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REMARK ON ELLIPTIC UNITS IN A $\mathbb{Z}_p$-EXTENSION OF AN IMAGINARY QUADRATIC FIELD

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Abstract. We shall study the group of units modulo the group of elliptic units in a $\mathbb{Z}_p$-extension of an imaginary quadratic field.

1. Main result

We fix an imaginary quadratic field k which is different from $\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, and an odd prime number p. Let $\mathfrak{p}$ be a prime ideal of k lying above p, and K/k a $\mathbb{Z}_p$-extension which is unramified outside $\mathfrak{p}$. Assume that $\mathfrak{p}$ is totally ramified in K/k. For a positive integer n, we denote by k_n the n^{th} layer of K/k. Let A_n be the Sylow p-subgroup of the ideal class group of k_n, E_n the group of units in k_n, and $\mathfrak{p}_n$ the unique prime ideal of k_n lying above $\mathfrak{p}$. Let $c(\mathfrak{p}_n)$ be the ideal class of k_n which contains $\mathfrak{p}_n$. We put $D_n = A_n \cap \langle c(\mathfrak{p}_n)\rangle$ and $A'_n = A_n/D_n$. Moreover, put $k_0 = k$, and define A_0, D_0, and A'_0 similarly. For a finite set S, we denote by $|S|$ the number of elements in S.

For any integer $n \geq 1$, let Φ_n be the group of certain elliptic units in k_n which is defined in Section 2. We will see later that Φ_n has finite index in E_n. Let B_n be the Sylow p-subgroup of E_n/Φ_n. In this paper, we shall show the following:

Theorem 1.1. *If $|A_n|$ is bounded as $n \to \infty$ (i.e. both of the Iwasawa λ- and μ-invariants of K/k are zero), then A'_n and B_n are isomorphic as $\mathrm{Gal}(k_n/k)$-modules for all sufficiently large n.*

We mention that a similar result is already given in [7] for the case that $p \geq 5$ splits in k, k_n is the ray class field of k modulo $\mathfrak{p}^{n+1}$, and $\mathfrak{p}$ does not split in the absolute class field of k. (When k is a real abelian field and K/k is the cyclotomic $\mathbb{Z}_p$-extension, similar results are previously known. See [11], [15], etc.)

In Section 5, we will give an additional result. This result is obtained as a corollary of known results.

2. Group of elliptic units

Fix an integer $n \geq 1$. In this section, we will define the group Φ_n of elliptic units in k_n. Our construction is similar to that of [7]. We use the same notation as given in Oukhaba [14].

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Let $\mathfrak{f}_n$ be the conductor of k_n/k, and f_n the minimal positive integer which is contained in $\mathfrak{f}_n \cap \mathbb{Z}$. Note that $\mathfrak{f}_n$ is a positive power of $\mathfrak{p}$. Let $k_{\mathfrak{f}_n}$ be the ray class field of k modulo $\mathfrak{f}_n$. We fix a $\mathbb{Z}$-basis (ω_1, ω_2) of $\mathfrak{f}_n$ satisfying $\mathrm{Im}(\omega_1/\omega_2) > 0$. Let

$$\varphi_{\mathfrak{f}_n} := \left(\kappa(1, \mathfrak{f}_n)\, \eta(\omega_1/\omega_2)^2\, \omega_2^{-1}\right)^{12 f_n}$$

be the Siegel-Ramachandra-Robert invariant defined in [14, Definition 2], where $\kappa(t, \mathfrak{f}_n)$ is the Klein form (see [13, p. 27]) and

$$\eta(\tau) = e^{2\pi i \tau/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

is the Dedekind eta function.

As noted in [14], $\varphi_{\mathfrak{f}_n}$ coincides with $E(\mathfrak{c}_0)$ in [19]. We also note that $\varphi_{\mathfrak{f}_n}$ is only dependent on $\mathfrak{f}_n$ (see [19, p. 223]). By [14, Proposition 2] or [19], we know that $\varphi_{\mathfrak{f}_n}$ is an algebraic integer in $k_{\mathfrak{f}_n}$ and any $12f_n^{\text{th}}$ root is contained in a certain abelian extension field of k. We put $\tilde{\varphi}_{k_n, \mathfrak{f}_n} = N_{k_{\mathfrak{f}_n}/k_n} \varphi_{\mathfrak{f}_n}^2$.

We mention that the roots of unity contained in k_n are only ± 1. Hence by [19, Lemma 6], there is a unique element u_n of k_n which satisfies

$$u_n^{3f_n} = \tilde{\varphi}_{k_n, \mathfrak{f}_n}.$$

(Note that f_n is odd.) We also note that u_n is a $\mathfrak{p}_n$-unit in k_n (which follows from, e.g., [14, Corollary 2]). Let E_n' be the group of $\mathfrak{p}_n$-units in k_n.

Definition 2.1. Let Φ_n' be the $\mathbb{Z}[\mathrm{Gal}(k_n/k)]$-submodule of E_n' generated by ± 1 and u_n. Similarly, let Ω_n' be the $\mathbb{Z}[\mathrm{Gal}(k_n/k)]$-submodule of E_n' generated by ± 1 and $\tilde{\varphi}_{k_n, \mathfrak{f}_n}$. Moreover, we put $\Phi_n = E_n \cap \Phi_n'$ and $\Omega_n = E_n \cap \Omega_n'$.

By the analytic class number formula ([13, Chapter 13, Theorem 2.1], [14, Theorem A and Proposition 16]), we obtain

$$(E_n : \Omega_n) = \frac{h(k_n)}{h(k)} (24 f_n)^{p^n - 1},$$

where $h(k_n)$ is the class number of k_n and $h(k)$ is the class number of k. By the definition of Φ_n, we obtain the following:

Lemma 2.2.

$$(E_n : \Phi_n) = \frac{h(k_n)}{h(k)} 8^{p^n - 1}.$$

Let f be a homomorphism $E_n/\Phi_n \to E_n'/\Phi_n'$ induced from the natural mapping. Since $E_n \cap \Phi_n' = \Phi_n$, we see that f is injective.

Lemma 2.3. *Assume that $|A_n|$ is bounded as $n \to \infty$. If n is sufficiently large, then the cokernel of f is finite and its order is prime to p.*

Proof. Let $\mathrm{Coker}(f)$ be the cokernel of f. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Phi_n & \to & E_n & \to & E_n/\Phi_n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Phi'_n & \to & E'_n & \to & E'_n/\Phi'_n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \Phi'_n/\Phi_n & \to & E'_n/E_n & \to & \mathrm{Coker}(f) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let d be the order of the ideal class of k_n which contains $\mathfrak{p}_n$. We fix an algebraic integer v of k_n which satisfies $\mathfrak{p}_n^d = (v)$. By using [14, Corollary 2], we can see that $\mathfrak{p}_n^{24h(k)} = (u_n)$ and hence $24h(k)$ is divisible by d.

Since $|A_n|$ is bounded as $n \to \infty$, we can see that $A_n^{\mathrm{Gal}(k_n/k)} = D_n$ for all sufficiently large n (cf. [5, Theorem 2], [3, Proposition 2.2]). By the genus formula, we have $|A_n^{\mathrm{Gal}(k_n/k)}| = |A_0|$. If n is sufficiently large, we get $|A_0| = |D_n|$ and then $24h(k)/d$ is prime to p. We note that $u_n\Phi_n$ is a generator of Φ'_n/Φ_n and vE_n is a generator of E'_n/E_n. Since $v^{24h(k)/d}E_n = u_nE_n$, we see that the order of the cokernel of the mapping $\Phi'_n/\Phi_n \to E'_n/E_n$ is finite and prime to p. □

3. Proof of Theorem 1.1

Assume that $|A_n|$ is bounded as $n \to \infty$. Let B_n be the Sylow p-subgroup of E_n/Φ_n, and B'_n the Sylow p-subgroup of E'_n/Φ'_n. By Lemma 2.3, we have $B_n \cong B'_n$ if n is sufficiently large.

The proof of Theorem 1.1 is given by using a well-known argument (cf. [11], [15], [7], etc.). Fix a positive integer n which satisfies $B_l \cong B'_l$ and $|A_l| = |A_n|$ for all $l \geq n$. We can take a positive integer $m > n$ which satisfies

$$\ker(A'_n \to A'_m) = A'_n.$$

We put $\Gamma_{m,n} = \mathrm{Gal}(k_m/k_n)$.

From the results given in Section 2, we see that $\Phi'_m/\{\pm 1\}$ is a free rank one $\mathbb{Z}[\mathrm{Gal}(k_m/k_0)]$-module. Hence both of the Tate cohomology groups $\hat{H}^0(\Gamma_{m,n}, \Phi'_m)$ and $H^1(\Gamma_{m,n}, \Phi'_m)$ are trivial. By using [14, Proposition 3], we can see that $N_{k_m/k_n}\Phi'_m = \Phi'_n$. From this, we see that $(\Phi'_m)^{\Gamma_{m,n}} = \Phi'_n$ because $\hat{H}^0(\Gamma_{m,n}, \Phi'_m)$ is trivial.

By taking the long cohomology sequence of the following exact sequence

$$0 \to \Phi'_m \to E'_m \to E'_m/\Phi'_m \to 0, \tag{3.1}$$

we obtain the following exact sequence:

$$0 \to \Phi'_n \to E'_n \to (E'_m/\Phi'_m)^{\Gamma_{m,n}} \to H^1(\Gamma_{m,n}, \Phi'_m).$$

Since $H^1(\Gamma_{m,n}, \Phi'_m)$ is trivial, we see that $B'_n \cong (B'_m)^{\Gamma_{m,n}}$. By using Lemma 2.2, we obtain

$$|B'_n| = |B_n| = \frac{|A_n|}{|A_0|} = \frac{|A_m|}{|A_0|} = |B_m| = |B'_m|,$$

and then we have an isomorphism $B'_n \cong B'_m$ induced from the natural injection. Hence the action of $\Gamma_{m,n}$ on B'_m is trivial. Consequently,

$$H^1(\Gamma_{m,n}, B'_m) \cong B'_m \cong B'_n \cong B_n.$$

On the other hand, we obtain the isomorphism

$$H^1(\Gamma_{m,n}, E'_m) \cong H^1(\Gamma_{m,n}, B'_m)$$

by taking the exact sequence of the Tate cohomology groups of (3.1). Moreover, we can see that

$$H^1(\Gamma_{m,n}, E'_m) \cong \ker(A'_n \to A'_m)$$

by using the same argument given in the proof of [9, Theorem 12]. Since $\ker(A'_n \to A'_m) = A'_n$, we have shown Theorem 1.1.

4. Consideration for Theorem 1.1

Assume that p splits in k. Let $\mathfrak{p}$ be a prime of k lying above p and K/k the unique $\mathbb{Z}_p$-extension unramified outside $\mathfrak{p}$. Moreover, we assume that $\mathfrak{p}$ is totally ramified in K/k. Fukuda and Komatsu studied this $\mathbb{Z}_p$-extension in [3]. By using [3, Proposition 2.2], we see that if $|A_0| = |D_0|$, then $|A_n|$ is bounded as $n \to \infty$. In this case, we can see that $|A'_n| = 1$ for all n, and hence Theorem 1.1 is trivially satisfied. However, Fukuda and Komatsu also found many imaginary quadratic fields such that $|A_0| \neq |D_0|$ and satisfy the assumption of Theorem 1.1 (see [3]). This implies there are nontrivial examples for Theorem 1.1 in this case.

Assume that p does not split in k. In this case, if p does not divide the class number of k (i.e. A_0 is trivial), then for any $\mathbb{Z}_p$-extension K/k and for any n, A_n is trivial. We also remark that if the class number of k is divisible by p, then the cyclotomic $\mathbb{Z}_p$-extension of k does not satisfy the assumption of Theorem 1.1 because $|A_n|$ is not bounded. However, Ozaki's result [17, Theorem 2] tells us that if "Greenberg's generalized conjecture" [6, Conjecture 3.5] holds for k and p, then there are infinitely many $\mathbb{Z}_p$-extensions of k which satisfy the assumption of Theorem 1.1.

5. Additional result

Let the notation be as in the previous sections. In this section, we assume that p does not split in k, and K/k is the cyclotomic $\mathbb{Z}_p$-extension. We noted in Section 4 that we cannot apply Theorem 1.1 for K/k except for the trivial case. However, we can obtain a similar type result. Kubert and Lang [12] pointed out that the group of "circular numbers" modulo the group of "modular numbers" relates to the Stickelberger ideal. We shall use their idea.

For a positive integer r, we put $\zeta_r = e^{2\pi i/r}$. Fix an integer $n \geq 1$. Let $\mathbb{Q}_n$ be the n^{th} layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Note that the maximal real subfield of k_n is $\mathbb{Q}_n$. We put $\Gamma_n = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. Let C'_n be the $\mathbb{Z}[\Gamma_n]$-module generated by ± 1 and

$$N_{\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q}_n}(1 - \zeta_{p^{n+1}}).$$

Let χ be the Dirichlet character corresponding to k and d the conductor of k. We denote by q_n the least common multiple of d and p^{n+1}. We put

$$\xi_n(\chi) = -\frac{1}{q_n} \sum_{0<a<q_n, (a,q_n)=1} a\chi(a)\, (\sigma_a|_{\mathbb{Q}_n})^{-1},$$

where σ_a is the element of $\mathrm{Gal}(\mathbb{Q}(\zeta_{q_n})/\mathbb{Q})$ defined by $\zeta_{q_n}^{\sigma_a} = \zeta_{q_n}^a$ (see, e.g., [20]). It is well known that

$$2 \sum_{0<a<q_n, (a,q_n)=1} \left(\frac{1}{2} - \frac{a}{q_n}\right) (\sigma_a|_{k_n})^{-1}$$

is contained in $\mathbb{Z}[\mathrm{Gal}(k_n/\mathbb{Q})]$ (see, e.g., [2, Theorem 1 (i)], [18, Theorem 7.2.2]). From this, we can see that $\xi_n(\chi)$ is contained in $\mathbb{Z}[\Gamma_n]$.

Let u_n be the element of k_n defined in Section 2. By using the result of Gillard [4] (which is a generalization of the result of Kubert and Lang [12]), we can show that

$$u_n = \left(N_{\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q}_n}(1 - \zeta_{p^{n+1}})\right)^{2\xi_n(\chi)}.$$

(See [4, p. 184, Corollaire]. See also [10].) Hence we see that Φ'_n is contained in C'_n, and

$$(C'_n/\Phi'_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p[\Gamma_n]/\xi_n(\chi)\mathbb{Z}_p[\Gamma_n].$$

On the other hand, if A_0 is a cyclic group, then

$$A_n \cong \mathbb{Z}_p[\Gamma_n]/\xi_n(\chi)\mathbb{Z}_p[\Gamma_n]$$

(see [1, Lemma 2.14 and Lemma 2.15]). Hence we have obtained the following:

Theorem 5.1. *If A_0 is a cyclic group, then*

$$A_n \cong (C'_n/\Phi'_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

for all $n \geq 1$.

We note that similar type results are given in [8], [16], and [7].

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