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MILNOR'S INVARIANTS AND SELF C_k-EQUIVALENCE

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ABSTRACT. It has long been known that a Milnor invariant with no repeated index is an invariant of link homotopy. We show that Milnor's invariants with repeated indices are invariants not only of isotopy, but also of self C_k-equivalence. Here self C_k-equivalence is a natural generalization of link homotopy based on certain degree k clasper surgeries, which provides a filtration of link homotopy classes.

1. INTRODUCTION

In his landmark 1954 paper [8], Milnor introduced his eponymous higher order linking numbers. For an n-component link, Milnor numbers are specified by a multi-index I, where the entries of I are chosen from $\{1,\dots,n\}$. In the paper [8], Milnor proved that when the multi-index I has no repeated entries, the numbers $\overline{\mu}(I)$ are invariants of link homotopy. In a follow-up paper [9], Milnor explored some of the properties of the other numbers and showed they are invariant under isotopy, but not link homotopy.

Milnor's invariants have been connected with finite-type invariants, and the connection between them is increasingly well understood. For string links, these invariants are known to be of finite-type [1, 6], and in fact related to (the tree part of) the Kontsevich integral in a natural and beautiful way [4].

By work of Habiro [5], the finite-type invariants of knots are intimately related to clasper surgery. Taking the view that link homotopy is generated by degree one clasper surgery on the link, where both disk-leaves of the clasper intersect the same component, is it possible that Milnor's isotopy invariants have some relation with clasper surgery?

The answer is yes. Let us consider the generalization of link homotopy known as self C_k-equivalence, introduced by Shibuya and the second author in [16]. These moves are defined for $k \in \mathbb{N}$ where self C_1-equivalence is link homotopy, and self C_2-equivalence is the self delta-equivalence of Shibuya [14]. Define a *self C_k-move* on a link L to be a degree k simple tree clasper surgery on L where all disk-leaves of the clasper intersect the same component. If L' is obtained from L by a sequence of self C_k-moves and ambient isotopies, we call L and L' *self C_k-equivalent*.

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Let $r(I)$ denote the maximum number of times that any index appears in I. Our main result is the following.

Theorem 1.1. *If $r(I) \leq k$, then $\overline{\mu}(I)$ is an invariant of self C_k-equivalence.*

Notice also that a self C_k-equivalence can be realized by self $C_{k'}$-moves when $k' < k$, and thus self C_k-equivalence classes form a filtration of link homotopy classes. Moving to larger and larger k provides more and more information about the structure of ambient isotopy classes of links.

The classification of links up to link homotopy has been completed by Habegger and Lin [3]. However, the structure of links under the higher order self C_k-moves remains mysterious. Nakanishi and Ohyama have classified two component links up to self C_2-equivalence [11, 12, 13], and Shibuya and the second author have shown that boundary links are self C_2-equivalent to trivial links [15]. Much of the work on self C_2-equivalence relies on invariants that are based on the coefficients of the Conway polynomial. In Example 1, we demonstrate a link that has trivial Conway polynomial, but a nontrivial Milnor invariant with $r(I) = 2$. Hence self C_2-invariants based on the Conway polynomial vanish, but the link is not self C_2-equivalent to the unlink.

We say L and L' are C_m^k-equivalent if L can be transformed into L' by ambient isotopy and degree m simple tree clasper surgery, where at least k disk-leaves of the clasper intersect the same component.

Lemma 1.2. *For $m \geq k$, if L is C_m^{k+1}-equivalent to L', then L is self C_k-equivalent to L'.*

More surprisingly, there is a partial converse to Lemma 1.2. A link $L = L_1 \cup L_2 \cup \cdots \cup L_n$ is called *Brunnian* if $L \setminus L_i$ is trivial for all $i = 1, 2, \ldots, n$.

Proposition 1.3. *Let L be an n-component Brunnian link, and U the n-component unlink. Then L is self C_k-equivalent to U if and only if L is C_{n+k-1}^{k+1}-equivalent to U.*

It is well known that Milnor's link homotopy invariants vanish if and only if the link is link homotopic to the unlink [8]. However, in Example 4.1 we make use of Proposition 1.3 to produce a boundary link that is not self C_3-equivalent to the unlink. As all Milnor numbers vanish for boundary links, this example demonstrates that self C_k-equivalence behaves very differently than we are used to when $k > 1$. Much work remains to be done before we have a clear understanding of self C_k-equivalence for higher k.

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2. Background

Given an n-component link L, we may compute *Milnor's numbers*, denoted $\mu_L(I)$ in the following way. Let G be the fundamental group of $S^3 \setminus L$, and let G_q be the qth subgroup of the lower central series of G. We have a presentation of G/G_q with n generators, given by the meridians m_i, $1 \leq i \leq n$, of the components of L. So for $1 \leq j \leq n$, a longitude l_j of the jth component of L is expressed modulo G_q as a word in the m_i's. The *Magnus expansion* $E(l_j)$ of l_j is the formal power series in noncommuting variables $X_1, ..., X_n$ obtained by substituting $1 + X_i$ for m_i and $1 - X_i + X_i^2 - X_i^3 + \cdots$ for m_i^{-1}, $1 \leq i \leq n$. Milnor's numbers are the coefficients of the monomials in the Magnus expansions of the longitudes l_j. Specifically, given a multi-index I with entries from $\{1, \ldots, n\}$, the number $\mu_L(i_1 \ldots i_r j)$ is the coefficient of $X_{i_1} \ldots X_{i_r}$ in the Magnus expansion of l_j. That is,

$$E(l_j) = 1 + \sum \mu_L(i_1 \ldots i_r j) X_{i_1} \ldots X_{i_r}.$$

These Milnor numbers are only defined up to an indeterminacy that depends on the values of the lower order invariants. For example, the Milnor "triple linking number" $\mu(123)$ is only well defined modulo $\gcd(\mu(12), \mu(23), \mu(13))$. Generally, one is interested in link invariants and so wants to study Milnor numbers modulo the indeterminacy, but unfortunately, determining the precise indeterminacy is difficult. Thus, one often works with the Milnor numbers modulo the greatest common divisor of all lower order invariants, and these numbers are denoted $\overline{\mu}(I)$. Note that for string links the indeterminacy does not arise and the $\mu(I)$ are well defined integral invariants of string link isotopy [3].

Two links L and L' are said to be *link homotopic* if L can be transformed to L' by a series of self-crossing changes and ambient isotopies. That is, we allow crossing changes in L as long as both strands of the crossing belong to the same component of L. If no index is repeated in the multi-index I, then $\overline{\mu}_L(I)$ is an invariant of link homotopy [9].

Clasper surgery was developed by Habiro in [5] for the study of finite-type invariants. We introduce some basic notions here, but the reader is advised to consult [5] for details. A clasper c is a trivalent graph denoting a surgery on a certain zero framed link L_c. Figure 1 demonstrates the conversion from the clasper to the link. Surgery on L_c does not change the ambient manifold, so the surgery may be considered a local move on the knot or link. This move is equivalent to a band sum of the original link with a link that is obtained from the Hopf link by iterated Bing doubling.

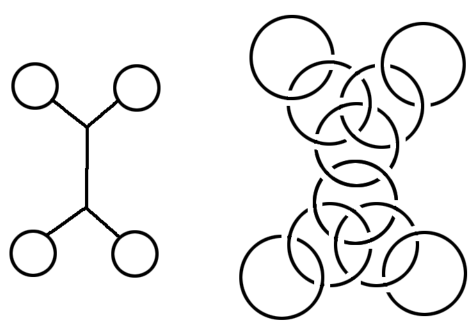

FIGURE 1. Converting from a clasper to the underlying link.

Loops in a clasper formed by a single edge are called *leaves*. For the purposes of this work, we will only consider claspers where each leaf forms an unknot. The disk bounded by a leaf is called a *disk-leaf*. We call a clasper *simple* if each disk-leaf intersects the link (or another clasper) exactly once. A clasper is said to be *tree-like* if the graph represented by the clasper (without its leaves) is a tree. A clasper c has *degree* k, where k is half the number of trivalent vertices in c. If c is tree-like, then its degree is also equal to the number of leaves, minus one. Claspers may contain *boxes*, which denote several disk-leaves intersecting each other. See Figure 2.

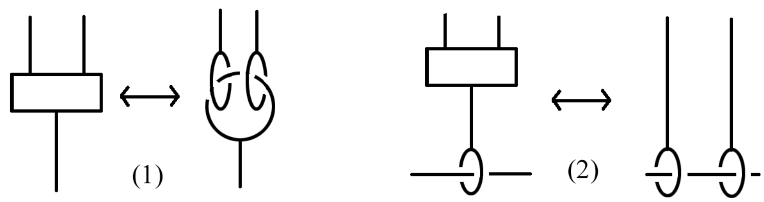

FIGURE 2. On the left, (1) depicts the meaning of a clasper with boxes. On the right, (2) shows Move 6 of [5].

It is possible to convert a tree-like clasper with boxes to a set of simple tree-like claspers by use of the *zip construction*. See Figure 3. Twists in the edges of a clasper are denoted by a small circle containing an s (positive half twist) or s^{-1} (negative half twist).

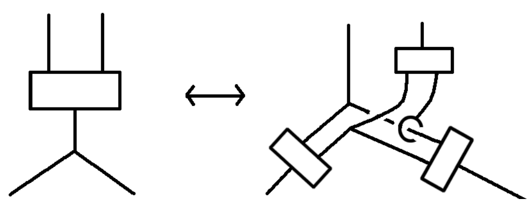

FIGURE 3. The zip construction. (Move 11 in [5].)

3. SELF C_k-INVARIANCE OF MILNOR NUMBERS

Proof of Theorem 1.1. Suppose that $r(I) \leq k$ and L is self C_k-equivalent to L'. Since Milnor's invariants are well known to be isotopy invariants [9], it suffices to check that the invariant agrees on L and L' when L' is obtained from L by a single self C_k-move on L represented by a clasper c.

The case $k = 1$ is that of link homotopy and was proven by Milnor in [8]. Thus, we need only consider the case $k \geq 2$.

To calculate $\overline{\mu}_L(I)$ (resp. $\overline{\mu}_{L'}(I)$) where I has repeated indices, we may instead study the invariants without repeated indices (link homotopy invariants) of $\overline{L}$ ($\overline{L'}$), where $\overline{L}$ ($\overline{L'}$) is the link obtained by taking zero-framed parallels of the components of L (L') [9]. If component i of $\overline{L}$ is a parallel of component j of L, let $h(i) = j$. Then $\overline{\mu}_{\overline{L}}(i_1, i_2, \ldots, i_m) = \overline{\mu}_L(h(i_1), h(i_2), \ldots, h(i_m))$ [9]. If the index i appears n_i times in I, when forming $\overline{L}$ ($\overline{L'}$) we will take n_i parallels of component i.

Taking a zero-framed parallel of a component of L and performing the clasper surgery c is the same as performing the clasper surgery and then taking a zero-framed parallel of the corresponding component of L', as for $k \geq 2$ a C_k-move does not affect linking numbers. More concisely, the clasper surgery carrying L to L' carries $\overline{L}$ to $\overline{L'}$.

The self C_k-move is realized by surgery on c, that is, a simple tree clasper with $k+1$ leaves, where all of these disk-leaves intersect the same component of L. Replace L with $\overline{L}$.

As $r(I) \leq k$, each index of I is repeated at most k times. Thus the link $\overline{L}$ has at most k copies of each component of L, and each disk-leaf of the clasper c intersects at most k components of $\overline{L}$. Further, as each disk-leaf of c intersects the same component of L, each disk-leaf of c intersects the same set of components of $\overline{L}$.

Use the zip construction on c to produce simple claspers on $\overline{L}$. That is, reduce the clasper c to a set of simple claspers c_j where each disk-leaf of each c_j intersects exactly one strand. Each disk-leaf of c intersects the same k components, so each disk-leaf of c_j intersects one of the strands that the corresponding disk-leaf of c intersected. Thus we must label each of the $k+1$ leaves of c_j with the component of the strand it intersects, but as there are only k components to choose from, at least two of the disk-leaves intersect the same component of $\overline{L}$. A surgery on a simple clasper with two disk-leaves that intersect the same component can be realized by a link homotopy. See Lemma 1.2.

We have shown that $\overline{L}$ is link homotopic to $\overline{L'}$, so $\overline{\mu}_{\overline{L}}(J) = \overline{\mu}_{\overline{L'}}(J)$ for all J with $r(J) = 1$. Hence, when $r(I) \leq k$, $\overline{\mu}_L(I) = \overline{\mu}_{L'}(I)$. □

Remark 3.1. Milnor invariants with $r(I) > k$ are not preserved by self C_k-equivalence.

Suppose that $r(I) > k$. If $\overline{\mu}(I) \neq 0$ for some link, by work of Cochran [2, Theorem 7.2], there exists a link L where $\overline{\mu}_L(I) \neq 0$, and all lower order Milnor invariants of L vanish. Such a link can be obtained by Bing doubling a Hopf link along a trivalent tree whose leaves are labeled by I, and then band summing components with the same label.

A Hopf link may be thought of as a simple degree one surgery on the two component unlink. Bing doubling a component of this link is the same as replacing the corresponding component of the unlink with two components and replacing the corresponding leaf of the clasper with a trivalent vertex and two new leaves. Each of the new leaves intersects one of the new components. In this way, we can see that Cochran's process is the same as a simple tree clasper surgery on the unlink, using the same tree, where disk-leaves labeled i intersect the ith component.

Since some index i in I is repeated at least $k+1$ times, this clasper has at least $k+1$ disk-leaves that intersect component i, and so surgery on this clasper can be realized by self C_k-moves. See Lemma 1.2. Thus, the link L we have constructed is self C_k-equivalent to the unlink. As all Milnor invariants of the unlink vanish, we have that $\overline{\mu}_L(I)$ is not a self C_k-invariant when $r(I) > k$.

We end this section with an example that demonstrates the power of our methods. Self C_2-equivalence of links is the same as self delta-equivalence, a relation that has been extensively studied. Much of this work relies on invariants of self C_2-(self-delta)-equivalence that are based on the coefficients of the Conway polynomial [13].

Example 3.2. Let L be a Bing double of the Whitehead link. See Figure 4. The Alexander polynomial of this link is trivial, but $\mu_L(123123) = 1$. Thus, L has trivial Conway polynomial[1] but is not self C_2-equivalent to the unlink.

[1] J. Hillman has pointed out that the three-variable Alexander polynomial of L is also 0.

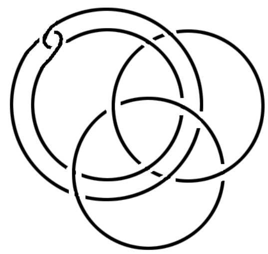

Figure 4. The Bing double of the Whitehead link.

4. Relations to C_k-equivalence

In the proof of Theorem 1.1 we used the fact that simple clasper surgery with $k+1$ disk-leaves that intersect a single component could be converted to a self C_k-equivalence. We will find a partial converse to that fact and demonstrate its usefulness in the study of self C_k-equivalence.

Proof of Lemma 1.2. Fix k and work by induction on $m-k$. The base case is $m=k$, and as the definition of C_k^{k+1}-equivalence is the same as that of self C_k-equivalence, it is trivial.

Suppose now that c is a clasper representing a C_{m+1}^{k+1}-move, where $k+1$ of the disk-leaves intersect component i. We will show that c is C_m^{k+1}-equivalent to a clasper representing a C_m^{k+1}-move. Let c' be a C_m^{k+1}-clasper obtained from a copy of c by deleting a disk-leaf that does not intersect component i; see Figure 5 (2). We choose c' so that by reversing the zip construction, we obtain the clasper with boxes shown in Figure 5 (3). By Move 12 of [5], we obtain the clasper c'', which is degree m and has $k+1$ disk-leaves intersecting component i. One may find it easier to read this figure from right to left.

That is, we have removed a leaf from c without reducing the number of disk-leaves that intersect component i, so it now represents a C_m^{k+1}-move, and so by induction, it can be realized by a self C_k-equivalence. We introduced a new clasper c' in the reduction, but for the zip construction to work, c' must have its leaves parallel to those of c (except for the leaf we are removing). Thus, c' represents a C_m^{k+1}-move and so can also be realized by self C_k-moves. $\square$

Since we are giving up degrees, Lemma 1.2 is an elementary, though useful, application of the properties of claspers. Less obvious, however, is that in certain circumstances we have a converse.

Proof of Proposition 1.3. The 'if' direction follows from Lemma 1.2. We will now show the 'only if' direction. The idea of this proof is similar to that of Theorem 1.2 in [10]. While the '*band description*' defined in [17] is used in [10], we will use clasper techniques. Band description and clasper surgery are simply different sets of terminology used to describe the same process of altering links.

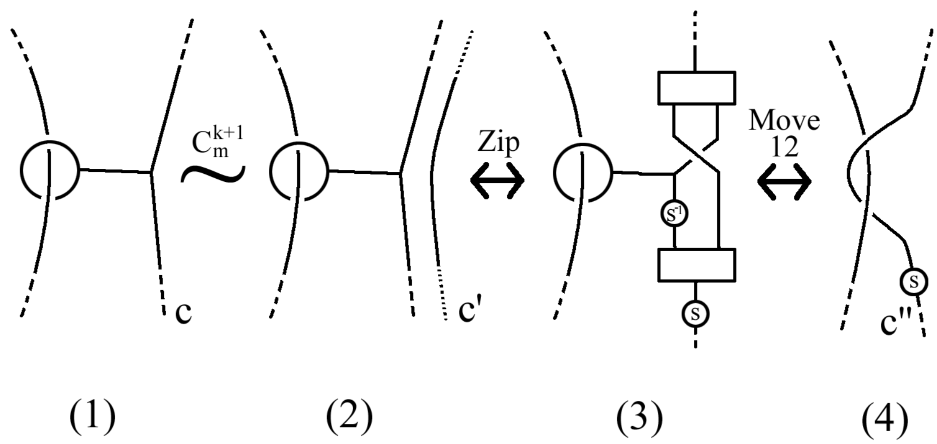

FIGURE 5. Reducing a C_{m+1}^{k+1}-clasper to C_m^{k+1}-claspers. The symbol s (s^{-1}) represents a positive (negative) half twist.

Assume that L and U are self C_k-equivalent. We can describe L as the unlink $U_1 \cup \cdots \cup U_n$ with claspers $c_{\{i\},j}$ representing the self C_k-moves on the ith component.

Let Δ_1 be the disk bounded by U_1 such that $\Delta_1 \cap U_i = \emptyset$ ($i \neq 1$).

If for some $i \neq 1$ a clasper $c_{\{i\},j}$ intersects Δ_1, we may remove this intersection point by a crossing change between that $c_{\{i\},j}$ and U_1. Passing a clasper of degree k through U_1 results in two new claspers, one of degree $k+1$ which has a new leaf that intersects Δ_1, and one of degree k which does not. Call the former $c_{\{1i\},j}$, and the latter $c'_{\{i\},j}$. See Figure 6. Repeat this process until Δ_1 intersects only the leaves of claspers $c_{\{1i\},j}$ and $c_{\{1\},j}$.

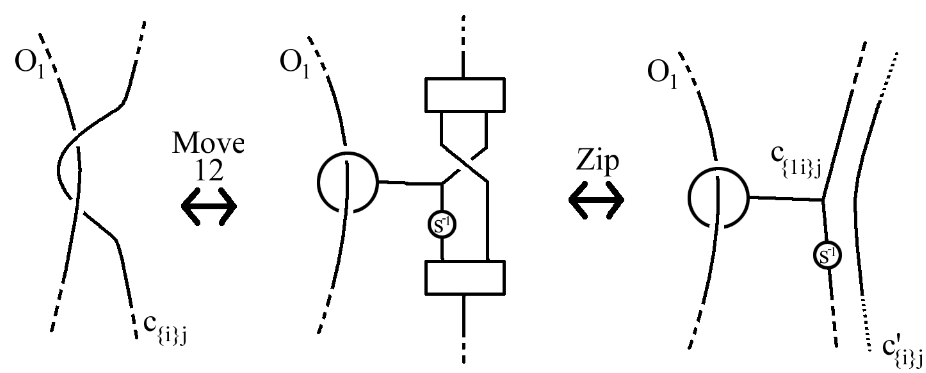

FIGURE 6. Altering the clasper $c_{\{i\}j}$ to reflect a crossing change between that clasper and U_1.

If a clasper $c_{\{i\},j}$ does not intersect Δ_1, relabel it $c'_{\{i\},j}$.

We can now express L as the unlink $U_1 \cup \cdots \cup U_n$ with claspers $c_{\{1\},j}$ of degree k, claspers $c'_{\{i\},j}$ $(i \geq 2)$ and claspers $c_{\{1i\},j}$ $(i \geq 2)$ of degree at least $k+1$.

Let L' be the link obtained by surgering the unlink $U_1 \cup \cdots \cup U_n$ only on the $c'_{\{i\},j}$. Note that $L' \setminus L'_1 = L \setminus L_1$. The $c'_{\{i\},j}$ do not intersect Δ_1. Thus, $L' = (L \setminus L_1) \amalg U_1$. Since L is Brunnian, L' is the unlink. Thus we can express L as the unlink L' with claspers $c_{\{1\},j}$ of degree k and claspers $c_{\{1i\},j}$ $(i \geq 2)$ of degree at least $k+1$ that have disk-leaves intersecting $U_1 \cup U_i$.

Since $L \setminus L_2$ is the unlink, repeating the argument above shows that L is expressed as an unlink L'' with claspers $c_{\{12\},j}$ of degree at least $k+1$ that have disk-leaves intersecting $U_1 \cup U_2$ and claspers $c_{\{12i\},j}$ $(i \geq 3)$ of degree at least $k+2$ that have disk-leaves intersecting $U_1 \cup U_2 \cup U_i$.

Repeating this process for each L_i, we can express L as the unlink with claspers $c_{\{12,\ldots,n\},j}$ of degree at least $k+n-1$ that have disk-leaves intersecting $U_1 \cup U_2 \cup \cdots \cup U_n$. Hence L is C^{k+1}_{k+n-1}-equivalent to the unlink.

□

Figure 7 illustrates the argument of Proposition 1.3 for the Whitehead link. The Whitehead link is Brunnian and link homotopic to the unlink. The center image of Figure 7 shows it as the unlink with one self C_1-clasper. The right-hand image shows the Whitehead link as the unlink with one C^2_2-clasper.

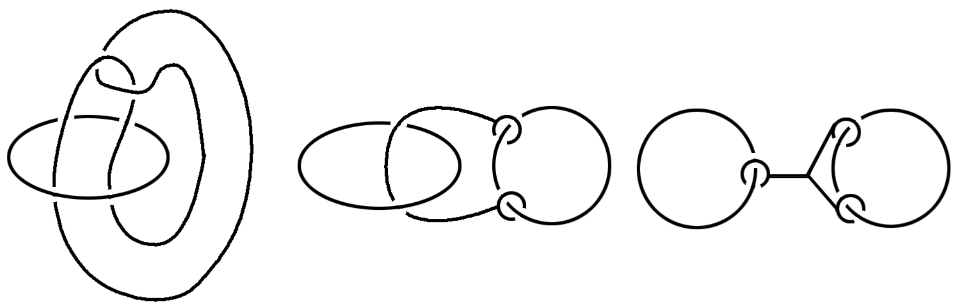

FIGURE 7. Proposition 1.3 for the Whitehead link.

Example 4.1. The link in Figure 8 is obtained from Whitehead doubling both components of the Hopf link and is a boundary link, so all Milnor invariants vanish. However, examine the Jones polynomial

$$J(L) = q^{-\frac{9}{2}} - 2q^{-\frac{7}{2}} + q^{-\frac{5}{2}} - q^{-\frac{3}{2}} - q^{\frac{3}{2}} + q^{\frac{5}{2}} - 2q^{\frac{7}{2}} + q^{\frac{9}{2}}.$$

We obtained the Jones polynomial from the Knot Atlas[2] and by a calculation using Knot.[3] Evaluating the fourth derivative of $J(L)$ at 1, we obtain a value distinct from that of the fourth derivative of the Jones polynomial of the unlink at 1. Since this is a finite-type invariant of degree four, L is not C_5-equivalent to the trivial link and

[2]http://www.math.toronto.edu/∼drorbn/KAtlas, link L11n247
[3]http://www.math.kobe-u.ac.jp/∼kodama/knot.html

hence by Lemma 6.1 of [7], not C_4^4-equivalent to the unlink. Using Proposition 1.3 we see that it is not self C_3-equivalent to the trivial link.

In fact, since L is Brunnian, if L is self C_k-equivalent to a split link, it is self C_k-equivalent to the trivial link. In order to see this, note that the components of L are unknots, so suppose L is self C_k-equivalent to a split link L' where some component L'_i of L' is a nontrivial knot. The knot L'_i is obtained from L_i by self C_k-moves, and so is self C_k-equivalent to the unknot. Thus, once we have transformed L to a split link L', it is possible to unknot any knotted component of L' by further self C_k-moves.

Thus, L is not self C_3-equivalent to a split link, even though all Milnor numbers of L vanish.

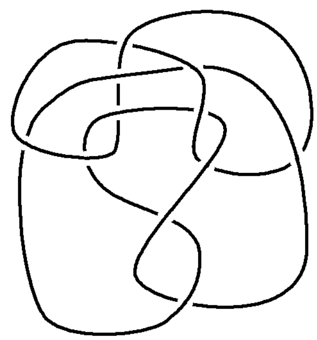

FIGURE 8. The Hopf link with both components Whitehead doubled.

Example 4.1 is interesting for other reasons. Let *Milnor's self C_k-invariants* denote Milnor's invariants $\overline{\mu}(I)$ with $r(I) \leq k$. Recall that Milnor's link homotopy invariants vanish if and only if L is link homotopic to the unlink. The example above shows that a similar statement is not true for Milnor's self C_3-invariants. While the link in Example 4.1 is a boundary link, it is not self C_3-equivalent to the trivial link or even a split link. It would be interesting to know what the vanishing of Milnor's self C_k-invariants implies about the self C_k-equivalence class of L.

For two-component links, vanishing $\mu_L(12)$ and $\mu_L(1122)$ imply that the link is self C_2-equivalent to the unlink [13]. Shibuya and the second author have recently shown that all boundary links are self C_2-equivalent to the unlink [15]. This leads one to ask whether the vanishing of Milnor's self C_2-invariants is sufficient to show that a link is self C_2-equivalent to the unlink.

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