FINITENESS PROPERTIES OF LOCAL COHOMOLOGY MODULES FOR $a$-MINIMAX MODULES

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(Communicated by Bernd Ulrich)

Abstract. Let $R$ be a commutative Noetherian ring and $a$ an ideal of $R$. In this paper we introduce the concept of $a$-minimax $R$-modules, and it is shown that if $M$ is an $a$-minimax $R$-module and $t$ a non-negative integer such that $H^i_a(M)$ is $a$-minimax for all $i < t$, then for any $a$-minimax submodule $N$ of $H^t_a(M)$, the $R$-module $\text{Hom}_R(R/a, H^t_a(M)/N)$ is $a$-minimax. As a consequence, it follows that the Goldie dimension of $H^t_a(M)/N$ is finite, and so the associated primes of $H^t_a(M)/N$ are finite. This generalizes the main result of Brodmann and Lashgari (2000).

1. Introduction

Let $R$ be a commutative Noetherian ring, $a$ an ideal of $R$, and $M$ a finitely generated $R$-module. An important problem in commutative algebra is determining when the set of associated primes of the $i^{th}$ local cohomology module $H^i_a(M)$ of $M$ with support in $V(a)$ is finite (see [11, Problem 4]). A. Singh [21] and M. Katzman [12] have given counterexamples to this conjecture. However, it is known that this conjecture is true in many situations; see [1, 2, 9, 10, 13, 14, 15]. In particular, Brodmann and Lashgari [1, Theorem 2.2] showed that if, for a finitely generated $R$-module $M$ and an integer $t$, the local cohomology modules $H^0_a(M), H^1_a(M), \ldots, H^{t-1}_a(M)$ are finitely generated, then the set $\text{Ass}_R H^t_a(M)/N$ is finite for every finitely generated submodule $N$ of $H^t_a(M)$. For a survey of recent developments on finiteness properties of local cohomology modules, see Lyubeznik’s interesting paper [15].

This paper is concerned with what might be considered a generalization of the above-mentioned result of Brodmann and Lashgari to the class of $a$-minimax modules. More precisely, we shall show that:

Theorem 1.1. Let $R$ be a Noetherian ring, $a$ an ideal of $R$ and $M$ an $a$-minimax $R$-module. Let $t$ be a non-negative integer such that $H^t_a(M)$ is $a$-minimax for all
Then for any $a$-minimax submodule $N$ of $H^t_a(M)$, the $R$-module $\text{Hom}_R(R/a, H^t_a(M)/N)$ is $a$-minimax. In particular, the Goldie dimension of $H^t_a(M)/N$ is finite, and so the set $\text{Ass}_R H^t_a(M)/N$ is finite.

Recall that an $R$-module $M$ is said to have finite Goldie dimension (written $\text{Gdim} \ M < \infty$) if $M$ does not contain an infinite direct sum of non-zero submodules, or equivalently the injective hull $E(M)$ of $M$ decomposes as a finite direct sum of indecomposable (injective) submodules. Also, an $R$-module $M$ is said to have finite $a$-relative Goldie dimension if the Goldie dimension of the $a$-torsion submodule $\Gamma_a(M) := \bigcup_{n \geq 1} (0 : M a^n)$ of $M$ is finite.

We say that an $R$-module $M$ is $a$-minimax if the $a$-relative Goldie dimension of any quotient module of $M$ is finite. One of our tools for proving Theorem 1.1 is the following:

**Proposition 1.2.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ be a finitely generated $R$-module and $N$ an arbitrary $R$-module. Let $t$ be a non-negative integer such that $\text{Ext}^i_R(M, N)$ is $a$-minimax for all $i \leq t$. Then for any finitely generated $R$-module $L$ with $\text{Supp} L \subseteq \text{Supp} M$, $\text{Ext}^i_R(L, N)$ is $a$-minimax for all $i \leq t$.

Throughout this paper, $R$ will always be a commutative Noetherian ring with non-zero identity, and $a$ will be an ideal of $R$. The $i$th local cohomology module of an $R$-module $M$ with respect to $a$ is defined by

$$H^i_a(M) = \lim_{\rightarrow} \text{Ext}^i_R(R/a^n, M).$$

We refer the reader to [7] or [3] for the basic properties of local cohomology.

2. $a$-MINIMAX MODULES AND GOLDIE DIMENSION

For an $R$-module $M$, the **Goldie dimension of $M$** is defined as the cardinal of the set of indecomposable submodules of $E(M)$ which appear in a decomposition of $E(M)$ into a direct sum of indecomposable submodules. We shall use $\text{Gdim} \ M$ to denote the Goldie dimension of $M$. For a prime ideal $p$, let $\mu^0(p, M)$ denote the 0-th Bass number of $M$ with respect to the prime ideal $p$. It is known that $\mu^0(p, M) > 0$ if and only if $p \in \text{Ass}_R M$. It is clear by the definition of the Goldie dimension that

$$\text{Gdim} \ M = \sum_{p \in \text{Ass}_R M} \mu^0(p, M).$$

Also, for any ideal $a$ of $R$ and any $R$-module $M$, the **$a$-relative Goldie dimension of $M$** is defined as

$$\text{Gdim}_a M := \sum_{p \in \text{V}(a)} \mu^0(p, M).$$

The $a$-relative Goldie dimension of an $R$-module $M$ has been studied in [5].

In [21], H. Zöschinger introduced the interesting class of minimax modules, and he has in [21] and [25] given many equivalent conditions for a module to be minimax. The $R$-module $M$ is said to be a **minimax module** if there is a finitely generated submodule $N$ of $M$, such that $M/N$ is Artinian. It was shown by T. Zink [23] and by E. Enochs [6] that a module over a complete local ring is minimax if and only if it is Matlis reflexive. On the other hand, it is known that when $R$ is a
Noetherian ring, a module is minimax if and only if each of its quotients has finite Goldie dimension, [23] or [24]. This motivates the definition:

**Definition 2.1.** Let $\mathfrak{a}$ be an ideal of $R$. An $R$-module $M$ is said to be **minimax with respect to** $\mathfrak{a}$ or **$\mathfrak{a}$-minimax** if the $\mathfrak{a}$-relative Goldie dimension of any quotient module of $M$ is finite; i.e., for any submodule $N$ of $M$, $G\dim_\mathfrak{a} M/N < \infty$.

**Remark 2.2.** Let $\mathfrak{a}$ be an ideal of $R$ and let $M$ be an $R$-module.

(i) If $\mathfrak{a} = 0$, then $M$ is $\mathfrak{a}$-minimax if and only if $M$ is minimax.

(ii) If $M$ is $\mathfrak{a}$-torsion, then $M$ is $\mathfrak{a}$-minimax if and only if $M$ is minimax by [5, Lemma 2.6].

(iii) If $M$ is Noetherian or Artinian, then $M$ is $\mathfrak{a}$-minimax.

(iv) If $\mathfrak{b}$ is a second ideal of $R$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and $M$ is $\mathfrak{a}$-minimax, then $M$ is $\mathfrak{b}$-minimax. In particular, every minimax module is $\mathfrak{a}$-minimax.

The following proposition is needed in the proof of the main theorem of this paper.

**Proposition 2.3.** Let $\mathfrak{a}$ be an ideal of $R$. Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of $R$-modules. Then $M$ is $\mathfrak{a}$-minimax if and only if $M'$ and $M''$ are both $\mathfrak{a}$-minimax.

**Proof.** We may suppose for the proof that $M'$ is a submodule of $M$ and that $M'' = M/M'$. If $M$ is $\mathfrak{a}$-minimax, then it easily follows from the definition that $M'$ and $M/M'$ are $\mathfrak{a}$-minimax. Now, suppose that $M'$ and $M/M'$ are $\mathfrak{a}$-minimax. Let $N$ be an arbitrary submodule of $M$, and let $p \in \text{Ass}(M/N) \cap V(\mathfrak{a})$. Then the exact sequence

$$0 \to \frac{M' + N}{N} \to \frac{M}{N} \to \frac{M}{M' + N} \to 0$$

induces the exact sequence

$$0 \to \text{Hom}_R(k(p), \frac{M'_p}{M'_p \cap N_p}) \to \text{Hom}_R(k(p), \frac{M_p}{N_p}) \to \text{Hom}_R(k(p), \frac{M_p}{M'_p + N_p}),$$

where $k(p) = R_p/pR_p$. Moreover, since $\text{Ass}_R M/N \subseteq \text{Ass}_R \frac{M' + N}{N} \cup \text{Ass}_R \frac{M}{M' + N}$, and the sets $\text{Ass}_R \frac{M' + N}{N} \cap V(\mathfrak{a})$ and $\text{Ass}_R \frac{M}{M' + N} \cap V(\mathfrak{a})$ are finite, it follows that $G\dim_\mathfrak{a} M/N < \infty$, and so $M$ is $\mathfrak{a}$-minimax.

**Corollary 2.4.** Let $\mathfrak{a}$ be an ideal of $R$. Then any quotient of an $\mathfrak{a}$-minimax module, as well as any finite direct sum of $\mathfrak{a}$-minimax modules, is $\mathfrak{a}$-minimax.

**Proof.** The assertion follows from the definition and Proposition 2.3.

**Corollary 2.5.** Let $\mathfrak{a}$ be an ideal of $R$. Let $M$ be a finitely generated $R$-module and $N$ an $\mathfrak{a}$-minimax $R$-module. Then $\text{Ext}_R^i(M, N)$ and $\text{Tor}_R^i(M, N)$ are $\mathfrak{a}$-minimax modules for all $i$. In particular, the $R$-modules $\text{Ext}_R^i(R/\mathfrak{a}, N)$ and $\text{Tor}_R^i(R/\mathfrak{a}, N)$ are $\mathfrak{a}$-minimax for all $i$. 


Proof. As \( R \) is Noetherian and \( M \) is finitely generated, it follows that \( M \) possesses a free resolution
\[
F_\bullet : \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0,
\]
whose free modules have finite ranks.
Thus \( \text{Ext}^i_R(M, N) = H^i(\text{Hom}_R(F_\bullet, N)) \) is a subquotient of a direct sum of finitely many copies of \( N \). Therefore, it follows from Corollary 2.4 that \( \text{Ext}^i_R(M, N) \) is \( \mathfrak{a} \)-minimax for all \( i \geq 0 \). By using a similar proof as above we can deduce that \( \text{Tor}^i_R(M, N) \) is \( \mathfrak{a} \)-minimax for all \( i \geq 0 \). \( \Box \)

**Proposition 2.6.** Let \( \mathfrak{a} \) be an ideal of \( R \). Let \( M \) be an \( \mathfrak{a} \)-minimax \( R \)-module such that \( \text{Ass}_R M \subseteq V(\mathfrak{a}) \). Then \( H^i_\mathfrak{a}(M) \) is \( \mathfrak{a} \)-minimax for all \( i \geq 0 \).

Proof. If \( i = 0 \), then \( H^0_\mathfrak{a}(M) = \Gamma_\mathfrak{a}(M) \) is a submodule of \( M \), and so by Proposition 2.3, \( \Gamma_\mathfrak{a}(M) \) is \( \mathfrak{a} \)-minimax. As \( \text{Ass}_R M/\Gamma_\mathfrak{a}(M) \subseteq \text{Ass}_R M \), it easily follows from \( \text{Ass}_R M \subseteq V(\mathfrak{a}) \) that \( M = \Gamma_\mathfrak{a}(M) \). Consequently, by [3, Corollary 2.1.7(ii)], \( H^i_\mathfrak{a}(M) = 0 \) for all \( i > 0 \), and so \( H^i_\mathfrak{a}(M) \) is \( \mathfrak{a} \)-minimax for all \( i \geq 0 \), as required. \( \Box \)

We are now ready to state and prove the main result of this section, which will be used in the main result of Section 4.

**Theorem 2.7.** Let \( \mathfrak{a} \) be an ideal of \( R \). Let \( M \) be a finitely generated \( R \)-module and \( N \) an arbitrary \( R \)-module. Let \( t \) be a non-negative integer such that \( \text{Ext}^i_R(M, N) \) is \( \mathfrak{a} \)-minimax for all \( i \leq t \). Then for any finitely generated \( R \)-module \( L \) with \( \text{Supp} L \subseteq \text{Supp} M \), \( \text{Ext}^i_R(L, N) \) is \( \mathfrak{a} \)-minimax for all \( i \leq t \).

Proof. Since \( \text{Supp} L \subseteq \text{Supp} M \), according to Gruson’s Theorem [22, Theorem 4.1], there exists a chain
\[
0 = L_0 \subset L_1 \subset \cdots \subset L_k = L,
\]
such that the factors \( L_j/L_j-1 \) are homomorphic images of a direct sum of finitely many copies of \( M \). Now consider the exact sequences
\[
0 \to K \to M^n \to L_1 \to 0
\]
\[
0 \to L_1 \to L_2 \to L_2/L_1 \to 0
\]
\[
\vdots
\]
\[
0 \to L_{k-1} \to L_k \to L_k/L_{k-1} \to 0,
\]
for some positive integer \( n \).

Now from the long exact sequence
\[
\cdots \to \text{Ext}^{i-1}_R(L_{j-1}, N) \to \text{Ext}^i_R(L_{j}/L_{j-1}, N) \to \text{Ext}^i_R(L_j, N)
\]
\[
\to \text{Ext}^i_R(L_{j-1}, N) \to \cdots
\]
and an easy induction on \( k \), it suffices to prove the case when \( k = 1 \).

Thus there is an exact sequence
\[
(\ast) \quad 0 \to K \to M^n \to L \to 0
\]
for some \( n \in \mathbb{N} \) and some finitely generated \( R \)-module \( K \).

Now, we use induction on \( t \). First, \( \text{Hom}_R(L, N) \) is a submodule of \( \text{Hom}_R(M^n, N) \); hence in view of the assumption and Corollary 2.4, \( \text{Ext}^0_R(L, N) \) is \( \mathfrak{a} \)-minimax. So assume that \( t > 0 \) and that \( \text{Ext}^i_R(L', N) \) is \( \mathfrak{a} \)-minimax for every finitely generated
$R$-module $L'$ with $\text{Supp} L' \subseteq \text{Supp} M$ and all $j \leq t - 1$. Now, the exact sequence (*) induces the long exact sequence
\[
\cdots \to \text{Ext}^{i-1}_{R}(K, N) \to \text{Ext}^{i}_{R}(L, N) \to \text{Ext}^{i}_{R}(M^n, N) \to \cdots ,
\]
so that, by the inductive hypothesis, $\text{Ext}^{i-1}_{R}(K, N)$ is $a$-minimax for all $i \leq t$.

On the other hand, according to Corollary 2.4, $\text{Ext}^{1}_{R}(M^n, N) \cong \oplus \text{Ext}^{1}_{R}(M, N)$ is $a$-minimax. Therefore, it follows from Proposition 2.5 that $\text{Ext}^{i}_{R}(L, N)$ is $a$-minimax for all $i \leq t$, and the inductive step is complete. \hfill \Box

**Corollary 2.8.** Let $a$ be an ideal of $R$, and let $t$ be a non-negative integer. Then, for any $R$-module $M$ the following conditions are equivalent:

(i) $\text{Ext}^{i}_{R}(R/a, M)$ is $a$-minimax for all $i \leq t$.

(ii) For any ideal $b$ of $R$ with $b \supseteq a$, $\text{Ext}^{i}_{R}(R/b, M)$ is $a$-minimax for all $i \leq t$.

(iii) For any finitely generated $R$-module $N$ with $\text{Supp} N \subseteq V(a)$, $\text{Ext}^{i}_{R}(N, M)$ is $a$-minimax for all $i \leq t$.

(iv) For any minimal prime ideal $p$ over $a$, $\text{Ext}^{i}_{R}(R/p, M)$ is $a$-minimax for all $i \leq t$.

**Proof.** In view of Theorem 2.7, it is enough to show that (iv) implies (i). To do this, let $p_1, \ldots, p_n$ be the minimal primes of $a$. Then, by assumption, the $R$-modules $\text{Ext}^{i}_{R}(R/p_j, M)$ are $a$-minimax for all $j = 1, 2, \ldots, n$. Hence by Corollary 2.4, $\bigoplus_{j=1}^{n} \text{Ext}^{i}_{R}(R/p_j, M) \cong \text{Ext}^{i}_{R}(\bigoplus_{j=1}^{n} R/p_j, M)$ is $a$-minimax. Since $\text{Supp}(\bigoplus_{j=1}^{n} R/p_j) = \text{Supp} R/a$, it follows from Theorem 2.7 that $\text{Ext}^{i}_{R}(R/a, M)$ is $a$-minimax, as required. \hfill \Box

### 3. $a$-Cominimax Modules and Local Cohomology

Let $R$ be a Noetherian ring, $a$ an ideal of $R$ and $M$ an $R$-module. Recall that $M$ is said to be $a$-cofinite if $M$ has support in $V(a)$ and $\text{Ext}^{i}_{R}(R/a, M)$ is a finitely generated $R$-module for each $i$ (see [8]). This motivates the following definition:

**Definition 3.1.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. We say that an $R$-module $M$ is $a$-cominimax if the support of $M$ is contained in $V(a)$ and $\text{Ext}^{i}_{R}(R/a, M)$ is $a$-minimax for all $i \geq 0$.

**Example 3.2.** (i) Let $a$ be an ideal of $R$ and let $M$ be an $a$-minimax $R$-module such that $\text{Supp} M \subseteq V(a)$. Then it follows from Corollary 2.5 that $M$ is $a$-cominimax. In particular, every minimax $R$-module with support in $V(a)$ is $a$-cominimax.

(ii) Let $a$ be an ideal of $R$. Then every $a$-cofinite $R$-module is $a$-cominimax. In particular, any Noetherian module with support in $V(a)$ is $a$-cominimax.

(iii) Let $a$ be an ideal of $R$ and let $N$ be a pure submodule of an $R$-module $M$. Then $M$ is $a$-cominimax if and only if $N$ and $M/N$ are $a$-cominimax. In fact, P.M. Cohn’s characterization of purity (see [20] Theorem 3.65) implies that the sequence
\[
0 \to \text{Ext}^{i}_{R}(R/a, N) \to \text{Ext}^{i}_{R}(R/a, M) \to \text{Ext}^{i}_{R}(R/a, M/N) \to 0
\]
is exact for all $i$ (see also the proof of [18 Proposition 2.7]). Hence the result follows from Proposition 2.3.
Proposition 3.3. Let \( a \) be an ideal of \( R \). Let
\[
0 \to M' \to M \to M'' \to 0
\]
be an exact sequence of \( R \)-modules such that two of the modules are \( a \)-cominimax. Then so is the third one.

**Proof.** The exact sequence
\[
0 \to M' \to M \to M'' \to 0
\]
induces a long exact sequence
\[
\cdots \to \text{Ext}^i_R(R/a, M) \to \text{Ext}^i_R(R/a, M') \to \text{Ext}^{i+1}_R(R/a, M') \\
\to \text{Ext}^{i+1}_R(R/a, M) \to \cdots .
\]
Now the result follows easily from Proposition 3.3. \( \square \)

**Corollary 3.4.** Let \( a \) be an ideal of \( R \). Let \( f : M \to N \) be a homomorphism between two \( a \)-cominimax modules such that one of the three modules Ker \( f \), Im \( f \) and Coker \( f \) is \( a \)-cominimax. Then all three of them are \( a \)-cominimax.

**Proof.** The result follows from Proposition 3.3 and the following exact sequences:
\[
0 \to \text{Ker} f \to M \to \text{Im} f \to 0 , \\
0 \to \text{Im} f \to N \to \text{Coker} f \to 0 .
\] \( \square \)

**Proposition 3.5.** Let \( a \) be an ideal of \( R \). Let \( M \) be an \( R \)-module such that \( \text{Supp} M \subseteq V(a) \) and \( 0 :_M a \) has finite Goldie dimension. Then \( M \) has finite Goldie dimension.

**Proof.** Since \( 0 :_M a \) has finite Goldie dimension and \( \text{Supp} M \subseteq V(a) \), it follows from Bourbaki’s Theorem (see [4, Exercise 1.2.27]) that \( \text{Ass} R M \) is finite. On the other hand, for any \( p \in \text{Ass} R M \) we have
\[
\text{Hom}_{R_p}(k(p), M_p) \cong \text{Hom}_{R_p}(k(p), 0 :_{M_p} aR_p) ,
\]
as \( k(p) \)-vector spaces, where \( k(p) = R_p/pR_p \). Therefore \( \mu^0(p, M) \) is finite, and so \( G \dim M < \infty \). \( \square \)

**Corollary 3.6.** Let \( a \) be an ideal of \( R \), and let \( M \) be an \( a \)-cominimax \( R \)-module. Then \( M \) has finite Goldie dimension. In particular the set of associated primes of \( M \) is finite.

**Proof.** This is immediate from Proposition 3.5. \( \square \)

**Proposition 3.7.** Let \( a \) be an ideal of \( R \). Let \( M \) be an \( R \)-module such that \( H^i_a(M) \) is \( a \)-cominimax for all \( i \). Then \( \text{Ext}^i_R(R/a, M) \) is \( a \)-minimax for all \( i \).

**Proof.** The case \( i = 0 \) is clear, so let \( i > 0 \) and do induction on \( i \). We first reduce to the case \( \Gamma_a(M) = 0 \). To do this, let \( \bar{M} = M/\Gamma_a(M) \). Then we have the long exact sequence
\[
\cdots \to \text{Ext}^i_R(R/a, \Gamma_a(M)) \to \text{Ext}^i_R(R/a, M) \to \text{Ext}^i_R(R/a, \bar{M}) \to \cdots
\]
and the isomorphism \( H^i_a(M) \cong H^i_a(\bar{M}) \) for \( i > 0 \). So in view of Proposition 3.3, we may assume that \( M \) is \( a \)-torsion free. Let \( E \) be the injective envelope of \( M \) and put \( L = E/M \). Then \( \text{Hom}_R(R/a, E) = 0 \), and we therefore get the isomorphisms \( H^i_a(L) \cong H^{i+1}_a(M) \) and \( \text{Ext}^i_R(R/a, L) \cong \text{Ext}^{i+1}_R(R/a, M) \) for all \( i \geq 0 \). Now the assertion follows by induction. \( \square \)
Proposition 3.8. Let $a$ be an ideal of $R$. Let $M$ be an $R$-module such that $\text{Ext}_R^i(R/a, M)$ is $a$-minimax for all $i$. If $t$ is a non-negative integer such that $H^i_a(M)$ is $a$-cominimax for all $i \neq t$, then $H^t_a(M)$ is $a$-cominimax.

Proof. We use induction on $t$. Let $\bar{M} = M/\Gamma_a(M)$. Then $H^i_a(M) \cong H^i_a(\bar{M})$ for all $i > 0$. If $t = 0$, then $H^i_a(M)$ is $a$-cominimax for all $i$. Hence by Proposition 3.7, $\text{Ext}_R^i(R/a, M)$ is $a$-minimax for all $i$. It follows that $\Gamma_a(M)$ is $a$-cominimax. So let $t > 0$ and suppose that the result has been proved for $t - 1$. Since $\Gamma_a(M)$ is $a$-cominimax, the exact sequence

$$
\cdots \to \text{Ext}_R^t(R/a, \Gamma_a(M)) \to \text{Ext}_R^t(R/a, M) \to \text{Ext}_R^t(R/a, \bar{M}) \to \cdots
$$

allows us to assume that $M$ is $a$-torsion free. Let $E$ be the injective envelope of $M$ and put $L = E/M$. Then $\text{Hom}_R(R/a, E) = 0$ and $\Gamma_a(E) = 0$, and we therefore get the isomorphisms $H^i_a(L) \cong H^i_a(M)$ and $\text{Ext}_R^t(R/a, L) \cong \text{Ext}_R^{t+1}(R/a, M)$ for all $i \geq 0$. Now the assertion follows by induction. □

Corollary 3.9. Let $a$ be an ideal of $R$ and $M$ an $a$-minimax $R$-module. If $t$ is a non-negative integer such that $H^t_a(M)$ is $a$-cominimax for all $i \neq t$, then $H^t_a(M)$ is $a$-cominimax.

Proof. This follows from Corollary 2.5 and Proposition 3.8. □

Corollary 3.10. Let $a$ be a principal ideal of $R$ and $M$ an $a$-minimax $R$-module. Then $H^i_a(M)$ is $a$-cominimax for all $i \geq 0$.

Proof. Since $H^0_a(M)$ is a submodule of $M$, it turns out that $H^0_a(M)$ is $a$-cominimax by Proposition 2.3 and Example 3.2(i). Also $H^i_a(M) = 0$ for all $i > 1$. Therefore, the result follows from Corollary 3.9. □

4. Finiteness of Associated Primes

It will be shown in this section that the subjects of the previous sections can be used to prove a finiteness result about local cohomology modules. In fact, we will generalize the main result of Brodmann and Lashgari to $a$-minimax modules. The main result is Theorem 4.2. The following theorem will serve to shorten the proof of the main theorem.

Theorem 4.1. Let $a$ be an ideal of $R$ and let $M$ be an $R$-module. Let $t$ be a non-negative integer such that $H^i_a(M)$ is $a$-cominimax for all $i < t$, and $\text{Ext}_R^t(R/a, M)$ is $a$-minimax. Then for any $a$-minimax submodule $N$ of $H^t_a(M)$ and for any finitely generated $R$-module $L$ with $\text{Supp } L \subseteq V(a)$, the $R$-module $\text{Hom}_R(L, H^t_a(M)/N)$ is $a$-minimax.

Proof. The exact sequence

$$
0 \to N \to H^t_a(M) \to H^t_a(M)/N \to 0
$$

provides the following exact sequence:

$$
\text{Hom}_R(L, H^t_a(M)) \to \text{Hom}_R(L, H^t_a(M)/N) \to \text{Ext}_R^t(L, N) \to \cdots.
$$

Since by Corollary 2.5, $\text{Ext}_R^t(L, N)$ is $a$-minimax, so in view of Proposition 2.3 it is thus sufficient for us to show that the $R$-module $\text{Hom}_R(L, H^t_a(M))$ is $a$-minimax. To this end, in view of Corollary 2.8, it is enough for us to show that the $R$-module $\text{Hom}_R(R/a, H^t_a(M))$ is $a$-minimax.
We use induction on \( t \). When \( t = 0 \), the \( R \)-module \( \text{Hom}_R(R/a, M) \) is \( a \)-minimax, by assumption. Since
\[
\text{Hom}_R(R/a, H^0_a(M)) \cong \text{Hom}_R(R/a, \Gamma_a(M)) \cong \text{Hom}_R(R/a, M),
\]

it follows that \( \text{Hom}_R(R/a, H^0_a(M)) \) is \( a \)-minimax.

Now suppose, inductively, that \( t > 0 \) and that the result has been proved for \( t-1 \). Since \( \Gamma_a(M) \) is \( a \)-cominimax, it follows that \( \text{Ext}^i_R(R/a, \Gamma_a(M)) \) is \( a \)-minimax for all \( i \geq 0 \). On the other hand, the exact sequence
\[
0 \to \Gamma_a(M) \to M \to M/\Gamma_a(M) \to 0
\]
induces the exact sequence
\[
\text{Ext}^i_R(R/a, M) \to \text{Ext}^i_R(R/a, M/\Gamma_a(M)) \to \text{Ext}^{i+1}_R(R/a, \Gamma_a(M)).
\]

Hence, by Proposition 2.3 and the assumption, the \( R \)-module \( \text{Ext}^i_R(R/a, M/\Gamma_a(M)) \) is \( a \)-minimax. Also since \( H^0_a(M/\Gamma_a(M)) = 0 \) and \( H^i_a(M/\Gamma_a(M)) \cong H^i_a(M) \) for all \( i > 0 \), it follows that \( H^i_a(M/\Gamma_a(M)) \) is \( a \)-cominimax for all \( i < t \). Therefore we may assume that \( M \) is \( a \)-torsion free. Let \( E \) be an injective envelope of \( M \) and put \( M_1 = E/\Gamma_a(M) \). Then also \( \Gamma_a(E) = 0 \) and \( \text{Hom}_R(R/a, E) = 0 \). Consequently, \( \text{Ext}^i_R(R/a, M_1) \cong \text{Ext}^{i+1}_R(R/a, M) \) and \( H^i_2(M_1) \cong H^i_2(M) \) for all \( i \geq 0 \) (including the case \( i = 0 \)). The induction hypothesis applied to \( M_1 \) yields that \( \text{Hom}_R(R/a, H^i_2(M_1)) \) is \( a \)-minimax. Hence \( \text{Hom}_R(R/a, H^i_a(M)) \) is \( a \)-minimax.

Now we are prepared to prove the main theorem of this section, which is a generalization of the main result of Brodmann and Lashgari.

**Theorem 4.2.** Let \( a \) be an ideal of \( R \) and let \( M \) be an \( a \)-minimax \( R \)-module. Let \( t \) be a non-negative integer such that \( H^i_a(M) \) is \( a \)-minimax for all \( i < t \). Then for any \( a \)-minimax submodule \( N \) of \( H^i_a(M) \), the \( R \)-module \( \text{Hom}_R(R/a, H^i_a(M)/N) \) is \( a \)-minimax. In particular, the Goldie dimension of \( H^1_a(M)/N \) is finite, and so the set \( \text{Ass}_R H^1_a(M)/N \) is finite.

**Proof.** Apply Theorem 4.1 and Corollary 2.3. \( \square \)

Nhan, in [19, Proposition 5.5], established the following corollary in the case \( R \) is local. The following result provides a slight generalization of [19, Proposition 5.5] and [11, Theorem 2.2].

**Corollary 4.3.** Let \( R \) be a Noetherian ring, \( a \) an ideal of \( R \) and \( M \) a finitely generated \( R \)-module. Let \( \text{Obj}(\mathcal{N}) \) (resp. \( \text{Obj}(\mathcal{A}) \)) denote the category of all Noetherian (resp. Artinian) \( R \)-modules and \( R \)-homomorphisms. Let \( t \) be a non-negative integer such that \( H^i_a(M) \in \text{Obj}(\mathcal{N}) \cup \text{Obj}(\mathcal{A}) \) for all \( i < t \). Then the \( R \)-module \( \text{Hom}_R(R/a, H^i_a(M)) \) is \( a \)-minimax, and so the set \( \text{Ass}_R H^i_a(M) \) is finite.

**Proof.** Apply Theorem 4.1 and the fact that the class of \( a \)-minimax modules contains all Noetherian and Artinian modules. \( \square \)

**Corollary 4.4.** Let \( (R, m) \) be a local (Noetherian) ring, \( a \) an ideal of \( R \) and \( M \) a finitely generated \( R \)-module. Assume that \( a \) contains an \( M \)-filter regular sequence of length \( t \). Then \( H^i_a(M) \) has finite Goldie dimension.

**Proof.** According to Melkersson [17, Theorem 3.1], \( H^i_a(M) \) is Artinian for all \( i < t \). Hence, it follows from Corollary 4.3 that \( \text{Hom}_R(R/a, H^i_a(M)) \) is \( a \)-minimax, and so \( G \text{ dim} H^i_a(M) \) is finite. \( \square \)
Acknowledgments

The authors are deeply grateful to the referee for valuable suggestions on the paper and for leading the authors’ attention to a short proof of Proposition 2.6. Also, we would like to thank Prof. Hossein Zakeri for his useful suggestions and many helpful discussions during the preparation of this paper.

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