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FIXED POINT PROPERTIES OF NILPOTENT GROUP ACTIONS ON 1-ARCWISE CONNECTED CONTINUA

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Abstract. We show that every continuous action of a nilpotent group on a 1-arcwise connected continuum has at least one fixed point.

1. Introduction

Let X be a topological space, G a topological group, and $\phi : G \times X \longrightarrow X$ a continuous action of G on X. We call $x \in X$ a *fixed point* of G if

$$\phi(g, x) = x, \quad \text{for all } g \in G.$$

Denote by $\mathrm{Fix}_X(G)$ (or simply $\mathrm{Fix}(G)$) all fixed points of G, which is a closed subset of X. The following question has wide interest.

Under which conditions on G and X is the set $\mathrm{Fix}_X(G)$ nonempty regardless of ϕ?

By a *continuum*, we mean a nonempty, connected, compact and metrizable topological space. A continuum is said to be *1-arcwise connected* (or *uniquely arcwise connected*) if for any two different points x, y of it, there is a unique arc in it with endpoints x and y. This is equivalent to saying that the continuum is arcwise connected and contains no circle.

In 1957, Isbell proved in [3] that $\mathrm{Fix}_X(G)$ is nonempty if G is commutative, and X is a *dendrite*, i.e., a locally connected, 1-arcwise connected continuum. In 1975, Mohler answered a question raised by Bing in [1], by proving in [6] that $\mathrm{Fix}_X(G)$ is nonempty if G is the discrete cyclic group $\mathbb{Z}$, and X is a 1-arcwise connected continuum. For further studies of fixed point theory of 1-arcwise connected spaces, one may consult [2, 4, 5].

The purpose of this paper is to prove the following common generalization of the above results of Isbell and Mohler.

Theorem 1.1. *If X is a 1-arcwise connected continuum and G is nilpotent as an abstract group, then $Fix_X(G)$ is nonempty.*

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If X is an arc and G is the solvable group $(\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}$, then there is a continuous action of G on X such that $\mathrm{Fix}_X(G)$ is empty (See Remark 2.5). Therefore nilpotency of G is also necessary. Young constructed a 1-arcwise connected continuum and a continuous self map of it without fixed points (see [8]). Therefore in Theorem 1.1, we cannot replace G by the semigroup $\mathbb{N} = \{0, 1, 2, \cdots\}$.

2. The proof

Let G be a group. Recall that the *commutator* of two elements a, b of G is by definition

$$[a, b] = a^{-1}b^{-1}ab.$$

For any two subsets A and B of G, define $[A, B]$ to be the subgroup generated by the set $\{[a, b] : a \in A, b \in B\}$. Set $G_0 = G$ and $G_{i+1} = [G_i, G]$, for $i = 0, 1, 2, \cdots$. Then we get a sequence

$$G_0 = G \rhd G_1 \rhd G_2 \rhd \cdots$$

of normal subgroups of G. If there is some $n \in \mathbb{N}$ such that $G_n = \{e\}$, then G is called *nilpotent* and

$$G_0 = G \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = \{e\}$$

is called the *lower center sequence* of G, where e is the identity of G.

When X is a 1-arcwise connected continuum and $x, y \in X$, we use the symbol $[x, y]$ to denote the unique arc in X from x to y (if $x = y$, then $[x, y]$ is defined to be the point set $\{x\}$).

We should note that, though the symbol [,] has two different meanings as above, it is easy to distinguish them in context. For simplicity, we write

$$gx = \phi(g, x), \quad \text{for all } g \in G,\ x \in X.$$

Lemma 2.1. *Let G be a nilpotent group, X the unit closed interval $[0, 1]$ and let $\phi : G \times X \longrightarrow X$ be a continuous action. Then $Fix_X(G)$ is nonempty.*

Proof. If for every $g \in G$ we have $g(0) = 0$ and $g(1) = 1$, then 0 and 1 are both fixed points of G. Otherwise there is some $g_0 \in G$ such that $g_0(0) = 1$ and $g_0(1) = 0$. Then g_0 has a unique fixed point $x_0 \in I$, and clearly g_0 is not the identity e. In the following, we will show that x_0 is also a fixed point of G.

Define inductively a sequence of subsets $N_i \subset G$ as follows. Let $N_0 = \{e\}$, where e is the identity of G. Suppose that N_i has been defined; then define $N_{i+1} = \{g \in G : [g, g_0] = g^{-1}g_0^{-1}gg_0 \in N_i\}$. Since G is a nilpotent group, there is a natural number m such that $N_m = G$. Thus we get a sequence of subsets: $\{e\} = N_0 \subset N_1 \subset \cdots \subset N_m = G$. If $g(x_0) = x_0$ for all $g \in N_i$, then for $g \in N_{i+1}$ we have

$$g_0^{-1}gx_0 = g(g^{-1}g_0^{-1}gg_0)x_0 = g[g, g_0]x_0 = gx_0.$$

So gx_0 is also a fixed point of g_0. But x_0 is the unique fixed point of g_0, so $gx_0 = x_0$. Thus x_0 is also a common fixed point of elements in N_{i+1}. Inductively, we get at last that for any $g \in G = N_m$, $gx_0 = x_0$. That is, x_0 is a fixed point of G. $\square$

Lemma 2.2. *Let G be a nilpotent group, H a normal subgroup of G and suppose that G/H is a cyclic group. Let X be a 1-arcwise connected space, and let $\phi : G \times X \longrightarrow X$ be a group action. If $Fix(H) \neq \emptyset$, then $Fix(G) \neq \emptyset$.*

Proof. Let $G/H =< \overline{g}_0 >$, where $g_0 \in G$ and $\overline{g}_0$ denotes the coset class of g_0 in G/H. Let $Y = \mathrm{Fix}(H)$. For each $y \in Y$ and $h \in H$, since $g_0^{-1}hg_0 \in H$, we have that $hg_0y = g_0(g_0^{-1}hg_0)y = g_0y$. Thus $g_0(Y) \subseteq Y$. Replacing g_0 by g_0^{-1}, we get $g_0^{-1}(Y) \subseteq Y$ similarly. Hence $g_0(Y) = Y$. Let $\mathcal{A} = \{[x, g_0(x)] : x \in Y\}$. Define a partial order " $\prec$ " in $\mathcal{A}$: $[x, g_0(x)] \prec [x', g_0(x')]$ if and only if $[x, g_0(x)] \supseteq [x', g_0(x')]$. It is easy to see that if $\mathcal{B} = \{[x_\lambda, g_0(x_\lambda)] : \lambda \in \Lambda\}$ is a totally ordered subset of $\mathcal{A}$, then $\bigcap_{\lambda\in\Lambda}[x_\lambda, g_0(x_\lambda)]$ is an upper bound of $\mathcal{B}$. So, by Zorn's Lemma, there is a maximal element $[y_0, g_0(y_0)] \in \mathcal{A}$. Now we discuss this in three cases.

Case 1. $g_0(y_0) = y_0$. Then y_0 is a fixed point of g_0, and thus is a fixed point of G.

Case 2. $g_0^2(y_0) = y_0$. Then by the uniquely arcwise connected property we see that $g_0([y_0, g_0y_0]) = [y_0, g_0y_0]$, and then $[y_0, g_0y_0]$ is a G-invariant interval. From Lemma 2.1, there is a fixed point of G in $[y_0, g_0y_0]$.

Case 3. $g_0(y_0) \neq y_0$ and $g_0^2(y_0) \neq y_0$. We will show that $[y_0, g_0y_0] \cap [g_0y_0, g_0^2y_0] = \{g_0y_0\}$. First by the uniquely arcwise connected property, $[y_0, g_0y_0] \cap [g_0y_0, g_0^2y_0] = [x, g_0y_0]$ for some $x \in X$. Then for each $h \in H$, we have $[h(x), g_0y_0] = h([x, g_0y_0]) = h([y_0, g_0y_0]) \cap h([g_0y_0, g_0^2y_0]) = [x, g_0y_0]$. Thus $h(x) = x$, and hence $x \in \mathrm{Fix}(H)$. Since $x \in [g_0y_0, g_0^2y_0]$, there exists some $x' \in [y_0, g_0y_0]$ such that $g_0(x') = x$. Then $[x', x] = [x', g_0(x')] \in \mathcal{A}$. On the other hand, since $[x', x] \subseteq [y_0, g_0y_0]$ and $[y_0, g_0y_0]$ is maximal in $\mathcal{A}$, it can only be that $[x', x] = [y_0, g_0y_0]$. This implies that $x' = y_0, x = g_0y_0$ (Since $g_0^2(y_0) \neq y_0$, the case $x' = g_0(y_0)$ and $x = y_0$ will not occur.) This completes the proof of the claim. It follows that

$$[g_0^{n-1}y_0, g_0^ny_0] \cap [g_0^ny_0, g_0^{n+1}y_0] = \{g_0^ny_0\}, \text{ for all } n \in \mathbb{Z}. \tag{2.1}$$

Denote $L = \bigcup_{n=-\infty}^{+\infty}[g_0^ny_0, g_0^{n+1}y_0]$, $L^+ = \bigcup_{n=0}^{+\infty}[g_0^ny_0, g_0^{n+1}y_0]$, and $L^- = \bigcup_{n=-\infty}^{-1}[g_0^ny_0, g_0^{n+1}y_0]$. Then $L = L^+ \cup L^-$. Noting that there is no circle in X, from (2.1) we see that L is an image of an injective continuous map defined over the real line $\mathbb{R}$. Now we discuss this in three cases.

Case 3.1. If there exists an arc $[a, b] \subseteq X$ such that $L^+ \subseteq [a, b]$, then $y = \lim\limits_{n\to\infty} g_0^{n+1}y_0 \in Y$ exists. Clearly, y is a fixed point of g_0, and moreover is a fixed point of G.

Case 3.2. If there exists an arc $[a, b] \subseteq X$ such that $L^- \subseteq [a, b]$, then similar to Case 3.1, we can also get a fixed point of G in Y.

Case 3.3. For any arc $[a, b] \subseteq X$, neither $L^+ \subseteq [a, b]$ nor $L^- \subseteq [a, b]$. This implies that for each $x \in X$, there exists a unique $p(x) \in L$ such that $[x, y_0] \cap L = [y_0, p(x)]$. For each $n \in \mathbb{Z}$, set $X_n = \{x \in X : p(x) \in [g_0^n(y_0), g_0^{n+1}(y_0)\}$. Then $\{X_n : n \in \mathbb{Z}\}$ becomes a partition of X. From [6] we know that each X_n is a Borel measurable set. Let μ be a g_0-invariant Borel probability measure. (For the existence of such a measure, one may consult [7], Corollary 6.9.1). Since $g_0(X_n) = X_{n+1}$ for all $n \in \mathbb{Z}$, we have $\mu(X_n) = \mu(X_m)$ for all $m, n \in \mathbb{Z}$. This contradicts $\mu(X) = 1$. So this case will not happen. $\square$

Lemma 2.3. *Let X be a 1-arcwise connected continuum and let $\phi : G \times X \to X$ be an action of G on X. Suppose H is a normal subgroup of G and G/H is a finitely generated abelian group. If $Fix(H) \neq \emptyset$, then $Fix(G) \neq \emptyset$.*

Proof. Since G/H is a finitely generated abelian group, $G/H \cong \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_n}$, where $\mathbb{Z}_{k_i} = \mathbb{Z}/ < k_i >$, for $1 \leq i \leq n$ (k_i may be 0). Let $\overline{H}_i$ be a subgroup of G/H defined by $\overline{H}_i = \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_i}$. Let $\phi : G \longrightarrow G/H$ be the quotient homomorphism, and let $H_i = \phi^{-1}(\overline{H}_i)$. Then $H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ is a sequence of normal subgroups of G and

$$H_{i+1}/H_i \cong (H_{i+1}/H_0)/(H_i/H_0) = (\mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_{i+1}})/(\mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_i}) \cong \mathbb{Z}_{k_{i+1}}.$$

So H_{i+1}/H_i is a cyclic group. Since $\mathrm{Fix}(H_0) = \mathrm{Fix}(H) \neq \emptyset$, using Lemma 2.2 repeatedly we obtain that $\mathrm{Fix}(G) \neq \emptyset$. □

Lemma 2.4. *Let X be a 1-arcwise connected continuum and let $\phi : G \times X \to X$ be an action of G on X. Suppose that H is a normal subgroup of G and G/H is abelian. If $Fix(H) \neq \emptyset$, then $Fix(G) \neq \emptyset$.*

Proof. For any finite subset S of G, we define A_S to be the group generated by H and S, that is, $A_S =< H, S >$. Since A_S/H is a finitely generated abelian group, it follows from Lemma 2.3 that $\mathrm{Fix}(A_S) \neq \emptyset$. As $\mathrm{Fix}(A_S) \cap \mathrm{Fix}(A_{S'}) = \mathrm{Fix}(A_{S \cup S'})$, we know $\mathcal{K} = \{\mathrm{Fix}(A_S) : S \text{ is a finite subset of } G\}$ has the finite intersection property. Since X is compact, we have $Y = \bigcap_{K \in \mathcal{K}} K \neq \emptyset$. Obviously $Y \subseteq \mathrm{Fix}(G)$, so $\mathrm{Fix}(G) \neq \emptyset$. □

Proof of Theorem 1.1. Consider the lower central sequence $G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{e\}$ of G. Since $G_{i+1} = [G, G_i]$, we know G_i/G_{i+1} is an Abelian group, for $0 \leq i \leq n-1$. Also, because $\mathrm{Fix}(G_n) = \mathrm{Fix}(\{e\}) = X \neq \emptyset$, it follows inductively from Lemma 2.4 that $\mathrm{Fix}(G) \neq \emptyset$. □

Remark 2.5. If the action group G is solvable, then Theorem 1.1 does not hold. For example, let f and g be the maps on the real line $\mathbb{R}$ defined by $f(x) = x+1$, $g(x) = -x$, for all $x \in \mathbb{R}$. It is well known that the group $< f, g >$ generated by f and g is solvable. Let $h : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be defined by $x \mapsto \tan x$. Now we define two homeomorphisms $\tilde{f}$ and $\tilde{g}$ on the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ by

$\tilde{f}(-\frac{\pi}{2}) = -\frac{\pi}{2}$, $\tilde{f}(\frac{\pi}{2}) = \frac{\pi}{2}$, and $\tilde{f}(x) = h^{-1} \circ f \circ h(x)$, for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$,
$\tilde{g}(-\frac{\pi}{2}) = \frac{\pi}{2}$, $\tilde{g}(\frac{\pi}{2}) = -\frac{\pi}{2}$, and $\tilde{g}(x) = h^{-1} \circ g \circ h(x)$, for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Then the group $< \tilde{f}, \tilde{g} >$ generated by $\tilde{f}$ and $\tilde{g}$ is isomorphic to the group $< f, g >$. So $< \tilde{f}, \tilde{g} >$ is a solvable group acting on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. It is obvious that there is no fixed point for this action.

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