ for $n \leqslant n_0$ and $a_n = b'_n$ for $n \geqslant n_0 + 1$. In both cases, we have $a_n \leqslant n g_n$ for each $n \geqslant 1$. This completes the proof of the theorem. □

Suppose that ξ is either a real algebraic number of degree $\geqslant 3$ or a real transcendental number. Is there is a slowly increasing sequence of positive integers $a_1 < a_2 < a_3 < \dots$ satisfying, for instance, $a_n \leqslant n[(\log n)^{\varepsilon}]$ for each $n \geqslant 3$, such that $\lim_{n\to\infty}\{a_n \xi\} = 0$? (For example, $\lim_{n\to\infty}\{a_n \sqrt[3]{2}\} = 0$ or $\lim_{n\to\infty}\{a_n \pi\} = 0$?) We conclude this section with the following construction of some special transcendental numbers.

Theorem 5. *For any sequence $1 \leqslant g_1 \leqslant g_2 \leqslant \dots$ satisfying $\lim_{n\to\infty} g_n = \infty$, there is a transcendental Liouville number γ for which there is a sequence of positive integers $(a_n)_{n=1}^{\infty}$ satisfying $a_n \leqslant n g_n$ for infinitely many $n \in \mathbb{N}$ such that $\{a_n \gamma\} \to 0$ as $n \to \infty$.*

Proof. Take $\gamma = \sum_{k=1}^{\infty} 2^{-d_k}$, where $(d_k)_{k=1}^{\infty}$ is a sequence of positive integers increasing so fast that $d_{k+1} > 3d_k$ and $g_{\ell_k} > 2^{d_k}$, where $\ell_k = [2^{d_{k+1}/2}]$. Then $0 < 2^{d_m}\alpha - u_m < 2^{-d_{m+1}+d_m+1}$ with some $u_m \in \mathbb{N}$. Therefore, $0 < \{\ell 2^{d_m}\gamma\} < \ell 2^{-d_{m+1}+d_m+1}$ for every $\ell \in \mathbb{N}$. Select

$$A_m = \{\ell 2^{d_m} \mid \ell = 1, 2, \dots, \ell_m\}$$

and define $A = \bigcup_{m=1}^{\infty} A_m = \{a_1 < a_2 < a_3 < \dots\}$.

By the choice of ℓ_m, it is easy to see that $\{a_n\gamma\} \to 0$ as $n \to \infty$. Furthermore, for each $n = \ell_m$, we have $a_n = a_{\ell_m} \leqslant \ell_m 2^{d_m} < \ell_m g_{\ell_m} = n g_n$, because the elements of A_m are distinct. So the inequality $a_n \leqslant n g_n$ holds for infinitely many $n \in \mathbb{N}$. The number γ is a transcendental Liouville number if $\limsup_{k\to\infty} d_{k+1}/d_k = \infty$. From $g_{\ell_k} > 2^{d_k}$, where $\ell_k = [2^{d_{k+1}/2}]$, we see that this is the case when the sequence $(g_n)_{n=1}^{\infty}$ is increasing slowly, for example, $g_n \leqslant \log n$. This can be assumed without loss of generality, by replacing the initial sequence $g_1, g_2, g_3, \dots$ by the sequence $g_1^* = g_2^* = 1$ and $g_n^* = \min\{g_n, \log n\}$ for $n \geqslant 3$. □

This result is, of course, weaker than the same inequality $a_n \leqslant n g_n$ of Theorem 1, which holds for all $n \in \mathbb{N}$.

4. Proof of Theorem 2

Set $g = \liminf_{n\to\infty} a_n/n < \infty$. Suppose that the sequence $\{a_n\xi\}_{n=1}^{\infty}$ has only t limit points for some $\xi \notin \mathbb{Q}$. Let us denote the number of elements of A lying in $[1, x]$ by $A(x)$. The condition $g = \liminf_{n\to\infty} a_n/n < \infty$ implies that $A(n) > n/(2g)$ for infinitely many $n \in \mathbb{N}$.

Put $L = \lceil 3gt \rceil$. We claim that the sequence $A = (a_n)_{n=1}^{\infty}$ contains at least $t+1$ elements in infinitely many intervals $[N+1, N+L]$, where $N \in \mathbb{N}$. Indeed, if at most t elements of A lie in each of the intervals $[kL+1, kL+L]$, $k = 0, 1, 2, \dots$, except for, say, l intervals, then the number of elements of A up to kL is $\leqslant lL + (k-l)t$, i.e., $A(kL) \leqslant lL + (k-l)t \leqslant kt + lL$. For a given $n \in \mathbb{N}$, take $k \in \mathbb{N}$ such that $(k-1)L < n \leqslant kL$. Then, using $L \geqslant 3gt$, we find that

$$A(n) \leqslant A(kL) \leqslant kt + lL < (n/L + 1)t + lL \leqslant n/(3g) + t + lL.$$

So the inequality $A(n) > n/(2g)$ cannot hold for infinitely many $n \in \mathbb{N}$, a contradiction. This proves our assertion.

Note that $||q\xi|| > 0$ for every $q \in \mathbb{N}$, because $\xi \notin \mathbb{Q}$. Here an below $||x||$ stands for the distance from a real number x to the nearest integer. Fix any ε satisfying

$$0 < 2\varepsilon < \min\{||\xi||, ||2\xi||, \dots, ||(L-1)\xi||\}.$$

By our assumption, the sequence $\{a_n\xi\}_{n=1}^{\infty}$ has only t limit points. Hence, for $n \geqslant n_0(\varepsilon)$, the fractional part $\{a_n\xi\}$ must lie in an ε-neighborhood of at least one of those t points. Take an interval $[N+1, N+L]$, where $N \geqslant n_0(\varepsilon)$, which contains at least $t+1$ elements of A. (We already proved that this happens for infinitely many $N \in \mathbb{N}$, so such an interval exists.) By Dirichlet's box principle, at least two fractional parts, say, $\{a_{N+u}\xi\}$ and $\{a_{N+v}\xi\}$, where $1 \leqslant u < v \leqslant L$, lie in an ε-neighborhood of the same limit point, say, w. Putting $r = a_{N+v} - a_{N+u}$, where $r \in \{1, \dots, L-1\}$, and using $||a_{N+u}\xi - w||, ||a_{N+v}\xi - w|| \leqslant \varepsilon$, we deduce that

$$2\varepsilon < ||r\xi|| = ||(a_{N+v} - a_{N+u})\xi|| \leqslant ||a_{N+v}\xi - w|| + ||w - a_{N+u}\xi|| \leqslant \varepsilon + \varepsilon = 2\varepsilon,$$

a contradiction. This completes the proof of the theorem. $\square$

By the same argument as above one can prove that if $(a_n)_{n=1}^{\infty}$ is an increasing sequence of positive integers satisfying $\liminf_{n\to\infty}(a_{n+1} - a_n) < \infty$ and ξ is an irrational real number, then the sequence of fractional parts $\{a_n\xi\}_{n=1}^{\infty}$ has at least two limit points.

Indeed, the condition $\liminf_{n\to\infty}(a_{n+1} - a_n) < \infty$ implies that there exists a positive integer r such that $a_{n+1} - a_n = r$ for infinitely many n's. Suppose that for some real irrational number ξ we have $\lim_{n\to\infty}\{a_n\xi\} = w$, where $0 \leqslant w \leqslant 1$. Then, for any $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that $||a_n\xi - w|| \leqslant \varepsilon$ for each $n \geqslant n_0(\varepsilon)$. Fix $\varepsilon > 0$ satisfying $0 < 2\varepsilon < ||r\xi||$. Take any $n \geqslant n_0(\varepsilon)$ for which $a_{n+1} - a_n = r$. Then

$$2\varepsilon < ||r\xi|| = ||(a_{n+1} - a_n)\xi|| = ||a_{n+1}\xi - w + w - a_n\xi|| \leqslant ||a_{n+1}\xi - w|| + ||a_n\xi - w|| \leqslant 2\varepsilon,$$

a contradiction.

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