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A METRIC SPACE WITH THE HAVER PROPERTY WHOSE SQUARE FAILS THIS PROPERTY

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ABSTRACT. Haver introduced the following property of metric spaces (X, d): for each sequence $\epsilon_1, \epsilon_2, \ldots$ of positive numbers there exist collections $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of open subsets of X, the union $\bigcup_i \mathcal{V}_i$ of which covers X, such that the members of $\mathcal{V}_i$ are pairwise disjoint and every member of $\mathcal{V}_i$ has diameter less than ϵ_i. We construct two separable complete metric spaces (X_0, d_0), (X_1, d_1) with the Haver property such that d_0, d_1 generate the same topology on $X_0 \cap X_1 \neq \emptyset$, but $(X_0 \cap X_1, \max(d_0, d_1))$ fails this property. In particular, the square of a separable complete metric space with the Haver property may fail this property. Our results answer some questions posed by Babinkostova in 2007.

1. Introduction

A metric space (X, d) has the Haver property if for each sequence $\epsilon_1, \epsilon_2, \ldots$ of positive numbers there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of collections of open subsets of X, the union $\bigcup_i \mathcal{V}_i$ of which covers X, such that, for $i = 1, 2, \ldots$, the members of $\mathcal{V}_i$ are pairwise disjoint and each member of $\mathcal{V}_i$ has diameter less than ϵ_i; see [6].

The property C of a space X means that for any sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of collections of open subsets of X, the union $\bigcup_i \mathcal{V}_i$ of which covers X, such that, for $i = 1, 2, \ldots$, the members of $\mathcal{V}_i$ are pairwise disjoint and each member of $\mathcal{V}_i$ is contained in a member of $\mathcal{U}_i$; see [1] and [5].

A metrizable space X has the property C if and only if for any metric d on X generating the topology, (X, d) has the Haver property (see sec. 4 (D) and [2] for more information).

The aim of this note is the following theorem, where $\vee$ stands for the maximum of two functions considered on their common domain.

Theorem 1.1. *There are separable complete metric spaces (X_0, d_0), (X_1, d_1) with the property C such that d_0, d_1 generate the same topology on $X_0 \cap X_1 \neq \emptyset$ and $(X_0 \cap X_1, d_0 \vee d_1)$ fails the Haver property.*

This readily yields the following:

Corollary 1.2. *There is a separable complete metric space with the property C whose square fails the Haver property.*

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Indeed, the free union (X,d) of the metric spaces (X_0,d_0) and (X_1,d_1) from Theorem 1.1 has these properties, since the metric space $(X_0 \cap X_1, d_0 \vee d_1)$ embeds isometrically into the square of (X,d) with the metric $\rho((x_1,y_1),(x_2,y_2)) = \max(d(x_1,x_2),d(y_1,y_2))$.

A completely metrizable separable space with the property C whose square fails this property was constructed by Jan van Mill and R. Pol [10], and this construction is a key element in our proof of Theorem 1.1.

Corollary 1.2 answers some questions asked by Liljana Babinkostova [2, Problems 1 and 2]; see also sec.4 (A).

Let us notice that by [2, Theorem 15], the product of a metric space with the Haver property and a countable-dimensional metric space (i.e., a countable union of zero-dimensional sets; see [5]) always has the Haver property, while the property C may be lost upon multiplication by a zero-dimensional space; see [11].

2. A construction by Jan van Mill and R. Pol

Our proof of Theorem 1.1 is based on a construction from [10], which we shall use to the following effect.

Proposition 2.1. *There exist a compact metrizable space E, a subspace $S \subset E$, σ-compact sets H_0, H_1 in $E \setminus S$ and a sequence (A_n, B_n), $n = 1,2,\ldots$, of pairs of disjoint closed sets in E such that*

1. *$H_0 \cap H_1 = \emptyset$ and $E \setminus S = H_0 \cup H_1$,*
2. *if $T \subset S$ is uncountable-dimensional, then its closure $\overline{T}$ in E intersects both H_0 and H_1,*
3. *the sequence $(A_n \cap S, B_n \cap S)$, $n = 1,2,\ldots$, is essential in S.*

The condition (3) means that whenever L_i is a partition in S between $A_i \cap S$ and $B_i \cap S$, then $\bigcap_i L_i \neq \emptyset$; see [5, Definition 6.1.1].

Let us explain how to derive this proposition from the results in [10]. We shall call a compact space L a Bing compactum if for any continua L_1, L_2 in L with $L_1 \cap L_2 \neq \emptyset$, either $L_1 \subset L_2$ or $L_2 \subset L_1$; see [9].

Let $\mathbb{K}$ be the Cantor set in $I = [0,1]$, let $I^{\mathbb{N}}$ be the Hilbert cube and let $C_i = \{(x_0,x_1,\ldots) : x_i = 0\}$, $D_i = \{(x_0,x_1,\ldots) : x_i = 1\}$ be the opposite faces in $I^{\mathbb{N}}$. Let us fix a partition L between C_0 and D_0 in $I^{\mathbb{N}}$ which is a Bing compactum (cf. [3] or [9, sec. 3.8]), and let us set $E = \mathbb{K} \times L$, $C_n^* = E \cap (\mathbb{K} \times C_n)$, $D_n^* = E \cap (\mathbb{K} \times D_n)$. Let $p : E \to \mathbb{K}$ be the projection onto the first coordinate. Recall that for any sequence $L_1, L_2, \ldots$, where L_n is a partition in $I \times I^{\mathbb{N}}$ between the faces $I \times C_n$ and $I \times D_n$, for $n = 1,2,\ldots$, there is a continuum in $(I \times L) \cap \bigcap_{n=1}^{\infty} L_n$ joining the faces $\{0\} \times I^{\mathbb{N}}$ and $\{1\} \times I^{\mathbb{N}}$. Therefore, by Rubin, Schori and Walsh [13] (see also [9, Theorem 3.9.3]), there is a first Baire class function $f : \mathbb{K} \to E$ with $p \circ f(t) = t$, for $t \in \mathbb{K}$, such that for any sequence of partitions L_n in E between C_n^* and D_n^*, the intersection $\bigcap_{n=1}^{\infty} L_n$ hits $S = f(\mathbb{K})$. Enlarging C_n^* and D_n^* to open sets V_n and W_n with disjoint closures, and taking $A_n = \overline{V_n}$, $B_n = \overline{W_n}$, we get (3), cf. [5, Lemma 1.2.9]. Now, the construction in [10, sec. 3], applied to $p : E \to \mathbb{K}$ and $f : \mathbb{K} \to E$ ($\mathbb{K} = F$, in the notation of [10]), provides G_δ-sets S_j, $j = 1,2,\ldots$, in E such that $S_j \setminus S$ are pairwise disjoint, $S \cup \bigcup_{j \geq n} S_j$ is a G_δ-set for $n = 1,2,\ldots$, and for any uncountable-dimensional $T \subset S$, the closure $\overline{T}$ in E intersects all but finitely many S_j.

The set $E \setminus S$ is σ-compact and $S_i \setminus S$ are pairwise disjoint G_δ-sets in $E \setminus S$ such that each union $\bigcup_{j \geq n}(S_j \setminus S)$ is also G_δ in $E \setminus S$. Using the separation theorem [7, §30, VII], one can find pairwise disjoint σ-compact sets $F_i \supset S_i \setminus S$ with $\bigcup_i F_i = E \setminus S$, and we define $H_\tau = \bigcup\{F_{2j+\tau} : j = 0, 1, \ldots\}$, $\tau = 0, 1$. This gives (1) and (2).

3. Proof of Theorem 1.1

We shall use the notation introduced in Proposition 2.1. Let us fix a metric ρ on E generating the topology, with the ρ-diameter of E not greater than 1, and let $\delta_1 \geq \delta_2 \geq \ldots$ be a sequence of positive numbers such that the ρ-distance between A_n and B_n is not less than δ_n. Let us write, for $\tau = 0, 1$,

(4) $H_\tau = \bigcup_i H_{\tau,i}$ with $H_{\tau,1} \subset H_{\tau,2} \subset \ldots$ compact.

We define continuous maps $p_\tau : H_\tau \to I^{\mathbb{N}}$ such that $p_\tau(H_\tau)$ is countable-dimensional and for any i and $y \in p_\tau(H_{\tau,i} \setminus H_{\tau,i-1})$, where $H_{\tau,0} = \emptyset$, the fiber $p_\tau^{-1}(y)$ is disjoint from $H_{\tau,i-1}$ and has ρ-diameter not greater than $\delta_i \cdot 2^{-i}$; cf. [7, §28, IX], [8] and [5, Lemma 5.3.1]. Then the decomposition of E into fibers $p_\tau^{-1}(p_\tau(x))$, for $x \in H_\tau$, and singletons $\{x\}$, for $x \in S$, is upper-semicontinuous, and let us denote the quotient map by

(5) $\pi : E \to E^*$, $A^* = \pi(A)$, for $A \subset E$.

Since π embeds S homeomorphically into E^*, we shall identify S with its image S^* (notice also that $S = \pi^{-1}(S^*)$). We have

(6) $H_0^* \cap H_1^* = \emptyset$ and $E^* \setminus S = H_0^* \cup H_1^*$.

Let us define

(7) $X_\tau = S \cup H_\tau^*$, for $\tau = 0, 1$.

Since, by (6), $E^* \setminus X_\tau = H_{1-\tau}^*$, we infer that

(8) X_τ is a G_δ-set in E^*.

Claim A. X_τ has the property C.

To see this, let us first notice that

(9) if $T \subset S$ is closed in X_τ, then T is countable-dimensional.

Indeed, the closure of T in E^* is disjoint from H_τ^*, and therefore the closure of T in E is disjoint from H_τ. By (2) in Proposition 2.1, this shows that T is countable-dimensional.

Now, let us consider a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X_τ. Since $X_\tau \setminus S$ is countable-dimensional, we have $X_\tau \setminus S = \bigcup_{n=1}^\infty Z_n$, where Z_n is 0-dimensional. For $n = 1, 2, \ldots$, choose a family $\mathcal{V}_{2n}$ of pairwise disjoint open subsets of X_τ refining $\mathcal{U}_{2n}$ and covering Z_n (cf. [5, hint to Problem 6.3.D(a)]). Then $L = X_\tau \setminus \bigcup_{n=1}^\infty \bigcup \mathcal{V}_{2n}$ is a subset of S which is closed in X_τ, hence by (9), it is countable-dimensional. By the same argument, for every $n = 1, 2, \ldots$ there is a disjoint family $\mathcal{V}_{2n+1}$ of open sets in X_τ refining $\mathcal{U}_{2n+1}$ such that $L \subset \bigcup_{n=1}^\infty \bigcup \mathcal{V}_{2n+1}$.

Claim B. For $\tau = 0, 1$, there is a complete metric d_τ on X_τ generating the topology such that for the metric $d_0 \vee d_1$ on $S = X_0 \cap X_1$, the $(d_0 \vee d_1)$-distance between $A_n^* \cap S$, $B_n^* \cap S$ is positive, for $n = 1, 2, \ldots$.

To check this claim we begin with the observation that, from (4) and (5),

(10) $A_m^* \cap B_m^* \subset H_{0,m}^* \cup H_{1,m}^*$.

Indeed, if $i > m$, the ρ-distance between A_m and B_m is not less than $\delta_m \geq \delta_i$ and the fibers $p_\tau^{-1}(p_\tau(x))$ with $x \in H_{\tau,i} \setminus H_{\tau,i-1}$ have diameters not greater than $\delta_i \cdot 2^{-i}$. In effect, no fiber $\pi^{-1}(y)$ with $y \notin H^*_{0,m} \cup H^*_{1,m}$ intersects both A_m and B_m.

For each m, fix open sets $V_0(m), V_1(m)$ in E^* such that

(11) $H^*_{\tau,m} \subset V_\tau(m)$ for $\tau = 0, 1$, and $\overline{V_0(m)} \cap \overline{V_1(m)} = \emptyset$,

where the closures are considered in E^*. From (10) and (11) it follows that $A^*_m \cap V_{1-\tau}(m) \cap X_\tau$ and $B^*_m \cap V_{1-\tau}(m) \cap X_\tau$ have disjoint closures in X_τ; hence there exist continuous maps $\varphi^\tau_m : X_\tau \to [0, 1]$ such that

(12) $\varphi^\tau_m(a) = 0$, $\varphi^\tau_m(b) = 1$, whenever $a \in A^*_m \cap V_{1-\tau}(m)$, $b \in B^*_m \cap V_{1-\tau}(m)$.

Let ρ^* be any metric on E^* generating the topology and let σ_τ be a complete metric on X_τ generating the topology, cf. (8). We set, for $\tau = 0, 1$ and $x, y \in X_\tau$,

(13) $d_\tau(x, y) = \rho^*(x, y) + \sigma_\tau(x, y) + \sum_{m=1}^\infty 2^{-m} \mid \varphi^\tau_m(x) - \varphi^\tau_m(y) \mid$.

Then d_τ is a complete metric generating the topology on X_τ. We will show that d_0 and d_1 satisfy the conditions of Claim B.

Let us fix m. For $\tau = 0, 1$ take an open set $W_\tau(m)$ in E^* such that

(14) $H^*_{\tau,m} \subset W_\tau(m) \subset \overline{W_\tau(m)} \subset V_\tau(m)$,

cf. (11). Let η_τ be the ρ^*-distance between the sets $\overline{W_\tau(m)}$ and $E^* \setminus V_\tau(m)$ in E^* and let $\eta > 0$ be the ρ^*-distance between the sets $A^*_m \setminus (W_0(m) \cup W_1(m))$ and $B^*_m \setminus (W_0(m) \cup W_1(m))$ in E^*. We shall check that setting $\epsilon_m = \min\{\eta_0, \eta_1, \eta, 2^{-m}\}$, we have

(15) $(d_0 \vee d_1)(a, b) \geq \epsilon_m$, whenever $a \in A^*_m \cap S$, $b \in B^*_m \cap S$.

To prove (15), take $a \in A^*_m \cap S$, $b \in B^*_m \cap S$. If both a, b are outside of $W_0(m) \cup W_1(m)$, then $(d_0 \vee d_1)(a, b) > \rho^*(a, b) \geq \eta$. Suppose now that $a \in W_0(m)$. If $b \notin V_0(m)$, then $(d_0 \vee d_1)(a, b) \geq \rho^*(a, b) \geq \eta_0$. If $a, b \in V_0(m)$, then by (12), $\varphi^1_m(a) = 0$, $\varphi^1_m(b) = 1$, and hence $(d_0 \vee d_1)(a, b) \geq 2^{-m}$, cf. (13). If $a \in W_1(m)$, we proceed similarly, with the index 0 replaced by 1, and we can also replace a by b in this reasoning. In any case, $(d_0 \vee d_1)(a, b) \geq \epsilon_m$, which justifies (15) and completes the proof of Claim B.

Claim C. The space $(X_0 \cap X_2, d_0 \vee d_1)$ fails the Haver property, where d_τ, $\tau = 0, 1$, are as in Claim B.

Indeed, let $\epsilon_m > 0$ be the $(d_0 \vee d_1)$-distance in $S = X_0 \cap X_1$ between the sets $A^*_m \cap S$ and $B^*_m \cap S$, for $m = 1, 2, \ldots$. Since $A^*_m \cap S = A_m \cap S$ and $B^*_m \cap S = B_m \cap S$, the sequence $(A^*_m \cap S, B^*_m \cap S)$ is essential in S, by Proposition 2.1 (3), and it follows that there are no disjoint open collections $\mathcal{V}_1, \mathcal{V}_2, \ldots$ in S such that the union $\bigcup_m \mathcal{V}_m$ covers S and every member of $\mathcal{V}_m$ has $(d_0 \vee d_1)$-diameter less than ϵ_m; cf. sec. 4 (C).

4. Comments

(A) There are two classical covering counterparts to σ-compactness: the Hurewicz property and the weaker Menger property; see [2]. Babinkostova [2] showed that the Hurewicz property of a metric space with the Haver property yields the property C of this space and guarantees the Haver property of its square.

One can show that under Martin's Axiom, if the Hurewicz property is replaced here by the Menger property, the first statement is no longer true, and the second one fails even if we assume that the space has property C and its square has the Menger property; see [12] (this answers Problem 4 from [2] and shows that the answers to Problems 1 and 2 in [2] are negative, even if we assume the Menger property of the space).

(B) Corollary 1.2 shows that the statement of Theorem 16 in [2] is incorrect.

(C) The Haver property of (X, d) yields the following: for any sequence (A_n, B_n), $n = 1, 2, \ldots$, of pairs of closed sets such that the d-distance between A_n and B_n is not less than ϵ_n for some $\epsilon_n > 0$, there are partitions L_n between A_n and B_n in X with $\bigcap_n L_n = \emptyset$, cf. [1]. Indeed, let $\mathcal{V}_n$, $n = 1, 2, \ldots$, be the collections given by the Haver property for the sequence $\epsilon_1, \epsilon_2, \ldots$. There are closed sets $F_n \subset \bigcup \mathcal{V}_n$ with $\bigcup_n F_n = X$, and the traces of the members of $\mathcal{V}_n$ on F_n form a discrete collection $\mathcal{F}_n$ of closed sets with diameters less than ϵ_n. Consider $A_n^* = A_n \cup \bigcup\{F \in \mathcal{F}_n : F \cap A_n \neq \emptyset\}$ and $B_n^* = B_n \cup \bigcup\{F \in \mathcal{F}_n : F \cap A_n = \emptyset\}$, and let L_n be any partition in X between the closed disjoint sets A_n^* and B_n^*. Then $\bigcap_n L_n = \emptyset$.

(D) Let X be a metrizable space without property C and let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a sequence of open covers of X witnessing the failure of this property. Let d be a metric generating the topology of X such that for each n, the family of balls $\mathcal{W}_n = \{B_d(x, 1/n) : x \in X\}$ refines $\mathcal{U}_n$, cf. [4, IX.9.4]. Then (X, d) fails the Haver property. Indeed, there are no disjoint open collections $\mathcal{V}_1, \mathcal{V}_2, \ldots$ such that the union $\bigcup_n \mathcal{V}_n$ covers X and every member of $\mathcal{V}_n$ has d-diameter less than $\frac{1}{n}$, as each such $\mathcal{V}_n$ would refine $\mathcal{W}_n$, and hence also $\mathcal{U}_n$.

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