

CONSTRUCTIVE DECOMPOSITION OF A FUNCTION OF TWO VARIABLES AS A SUM OF FUNCTIONS OF ONE VARIABLE

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(Communicated by Alexander N. Dranishnikov)

ABSTRACT. Given a compact set K in the plane, which does not contain any triple of points forming a vertical and a horizontal segment, and a map $f \in C(K)$, we give a construction of functions $g, h \in C(\mathbb{R})$ such that $f(x, y) = g(x) + h(y)$ for all $(x, y) \in K$. This provides a constructive proof for a part of Sternfeld's theorem on basic embeddings in the plane. In our proof the set K is approximated by a finite set of points.

1. INTRODUCTION

An embedding $\varphi: K \rightarrow \mathbb{R}^k$ of a compactum (compact metric space) K in the k -dimensional Euclidean space \mathbb{R}^k is called a *basic embedding* if for each continuous real-valued function $f \in C(K)$ there exist continuous real-valued functions of a single real variable $g_1, \dots, g_k \in C(\mathbb{R})$ such that $f(x_1, \dots, x_k) = g_1(x_1) + \dots + g_k(x_k)$ for each point $(x_1, \dots, x_k) \in \varphi(K)$. We also say that the set $\varphi(K)$ is *basically embedded* in \mathbb{R}^k .

The question of the existence of basic embeddings is related to Hilbert's 13th problem [5]: Hilbert conjectured that not all continuous functions of three variables are expressible as superpositions of continuous functions of a smaller number of variables.

Ostrand [10] proved that each n -dimensional compactum can be basically embedded in \mathbb{R}^{2n+1} for $n \geq 1$. His result is an easy generalization of results of Arnold [1, 3] and Kolmogorov [6, 7].

Sternfeld [14] proved that the $2n + 1$ is the best possible in a very strong sense: namely, no n -dimensional compactum can be basically embedded in \mathbb{R}^{2n} for $n \geq 2$. Ostrand's and Sternfeld's results thus characterize compacta basically embeddable in \mathbb{R}^k for $k \geq 3$. Basic embeddability in the real line is trivially equivalent to embeddability. The remaining problem of the characterization of compacta basically embedded in \mathbb{R}^2 was already raised by Arnold [2] and solved by Sternfeld [15]:

Theorem 1.1 (Sternfeld). *Let K be a compactum and let $\varphi: K \rightarrow \mathbb{R}^k$ be an embedding. Then*

(B) *φ is a basic embedding*

Received by the editors January 16, 2007, and, in revised form, January 31, 2008.

2000 *Mathematics Subject Classification*. Primary 26B40, 54C30; Secondary 54F99, 54C25.

Key words and phrases. Basic embedding, plane compactum, Kolmogorov representation theorem, Hilbert's 13th problem, finite approximation of sets.

The author was supported by grants VEGA 1/3002/06 and VEGA 1/3128/06.

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if and only if

(A) there exists an $m \in \mathbb{N}$ such that the set $\varphi(K)$ does not contain an array of length m .

Definition 1.2. An *array* is a sequence of points $\{(x_i, y_i) \in \mathbb{R}^2 \mid i \in I\}$, where $I = \{1, 2, \dots, m\}$ or $I = \mathbb{N}$, such that each two consecutive points (x_i, y_i) , (x_{i+1}, y_{i+1}) are different and either $x_{2j-1} = x_{2j}$ and $y_{2j} = y_{2j+1}$ for all j or $y_{2j-1} = y_{2j}$ and $x_{2j} = x_{2j+1}$ for all j . If $I = \{1, 2, \dots, m\}$, then the *length* of the array is $m - 1$.

Using the geometric description (A), Skopenkov [13] gave a characterization of continua basically embeddable in the plane by means of forbidden subsets resembling Kuratowski's characterization of planar graphs. In a similar way Kurlin [8] characterized finite graphs basically embeddable in $\mathbb{R} \times T_n$, where T_n is a star with n -rays. Repovš and Željko [11] proved a result concerning the smoothness of the functions in the decomposition on basically embedded subsets of the plane and gave a constructive decomposition on finite graphs in the plane which do not contain arrays of arbitrary length (i.e. those satisfying (A) of Theorem 1.1).

Sternfeld's proof of the equivalence (A) \Leftrightarrow (B) is not direct but uses a reduction to linear operators. In particular, it is not constructive. It is therefore desirable to find a straightforward, constructive proof which will consequently provide an elementary proof of Skopenkov's and Kurlin's characterizations. A constructive proof of (B) \Rightarrow (A) is given in [9]. In this paper we give an elementary construction proving the implication (A) \Rightarrow (B) provided that $m = 2$:

Theorem 1.3. *Let $\varphi: K \rightarrow \mathbb{R}^2$ be an embedding of a compactum K in the plane such that the set $\varphi(K)$ does not contain an array of length two. Then for every function $f \in C(\varphi(K))$ there exist functions $g, h \in C(\mathbb{R})$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in \varphi(K)$.*

So far, no constructive decomposition of f as $g + h$ on compacta in the plane satisfying (A) of Theorem 1.1 has been found, not even in the simplest case when the compactum satisfies (A) with $m = 2$. It turns out that even if a set $\varphi(K) \subseteq \mathbb{R}^2$ satisfies this simplest version of condition (A), a constructive decomposition of a function $f \in C(\varphi(K))$ is a non-trivial problem. We also believe that our argument can be modified to obtain a constructive proof (in the sense described below) of the implication (A) \Rightarrow (B) for an arbitrary $m \in \mathbb{N}$.

The main part of our proof (Theorem 2.2) consists of finding an approximate decomposition of a given function f as $g + h$. The functions g, h are defined on the projections of a finite approximation V of $\varphi(K)$. Then they are linearly extended to \mathbb{R} . Apart from two steps where we assert the existence of certain constants, this part of the proof is constructive. The existence of an exact decomposition $f = g + h$ follows by an elementary iterative procedure (Theorem 2.1).

In [11], the authors give the standard decomposition $f = g + h$ on finite graphs in the plane. According to the results of [13, 4], a finite graph can be basically embedded in the plane if and only if it can be embedded in a special graph R_n for some n . The authors of [11] inductively define an embedding $\varphi: R_n \rightarrow \mathbb{R}^2$. For a given function $f \in C(\varphi(R_n))$, they define the maps $g, h \in C(\mathbb{R})$ inductively again, starting from a well chosen subset of $\varphi(R_n)$. Although the sets in the plane we are dealing with do not contain arrays of length two, they can still be "arbitrarily bad". In particular, we are not able to choose a suitable subset on which we could start a similar construction of the functions g and h . However, we do use a similar

approach in the construction of the sequences of approximations which converge to the required functions g and h .

The proof of our result resembles, in a certain way, the proofs of Arnold [1, 3], Kolmogorov [6, 7] and Ostrand [10]. As in these proofs, we construct a sequence of finite families of squares. However, there the squares (or cubes in higher dimensions) are related only to the dimension of the set in question; here the squares are constructed using the property that the set does not contain an array of length two.

The paper is organized as follows: in Section 2 we formulate and prove the two statements, Theorem 2.1 and Theorem 2.2, which directly imply the main result, Theorem 1.3. Then we give the statements, Lemma 2.5 and Lemma 2.6, required in the proof of Theorem 2.2. Lemma 2.5 is proved in this section, and the proof of Lemma 2.6 is postponed to Section 3. Finally, in Section 4 we roughly explain where the proof breaks down in the case of an arbitrary m .

2. PROOF OF THE MAIN RESULT—THEOREM 1.3

It is convenient to omit the embedding φ from the notation $\varphi(K) \subseteq \mathbb{R}^2$ and assume that K is a subset of \mathbb{R}^2 .

We denote points in the plane by (x, y) , with $x, y \in \mathbb{R}$; intervals in \mathbb{R} by $[x; x']$, $[x; x')$, $(x; x')$, with $x, x' \in \mathbb{R}$; and segments in \mathbb{R}^2 by $[z; z']$, with $z, z' \in \mathbb{R}^2$. We define the distance in \mathbb{R}^2 as $|(x, y) - (x', y')| = \max\{|x - x'|, |y - y'|\}$ for $(x, y), (x', y') \in \mathbb{R}^2$. By $p, q: \mathbb{R}^2 \rightarrow \mathbb{R}$, we denote the vertical and horizontal orthogonal projections: $p(x, y) = x, q(x, y) = y$.

The main result, Theorem 1.3, clearly follows from the following two theorems.

Theorem 2.1. *Let $X \subseteq \mathbb{R}^2$ be a compact subset of the plane. Assume that there exists a positive integer $k \in \mathbb{N}$ such that for each function $f \in C(X)$ and each positive real $\varepsilon > 0$ there exist functions $g, h \in C(\mathbb{R})$ satisfying the following requirements:*

- (1) $|f(x, y) - g(x) - h(y)| < \varepsilon$ for each point $(x, y) \in X$,
- (2) $\|g\| \leq k\|f\|, \|h\| \leq k\|f\|$.

Then there exist functions $g, h \in C(\mathbb{R})$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in X$.

The above result is obtained by a slight modification of a part of Theorem 4.13 in [12]; we include the proof for the sake of completeness.

Proof. Let k and f be given. Let $f_1 = f$. Pick a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive reals such that $\sum_{n=1}^\infty \varepsilon_n < \infty$. Assume that $n \geq 1$ and f_n is constructed. Choose functions $g_n, h_n \in C(\mathbb{R})$ such that $\|f_n - g_n - h_n\| < \varepsilon_n, \|g_n\| \leq k\|f_n\|$ and $\|h_n\| \leq k\|f_n\|$. Let $f_{n+1} = f_n - g_n - h_n$.

Since $\|g_{n+1}\| \leq k\|f_{n+1}\| = k\|f_n - g_n - h_n\| < k\varepsilon_n$ for each n , the series $\sum_{n=1}^\infty g_n$ uniformly converges to a function $g \in C(\mathbb{R})$. Similarly $\sum_{n=1}^\infty h_n = h \in C(\mathbb{R})$. We can easily see that $g + h = f_1 = f$. □

The proof of the following theorem is postponed to the end of this section.

Theorem 2.2. *Suppose that a compactum $K \subseteq \mathbb{R}^2$ contains no array of length two and that $f \in C(K)$ is a continuous function on K . Then for each $\varepsilon > 0$ there exist functions $g, h \in C(\mathbb{R})$ satisfying the following requirements:*

- (1) $|f(x, y) - g(x) - h(y)| < \varepsilon$ for each point $(x, y) \in K$,
- (2) $\|g\| \leq \|f\|, \|h\| \leq 2\|f\|$.

The construction of the approximations g and h in Theorem 2.2 mimics the following construction of the functions g, h which works for certain types of sets K (for example finite graphs, considered in [11]). In particular, this idea is used in the proof of Lemma 2.6. Let K_x be the set of all points $(x, y) \in K$ which have a neighbor in the vertical direction in K , i.e. $K_x = \{(x, y) \in K \mid \exists(x, y') \in K, y \neq y'\}$. The set K_y is defined in a similar way. Assume that both sets K_x and K_y are closed.

Since K does not contain an array of length two, the functions p and q are injective on K_y and K_x , respectively, and the sets K_x and K_y are disjoint. For each point $x \in p(K_y)$ let $g(x) = f(x, p^{-1}(x))$, and for each point $x \in p(K_x)$ let $g(x) = 0$. Since $p(K_x \cup K_y)$ is closed, we can extend g continuously to \mathbb{R} . For each point $y \in q(K)$ pick an arbitrary point $(x, y) \in K$ and let $h(y) = f(x, y) - g(x)$. It is easily seen that h is continuous on $q(K)$, and thus it can be extended continuously to \mathbb{R} .

The function g was constructed on the set $p(K_x \cup K_y)$ first, and then, using the fact that $p(K_x \cup K_y)$ is closed, it was extended continuously to \mathbb{R} . In general, neither $p(K_x \cup K_y)$ nor $q(K_x \cup K_y)$ is closed, though, and the set K can be so “bad” that we cannot find suitable sets like K_x and K_y to begin the construction. Therefore we construct approximations of K .

Definition 2.3. Let $\alpha > 0$ be a positive real. Two points $z_1, z_2 \in \mathbb{R}^2$ are said to form an α -vertical pair if $|p(z_1) - p(z_2)| < \alpha$ and an α -horizontal pair if $|q(z_1) - q(z_2)| < \alpha$. A sequence of points $\{z_1, z_2, \dots, z_k\}$ from \mathbb{R}^2 such that $z_i \neq z_j$ for all $i \neq j$, with the exception that possibly $z_1 = z_k$, is said to form an α -array if each pair of consecutive points z_i, z_{i+1} forms either an α -vertical pair or an α -horizontal pair or both.

Note that, unlike an array, an α -array can contain two or more consecutive α -vertical pairs or two or more consecutive α -horizontal pairs.

Definition 2.4. Let $\alpha, \beta > 0$ be positive reals and let $l \in \mathbb{N}$ be a positive integer. A set $X \subseteq \mathbb{R}^2$ is said to be (α, β, l) -faithful if for each α -array $\{z_0, z_1, \dots, z_{k+1}\}$ in X such that $|p(z_0) - p(z_1)| \geq \beta$ and $|q(z_k) - q(z_{k+1})| \geq \beta$ we have $k > l$.

Clearly, if X is (α, β, l) -faithful, then for each positive $\alpha' \leq \alpha$ each subset $X' \subseteq X$ is (α', β, l) -faithful.

Lemma 2.5. *Suppose that a compactum $K \subseteq \mathbb{R}^2$ contains no array of length two. Then for each $\beta > 0$ and $l \in \mathbb{N}$ there exists an $\alpha > 0$ such that K is (α, β, l) -faithful.*

Proof. Assuming the contrary, there exists a $\beta > 0$ and an $l \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ the set K is not $(1/n, \beta, l)$ -faithful. That is, for each n there is a $1/n$ -array $\{z_0^n, z_1^n, \dots, z_{k_n}^n\} \subseteq K$ with $|p(z_0^n) - p(z_1^n)| \geq \beta$, $|q(z_{k_n}^n) - q(z_{k_n+1}^n)| \geq \beta$ and $k_n \leq l$. Without loss of generality we may assume that $k_n = l$ for all n . Since K is compact, there is a subsequence $\{\{z_0^{m_n}, z_1^{m_n}, \dots, z_{l+1}^{m_n}\}\}_n$ such that for each i the sequence $\{z_i^{m_n}\}_n$ converges to a point $z_i \in K$. For each i either $p(z_i) = p(z_{i+1})$ or $q(z_i) = q(z_{i+1})$ (or both in case z_i and z_{i+1} are equal). Moreover, $|p(z_0) - p(z_1)| \geq \beta$ and $|q(z_l) - q(z_{l+1})| \geq \beta$. Hence the set $\{z_0, z_1, \dots, z_{l+1}\}$, which is a subset of K , contains an array of length two, which is a contradiction. \square

The proof of the following lemma is postponed until the next section.

Lemma 2.6. *Let $f: V \rightarrow \mathbb{R}$ be a function defined on a finite set of points $V \subseteq \mathbb{R}^2$. Let $\varepsilon, \delta, \alpha > 0$ be positive reals such that $|f(z) - f(z')| < \varepsilon$ for all pairs $z, z' \in V$ with $|z - z'| < \delta$ and the set V is $(\alpha, \delta, [\|f\|/\varepsilon])$ -faithful. Then there exist functions $G: p(V) \rightarrow \mathbb{R}$ and $H: q(V) \rightarrow \mathbb{R}$ satisfying the following requirements:*

- (1) (a) $|f(x, y) - G(x) - H(y)| \leq 4\varepsilon$ for each point $(x, y) \in V$,
- (b) $|G(x_1) - G(x_2)| \leq \varepsilon$, for each α -vertical pair $(x_1, y_1), (x_2, y_2) \in V$,
- (c) $|H(y_1) - H(y_2)| \leq 2\varepsilon$ for each α -horizontal pair $(x_1, y_1), (x_2, y_2) \in V$;
- (2) $\|G\| \leq \|f\|, \|H\| \leq 2\|f\|$.

Proof of Theorem 2.2. Since f is continuous on the compact set K , the function f is uniformly continuous. Therefore there exists a positive real $\delta > 0$ such that $|f(z) - f(z')| < \varepsilon/8$ for all points $z, z' \in K$ with $|z - z'| < \delta$. According to Lemma 2.5, there exists an $\alpha > 0$ such that the set K is $(\alpha, \delta, [8\|f\|/\varepsilon])$ -faithful. Since the set K is $(\alpha', \delta, [8\|f\|/\varepsilon])$ -faithful for each $\alpha' \leq \alpha$, we may assume that $\alpha < \delta$. Pick a point from each non-empty intersection $K \cap ([i\alpha/2; (i + 1)\alpha/2) \times [j\alpha/2; (j + 1)\alpha/2)$, $i, j \in \mathbb{Z}$. Denote by V the union of these points. Since V is a subset of K , the set V is $(\alpha, \delta, [8\|f\|/\varepsilon])$ -faithful. According to Lemma 2.6, there exist functions $G: p(V) \rightarrow \mathbb{R}$ and $H: q(V) \rightarrow \mathbb{R}$ satisfying items (1a)–(1c) and (2) of that lemma.

Let $p(V) = \{x_1, \dots, x_k\}$, with $x_i < x_{i+1}$ for all i . Define g on $p(V)$ by letting $g(x_i) = G(x_i)$ for each i . On each interval $[x_i; x_{i+1}]$ such that $|x_i - x_{i+1}| < \alpha$, extend g linearly between the values $g(x_i)$ and $g(x_{i+1})$. If an interval $[x_i; x_{i+1}]$ is such that $|x_i - x_{i+1}| \geq \alpha$, then there exists an interval of the form $I = [j\alpha/2; (j + 1)\alpha/2)$ such that $x_i < j\alpha/2 < (j + 1)\alpha/2 \leq x_{i+1}$ and $p^{-1}(I) \cap K = \emptyset$. On the interval $[x_i; j\alpha/2]$ extend g as a constant, equal to $g(x_i)$. If the interval $[(j + 1)\alpha/2; x_{i+1}]$ is non-degenerate, i.e. if $(j + 1)\alpha/2 < x_{i+1}$, then extend g on this interval as a constant equal to $g(x_{i+1})$. On the interval $[j\alpha/2; (j + 1)\alpha/2]$ extend g linearly from $g(x_i)$ to $g(x_{i+1})$. On the intervals $(-\infty; x_1]$ and $[x_k; \infty)$ extend g as a constant.

Let $q(V) = \{y_1, \dots, y_l\}$ with $y_i < y_{i+1}$ for all i . Let $h(y_i) = H(y_i)$ for each i . Extend h to \mathbb{R} in a similar way as g .

By (1) of Lemma 2.6 and the assumption that $\alpha < \delta$, it follows that $|f(x, y) - g(x) - h(y)| < \varepsilon$. The bounds on the norms of the functions g and h follow from (2) of Lemma 2.6. □

3. PROOF OF LEMMA 2.6

We call a pair of points $z_1, z_2 \in V$ *distant* if $|z_1 - z_2| \geq \delta$; otherwise we call them *close*. Evidently $|f(z_1) - f(z_2)| < \varepsilon$ holds for every close pair of points $z_1, z_2 \in V$.

In this proof we use the idea given below the statement of Theorem 2.2 in Section 2. The analogue of the set K_x is the set of all points from V which belong to distant α -vertical pairs (i.e. it is the set $\{u \in V \mid \exists w \in V \text{ s.t. } u, w \text{ is a distant } \alpha\text{-vertical pair}\}$), while the analogue of the set K_y is the set of all points from V which belong to distant α -horizontal pairs. Accordingly, we define G to be approximately 0 on the first set and approximately f on the second set. In the rest of the points we only require that the difference between the values of G in the vertical projection of each α -vertical close pair does not exceed ε . We define H as $f - G$.

For technical reasons we start by defining a function $\gamma: V \rightarrow \mathbb{R}$ with the properties listed below. The function G shall be approximately γ .

- (i) (a) $|\gamma(w_1) - \gamma(w_2)| \leq \varepsilon$ for each α -vertical close pair and each α -horizontal close pair $w_1, w_2 \in V$,
 (b) $|f(w) - \gamma(w)| \leq \varepsilon$ for each $w \in V$ which belongs to a distant α -horizontal pair,
 (c) $|\gamma(w)| \leq \varepsilon$ for each $w \in V$ which belongs to a distant α -vertical pair;
 (ii) $\|\gamma\| \leq \|f\|$.

To construct γ , we define two abstract labeled graphs (V_+, E_+, l_+) and (V_-, E_-, l_-) , where both l_+ and l_- assign non-negative real values to the edges. The vertex set V_+ of the first graph is the set of all points $w \in V$ with $f(w) \geq 0$ with one vertex \mathbf{w}_+ added, i.e. $V_+ = \{w \in V \mid f(w) \geq 0\} \cup \{\mathbf{w}_+\}$. The vertex set V_- of the second graph is the set of all points $w \in V$ with $f(w) < 0$ with one vertex \mathbf{w}_- added, i.e. $V_- = \{w \in V \mid f(w) < 0\} \cup \{\mathbf{w}_-\}$. We define $f(\mathbf{w}_+) = \|f\| + \varepsilon$ and $f(\mathbf{w}_-) = -\|f\| - \varepsilon$.

The labeled edge set E_+ consists of edges

- $w_1 w_2$, where w_1, w_2 is a close α -vertical pair or a close α -horizontal pair, with label $l_+(w_1 w_2) = \varepsilon$
- $\mathbf{w}_+ w$, where w belongs to a distant α -horizontal pair, with label $l_+(\mathbf{w}_+ w) = f(\mathbf{w}_+) - f(w) = \|f\| + \varepsilon - f(w)$.

The edges E_- are defined analogously. The edges of the form $\mathbf{w}_- w$ are labeled by $-f(\mathbf{w}_-) + f(w) = \|f\| + \varepsilon + f(w)$, and the remaining edges are labeled by ε . Note that

$$(3.1) \quad \begin{aligned} |f(w_1) - f(w_2)| &\leq l_+(w_1 w_2) && \text{for all } w_1 w_2 \in E_+, \\ |f(w_1) - f(w_2)| &\leq l_-(w_1 w_2) && \text{for all } w_1 w_2 \in E_-. \end{aligned}$$

Let $d_+ : V_+ \rightarrow [0; \infty]$ be the function assigning to each vertex connected to \mathbf{w}_+ by a path in E_+ its weighed distance from \mathbf{w}_+ and assigning to each other vertex the value ∞ . For each vertex $w \in V_+$ let

$$\gamma(w) = \max \{ \|f\| + \varepsilon - d_+(w), 0 \}.$$

The function $d_- : V_- \rightarrow [0; \infty]$ is defined analogously, and γ on V_- is defined as $\gamma(w) = \min \{ -\|f\| - \varepsilon + d_-(w), 0 \}$.

Let us show that the function γ satisfies (ia)–(ic) and (ii).

(ia) Let w_1, w_2 be a close α -vertical pair or a close α -horizontal pair in V . If both w_1 and w_2 are in V_+ or both w_1 and w_2 are in V_- , then it follows directly from the definition of γ that $|\gamma(w_1) - \gamma(w_2)| \leq \varepsilon$. So, let $w_1 \in V_+$ and $w_2 \in V_-$. By an inductive argument we show that

$$(3.2) \quad \begin{aligned} 0 \leq \gamma(w) \leq f(w) &&& \text{for all } w \in V_+, \\ f(w) \leq \gamma(w) \leq 0 &&& \text{for all } w \in V_-. \end{aligned}$$

The points w_1, w_2 form a close pair; therefore $|f(w_1) - f(w_2)| < \varepsilon$. So $|\gamma(w_1) - \gamma(w_2)| = \gamma(w_1) - \gamma(w_2) \leq f(w_1) - f(w_2) < \varepsilon$.

(ib) Let w belong to a distant α -horizontal pair in V . For example, let $w \in V_+$. Since w is connected to \mathbf{w}_+ by an edge labeled by $\|f\| + \varepsilon - f(w)$, this expression is an upper bound on $d_+(w)$. On the other hand, equation (3.1) implies that the sum of the labels on each path connecting \mathbf{w}_+ to w is at least $\|f\| + \varepsilon - f(w)$, and (ib) follows.

(ic) Let w belong to a distant α -vertical pair in V . For instance, let $w \in V_+$. If w is not connected to \mathbf{w}_+ by a path, then, by definition, $\gamma(w) = 0$. So, let $\mathbf{w}_+ w_1 w_2 \dots w_k$ with $w_k = w$ be a path such that $d_+(w)$ is equal to the sum of its

labels. (Note that except for \mathbf{w}_+w_1 this path does not contain any other edge of the form \mathbf{w}_+w because this would yield a cycle.) By the definition of E_+ , the vertex w_1 belongs to some distant α -horizontal pair w_0, w_1 in V . By assumption the vertex $w_k = w$ belongs to some distant α -vertical pair w_k, w_{k+1} . The path $w_1w_2 \dots w_k$ is such that each of its edges corresponds to a close α -vertical or α -horizontal pair in V . Hence the vertices w_0, w_1, \dots, w_{k+1} form an α -array which satisfies the requirements of Definition 2.4. By assumption the set V is $(\alpha, \delta, \lceil \|f\|/\varepsilon \rceil)$ -faithful, so $k > \lceil \|f\|/\varepsilon \rceil$. Each of the edges $w_1w_2, \dots, w_{k-1}w_k$ is labeled by ε , so $d_+(w) = l_+(\mathbf{w}_+w_1) + (k - 1)\varepsilon > \|f\| - f(w_1) + \varepsilon + (\lceil \|f\|/\varepsilon \rceil - 1)\varepsilon \geq \|f\|$. Hence $\|f\| + \varepsilon - d_+(w) < \varepsilon$, so $|\gamma(w)| < \varepsilon$.

Item (ii) follows directly from equation (3.2).

The functions G and H are constructed in the following way. For each point $x \in p(V)$ fix an arbitrary point $(x, y_x) \in V$ and define $G(x) = \gamma(x, y_x)$. For each point $y \in q(V)$ fix an arbitrary point $(x_y, y) \in V$ and let $H(y) = f(x_y, y) - \gamma(x_y, y)$.

In order to prove item (1a) of Lemma 2.6, consider an arbitrary point $(x, y) \in V$. Then

$$(3.3) \quad |f(x, y) - G(x) - H(y)| = |f(x, y) - \gamma(x, y_x) - f(x_y, y) + \gamma(x_y, y)|.$$

The points $(x, y), (x, y_x)$ form an α -vertical pair (in fact a vertical pair), and the points $(x, y), (x_y, y)$ form an α -horizontal pair (in fact a horizontal pair).

If both pairs are close pairs, then item (ia) and the fact that $|f(x, y) - f(x_y, y)| < \varepsilon$ imply that the expression in equation (3.3) is at most 3ε .

Let $(x, y), (x, y_x)$ be a distant pair. From (ic) it follows that $|\gamma(x, y)| \leq \varepsilon$ and $|\gamma(x, y_x)| \leq \varepsilon$.

If $(x, y), (x_y, y)$ is a close pair, then $|f(x, y) - f(x_y, y)| < \varepsilon$ and, by (ia), it follows that $|\gamma(x_y, y) - \gamma(x, y)| \leq \varepsilon$. Therefore $|\gamma(x_y, y)| \leq 2\varepsilon$ and (3.3) is at most 4ε .

If $(x, y), (x_y, y)$ is a distant pair, then, by (ib), we have $|f(x_y, y) - \gamma(x_y, y)| \leq \varepsilon$ and $|f(x, y) - \gamma(x, y)| \leq \varepsilon$, so $|f(x, y)| \leq 2\varepsilon$. Hence (3.3) is at most 4ε .

If the pair $(x, y), (x, y_x)$ is close and the pair $(x, y), (x_y, y)$ is distant, then (3.3) can be rewritten as $|(f(x, y) - \gamma(x, y)) + (\gamma(x, y) - \gamma(x, y_x)) + (-f(x_y, y) + \gamma(x_y, y))|$ and we obtain the bound 3ε .

In order to prove (1b), consider an α -vertical pair $(x, y), (u, v)$. In this case

$$(3.4) \quad |G(x) - G(u)| = |\gamma(x, y_x) - \gamma(u, v_u)|.$$

The points $(x, y_x), (u, v_u)$ form an α -vertical pair as well. If it is a close pair, then the expression in equation (3.4) is bounded by ε , by (ia); and if it is a distant pair, then it is bounded by ε as well, by (ic).

Item (1c) is shown similarly, using the expression

$$|H(y) - H(v)| = |f(x_y, y) - \gamma(x_y, y) - f(u_v, v) + \gamma(u_v, v)|.$$

Item (2) follows directly from (ii).

4. THE CASE OF AN ARBITRARY m

Let us roughly explain where the proof of the implication (A) \Rightarrow (B) of Theorem 1.1 breaks down in the case of an arbitrary m . The analogue of the faithfulness property in the case when a given set K contains no array of length m , $m > 2$, is the following. A set $X \subseteq \mathbb{R}^2$ is (m, α, β, l) -faithful if for each α -array $\{z_0, z_1, \dots, z_{k+1}\}$ in X with the property that there are indices $0 \leq i_1 < j_1 \leq i_2 < j_2 \leq \dots \leq i_m < j_m \leq k + 1$ such that $|p(z_{i_1}) - p(z_{j_1})| \geq \beta$, $|q(z_{i_2}) - q(z_{j_2})| \geq \beta$, $|p(z_{i_3}) - p(z_{j_3})| \geq \beta$,

\dots , $|q(z_{i_m}) - q(z_{j_m})| \geq \beta$ for an even m and $|p(z_{i_m}) - p(z_{j_m})| \geq \beta$ for an odd m , or a similar series of inequalities but starting with $|q(z_{i_1}) - q(z_{j_1})| \geq \beta$, we have $k > l$.

However, we cannot prove an analogue of Lemma 2.5, not even in the case when the set K contains no array of length $m = 3$. For instance, there can be a sequence of $1/n$ -arrays $\{\{z_0^n, z_1^n, \dots, z_5^n\}\}_n$, where $|p(z_0^n) - p(z_1^n)| \geq \beta$, $|q(z_1^n) - q(z_2^n)| \geq \beta$, $|p(z_4^n) - p(z_5^n)| \geq \beta$, and $z_0^n \rightarrow z_0$, $z_1^n \rightarrow z_1$, $z_2^n \rightarrow z_2$, $z_3^n \rightarrow (z_1 + z_2)/2$, $z_4^n \rightarrow z_1$, $z_5^n \rightarrow z_0$ as $n \rightarrow \infty$. Hence this sequence converges to an array of length two and not three. So this does not lead to a contradiction as in the proof of Lemma 2.5.

ACKNOWLEDGMENTS

The author would like to thank Dušan Repovš and Arkadyi Skopenkov for the inspiration for this paper, Lev Bukovský for support, and especially Neža Mramor-Kosta for endless conversations on the topic and invaluable advice. The author would also like to thank the anonymous referee for valuable suggestions and comments.

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