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FINITENESS PROPERTIES OF LOCAL COHOMOLOGY MODULES FOR $\mathfrak{a}$-MINIMAX MODULES

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ABSTRACT. Let R be a commutative Noetherian ring and $\mathfrak{a}$ an ideal of R. In this paper we introduce the concept of $\mathfrak{a}$-minimax R-modules, and it is shown that if M is an $\mathfrak{a}$-minimax R-module and t a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-minimax for all $i < t$, then for any $\mathfrak{a}$-minimax submodule N of $\mathrm{H}^t_{\mathfrak{a}}(M)$, the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax. As a consequence, it follows that the Goldie dimension of $\mathrm{H}^t_{\mathfrak{a}}(M)/N$ is finite, and so the associated primes of $\mathrm{H}^t_{\mathfrak{a}}(M)/N$ are finite. This generalizes the main result of Brodmann and Lashgari (2000).

1. INTRODUCTION

Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, and M a finitely generated R-module. An important problem in commutative algebra is determining when the set of associated primes of the i^{th} local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(M)$ of M with support in $V(\mathfrak{a})$ is finite (see [11, Problem 4]). A. Singh [21] and M. Katzman [12] have given counterexamples to this conjecture. However, it is known that this conjecture is true in many situations; see [1], [2], [9], [10], [13], [14], [15], [16]. In particular, Brodmann and Lashgari [1, Theorem 2.2] showed that if, for a finitely generated R-module M and an integer t, the local cohomology modules $\mathrm{H}^0_{\mathfrak{a}}(M), \mathrm{H}^1_{\mathfrak{a}}(M), \ldots, \mathrm{H}^{t-1}_{\mathfrak{a}}(M)$ are finitely generated, then the set $\mathrm{Ass}_R \mathrm{H}^t_{\mathfrak{a}}(M)/N$ is finite for every finitely generated submodule N of $\mathrm{H}^t_{\mathfrak{a}}(M)$. For a survey of recent developments on finiteness properties of local cohomology modules, see Lyubeznik's interesting paper [15].

This paper is concerned with what might be considered a generalization of the above-mentioned result of Brodmann and Lashgari to the class of $\mathfrak{a}$-minimax modules. More precisely, we shall show that:

Theorem 1.1. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M an $\mathfrak{a}$-minimax R-module. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-minimax for all*

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$i < t$. Then for any $\mathfrak{a}$-minimax submodule N of $\mathrm{H}^t_{\mathfrak{a}}(M)$, the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax. In particular, the Goldie dimension of $\mathrm{H}^t_{\mathfrak{a}}(M)/N$ is finite, and so the set $\mathrm{Ass}_R\, \mathrm{H}^t_{\mathfrak{a}}(M)/N$ is finite.

Recall that an R-module M is said to have finite Goldie dimension (written $G\dim M < \infty$) if M does not contain an infinite direct sum of non-zero submodules, or equivalently the injective hull $E(M)$ of M decomposes as a finite direct sum of indecomposable (injective) submodules. Also, an R-module M is said to have finite $\mathfrak{a}$-relative Goldie dimension if the Goldie dimension of the $\mathfrak{a}$-torsion submodule $\Gamma_{\mathfrak{a}}(M) := \bigcup_{n\geq 1}(0 :_M \mathfrak{a}^n)$ of M is finite.

We say that an R-module M is $\mathfrak{a}$-minimax if the $\mathfrak{a}$-relative Goldie dimension of any quotient module of M is finite. One of our tools for proving Theorem 1.1 is the following:

Proposition 1.2. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module and N an arbitrary R-module. Let t be a non-negative integer such that $\mathrm{Ext}^i_R(M,N)$ is $\mathfrak{a}$-minimax for all $i \leq t$. Then for any finitely generated R-module L with $\mathrm{Supp}\, L \subseteq \mathrm{Supp}\, M$, $\mathrm{Ext}^i_R(L,N)$ is $\mathfrak{a}$-minimax for all $i \leq t$.*

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity, and $\mathfrak{a}$ will be an ideal of R. The i^{th} local cohomology module of an R-module M with respect to $\mathfrak{a}$ is defined by

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim_{n\geq 1} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

We refer the reader to [7] or [3] for the basic properties of local cohomology.

2. $\mathfrak{a}$-Minimax modules and Goldie dimension

For an R-module M, the *Goldie dimension of* M is defined as the cardinal of the set of indecomposable submodules of $E(M)$ which appear in a decomposition of $E(M)$ into a direct sum of indecomposable submodules. We shall use $G\dim M$ to denote the Goldie dimension of M. For a prime ideal $\mathfrak{p}$, let $\mu^0(\mathfrak{p}, M)$ denote the 0-th Bass number of M with respect to the prime ideal $\mathfrak{p}$. It is known that $\mu^0(\mathfrak{p}, M) > 0$ if and only if $\mathfrak{p} \in \mathrm{Ass}_R\, M$. It is clear by the definition of the Goldie dimension that

$$G\dim M = \sum_{\mathfrak{p}\in \mathrm{Ass}_R M} \mu^0(\mathfrak{p}, M).$$

Also, for any ideal $\mathfrak{a}$ of R and any R-module M, the $\mathfrak{a}$-*relative Goldie dimension of* M is defined as

$$G\dim_{\mathfrak{a}} M := \sum_{\mathfrak{p}\in V(\mathfrak{a})} \mu^0(\mathfrak{p}, M).$$

The $\mathfrak{a}$-relative Goldie dimension of an R-module M has been studied in [5].

In [24], H. Zöschinger introduced the interesting class of minimax modules, and he has in [24] and [25] given many equivalent conditions for a module to be minimax. The R-module M is said to be a *minimax module* if there is a finitely generated submodule N of M, such that M/N is Artinian. It was shown by T. Zink [23] and by E. Enochs [6] that a module over a complete local ring is minimax if and only if it is Matlis reflexive. On the other hand, it is known that when R is a

Noetherian ring, a module is minimax if and only if each of its quotients has finite Goldie dimension, [23] or [25]. This motivates the definition:

Definition 2.1. Let $\mathfrak{a}$ be an ideal of R. An R-module M is said to be *minimax with respect to* $\mathfrak{a}$ or $\mathfrak{a}$-*minimax* if the $\mathfrak{a}$-relative Goldie dimension of any quotient module of M is finite; i.e., for any submodule N of M, $G\dim_{\mathfrak{a}} M/N < \infty$.

Remark 2.2. Let $\mathfrak{a}$ be an ideal of R and let M be an R-module.

(i) If $\mathfrak{a} = 0$, then M is $\mathfrak{a}$-minimax if and only if M is minimax.

(ii) If M is $\mathfrak{a}$-torsion, then M is $\mathfrak{a}$-minimax if and only if M is minimax by [5, Lemma 2.6].

(iii) If M is Noetherian or Artinian, then M is $\mathfrak{a}$-minimax.

(iv) If $\mathfrak{b}$ is a second ideal of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M is $\mathfrak{a}$-minimax, then M is $\mathfrak{b}$-minimax. In particular, every minimax module is $\mathfrak{a}$-minimax.

The following proposition is needed in the proof of the main theorem of this paper.

Proposition 2.3. *Let* $\mathfrak{a}$ *be an ideal of* R. *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of R-*modules. Then* M *is* $\mathfrak{a}$-*minimax if and only if* M' *and* M'' *are both* $\mathfrak{a}$-*minimax.*

Proof. We may suppose for the proof that M' is a submodule of M and that $M'' = M/M'$. If M is $\mathfrak{a}$-minimax, then it easily follows from the definition that M' and M/M' are $\mathfrak{a}$-minimax. Now, suppose that M' and M/M' are $\mathfrak{a}$-minimax. Let N be an arbitrary submodule of M, and let $\mathfrak{p} \in \operatorname{Ass}(M/N) \cap V(\mathfrak{a})$. Then the exact sequence

$$0 \to \frac{M' + N}{N} \to \frac{M}{N} \to \frac{M}{M' + N} \to 0$$

induces the exact sequence

$$0 \to \operatorname{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), \frac{M'_{\mathfrak{p}}}{M'_{\mathfrak{p}} \cap N_{\mathfrak{p}}}) \to \operatorname{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), \frac{M_{\mathfrak{p}}}{N_{\mathfrak{p}}}) \to \operatorname{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), \frac{M_{\mathfrak{p}}}{M'_{\mathfrak{p}} + N_{\mathfrak{p}}}),$$

where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Moreover, since $\operatorname{Ass}_R M/N \subseteq \operatorname{Ass}_R \frac{M'+N}{N} \cup \operatorname{Ass}_R \frac{M}{M'+N}$, and the sets $\operatorname{Ass}_R \frac{M'+N}{N} \cap V(\mathfrak{a})$ and $\operatorname{Ass}_R \frac{M}{M'+N} \cap V(\mathfrak{a})$ are finite, it follows that $G\dim_{\mathfrak{a}} M/N < \infty$, and so M is $\mathfrak{a}$-minimax. □

Corollary 2.4. *Let* $\mathfrak{a}$ *be an ideal of* R. *Then any quotient of an* $\mathfrak{a}$-*minimax module, as well as any finite direct sum of* $\mathfrak{a}$-*minimax modules, is* $\mathfrak{a}$-*minimax.*

Proof. The assertion follows from the definition and Proposition 2.3. □

Corollary 2.5. *Let* $\mathfrak{a}$ *be an ideal of* R. *Let* M *be a finitely generated* R-*module and* N *an* $\mathfrak{a}$-*minimax* R-*module. Then* $\operatorname{Ext}^i_R(M, N)$ *and* $\operatorname{Tor}^R_i(M, N)$ *are* $\mathfrak{a}$-*minimax modules for all* i. *In particular, the* R-*modules* $\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ *and* $\operatorname{Tor}^R_i(R/\mathfrak{a}, N)$ *are* $\mathfrak{a}$-*minimax for all* i.

Proof. As R is Noetherian and M is finitely generated, it follows that M possesses a free resolution

$$\mathbb{F}_\bullet : \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0,$$

whose free modules have finite ranks.

Thus $\operatorname{Ext}^i_R(M,N) = H^i(\operatorname{Hom}_R(\mathbb{F}_\bullet, N))$ is a subquotient of a direct sum of finitely many copies of N. Therefore, it follows from Corollary 2.4 that $\operatorname{Ext}^i_R(M,N)$ is $\mathfrak{a}$-minimax for all $i \geq 0$. By using a similar proof as above we can deduce that $\operatorname{Tor}^R_i(M,N)$ is $\mathfrak{a}$-minimax for all $i \geq 0$. □

Proposition 2.6. *Let $\mathfrak{a}$ be an ideal of R. Let M be an $\mathfrak{a}$-minimax R-module such that $\operatorname{Ass}_R M \subseteq V(\mathfrak{a})$. Then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-minimax for all $i \geq 0$.*

Proof. If $i = 0$, then $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M)$ is a submodule of M, and so by Proposition 2.3, $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-minimax. As $\operatorname{Ass}_R M/\Gamma_{\mathfrak{a}}(M) \subseteq \operatorname{Ass}_R M$, it easily follows from $\operatorname{Ass}_R M \subseteq V(\mathfrak{a})$ that $M = \Gamma_{\mathfrak{a}}(M)$. Consequently, by [3, Corollary 2.1.7(ii)], $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$, and so $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-minimax for all $i \geq 0$, as required. □

We are now ready to state and prove the main result of this section, which will be used in the main result of Section 4.

Theorem 2.7. *Let $\mathfrak{a}$ be an ideal of R. Let M be a finitely generated R-module and N an arbitrary R-module. Let t be a non-negative integer such that $\operatorname{Ext}^i_R(M,N)$ is $\mathfrak{a}$-minimax for all $i \leq t$. Then for any finitely generated R-module L with $\operatorname{Supp} L \subseteq \operatorname{Supp} M$, $\operatorname{Ext}^i_R(L,N)$ is $\mathfrak{a}$-minimax for all $i \leq t$.*

Proof. Since $\operatorname{Supp} L \subseteq \operatorname{Supp} M$, according to Gruson's Theorem [22, Theorem 4.1], there exists a chain

$$0 = L_0 \subset L_1 \subset \cdots \subset L_k = L,$$

such that the factors L_j/L_{j-1} are homomorphic images of a direct sum of finitely many copies of M. Now consider the exact sequences

$$\begin{gathered} 0 \to K \to M^n \to L_1 \to 0 \\ 0 \to L_1 \to L_2 \to L_2/L_1 \to 0 \\ \vdots \\ 0 \to L_{k-1} \to L_k \to L_k/L_{k-1} \to 0, \end{gathered}$$

for some positive integer n.

Now from the long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}_R(L_{j-1},N) \to \operatorname{Ext}^i_R(L_j/L_{j-1},N) \to \operatorname{Ext}^i_R(L_j,N) \\ \to \operatorname{Ext}^i_R(L_{j-1},N) \to \cdots$$

and an easy induction on k, it suffices to prove the case when $k = 1$.

Thus there is an exact sequence

$$(*) \qquad 0 \to K \to M^n \to L \to 0$$

for some $n \in \mathbb{N}$ and some finitely generated R-module K.

Now, we use induction on t. First, $\operatorname{Hom}_R(L,N)$ is a submodule of $\operatorname{Hom}_R(M^n,N)$; hence in view of the assumption and Corollary 2.4, $\operatorname{Ext}^0_R(L,N)$ is $\mathfrak{a}$-minimax. So assume that $t > 0$ and that $\operatorname{Ext}^j_R(L',N)$ is $\mathfrak{a}$-minimax for every finitely generated

R-module L' with $\operatorname{Supp} L' \subseteq \operatorname{Supp} M$ and all $j \leq t-1$. Now, the exact sequence $(*)$ induces the long exact sequence

$$\cdots \to \operatorname{Ext}_R^{i-1}(K, N) \to \operatorname{Ext}_R^i(L, N) \to \operatorname{Ext}_R^i(M^n, N) \to \cdots,$$

so that, by the inductive hypothesis, $\operatorname{Ext}_R^{i-1}(K, N)$ is $\mathfrak{a}$-minimax for all $i \leq t$. On the other hand, according to Corollary 2.4, $\operatorname{Ext}_R^i(M^n, N) \cong \overset{n}{\oplus} \operatorname{Ext}_R^i(M, N)$ is $\mathfrak{a}$-minimax. Therefore, it follows from Proposition 2.3 that $\operatorname{Ext}_R^i(L, N)$ is $\mathfrak{a}$-minimax for all $i \leq t$, and the inductive step is complete. $\square$

Corollary 2.8. *Let $\mathfrak{a}$ be an ideal of R, and let t be a non-negative integer. Then, for any R-module M the following conditions are equivalent:*

(i) $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ *is $\mathfrak{a}$-minimax for all $i \leq t$.*

(ii) *For any ideal $\mathfrak{b}$ of R with $\mathfrak{b} \supseteq \mathfrak{a}$,* $\operatorname{Ext}_R^i(R/\mathfrak{b}, M)$ *is $\mathfrak{a}$-minimax for all $i \leq t$.*

(iii) *For any finitely generated R-module N with* $\operatorname{Supp} N \subseteq V(\mathfrak{a})$, $\operatorname{Ext}_R^i(N, M)$ *is $\mathfrak{a}$-minimax for all $i \leq t$.*

(iv) *For any minimal prime ideal $\mathfrak{p}$ over $\mathfrak{a}$,* $\operatorname{Ext}_R^i(R/\mathfrak{p}, M)$ *is $\mathfrak{a}$-minimax for all $i \leq t$.*

Proof. In view of Theorem 2.7, it is enough to show that (iv) implies (i). To do this, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of $\mathfrak{a}$. Then, by assumption, the R-modules $\operatorname{Ext}_R^i(R/\mathfrak{p}_j, M)$ are $\mathfrak{a}$-minimax for all $j = 1, 2, \ldots, n$. Hence by Corollary 2.4, $\bigoplus_{j=1}^n \operatorname{Ext}_R^i(R/\mathfrak{p}_j, M) \cong \operatorname{Ext}_R^i(\bigoplus_{j=1}^n R/\mathfrak{p}_j, M)$ is $\mathfrak{a}$-minimax. Since $\operatorname{Supp}(\bigoplus_{j=1}^n R/\mathfrak{p}_j) = \operatorname{Supp} R/\mathfrak{a}$, it follows from Theorem 2.7 that $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax, as required. $\square$

3. $\mathfrak{a}$-Cominimax modules and local cohomology

Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M an R-module. Recall that M is said to be $\mathfrak{a}$-cofinite if M has support in $V(\mathfrak{a})$ and $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ is a finitely generated R-module for each i (see [8]). This motivates the following definition:

Definition 3.1. Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. We say that an R-module M is $\mathfrak{a}$-*cominimax* if the support of M is contained in $V(\mathfrak{a})$ and $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax for all $i \geq 0$.

Example 3.2. (i) Let $\mathfrak{a}$ be an ideal of R and let M be an $\mathfrak{a}$-minimax R-module such that $\operatorname{Supp} M \subseteq V(\mathfrak{a})$. Then it follows from Corollary 2.5 that M is $\mathfrak{a}$-cominimax. In particular, every minimax R-module with support in $V(\mathfrak{a})$ is $\mathfrak{a}$-cominimax.

(ii) Let $\mathfrak{a}$ be an ideal of R. Then every $\mathfrak{a}$-cofinite R-module is $\mathfrak{a}$-cominimax. In particular, any Noetherian module with support in $V(\mathfrak{a})$ is $\mathfrak{a}$-cominimax.

(iii) Let $\mathfrak{a}$ be an ideal of R and let N be a pure submodule of an R-module M. Then M is $\mathfrak{a}$-cominimax if and only if N and M/N are $\mathfrak{a}$-cominimax. In fact, P.M. Cohn's characterization of purity (see [20, Theorem 3.65]) implies that the sequence

$$0 \to \operatorname{Ext}_R^i(R/\mathfrak{a}, N) \to \operatorname{Ext}_R^i(R/\mathfrak{a}, M) \to \operatorname{Ext}_R^i(R/\mathfrak{a}, M/N) \to 0$$

is exact for all i (see also the proof of [18, Proposition 2.7]). Hence the result follows from Proposition 2.3.

Proposition 3.3. *Let $\mathfrak{a}$ be an ideal of R. Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of R-modules such that two of the modules are $\mathfrak{a}$-cominimax. Then so is the third one.

Proof. The exact sequence

$$0 \to M' \to M \to M'' \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, M'') \to \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M') \to \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M) \to \cdots .$$

Now the result follows easily from Proposition 2.3. □

Corollary 3.4. *Let $\mathfrak{a}$ be an ideal of R. Let $f : M \to N$ be a homomorphism between two $\mathfrak{a}$-cominimax modules such that one of the three modules* $\mathrm{Ker} f$*,* $\mathrm{Im} f$ *and* $\mathrm{Coker} f$ *is $\mathfrak{a}$-cominimax. Then all three of them are $\mathfrak{a}$-cominimax.*

Proof. The result follows from Proposition 3.3 and the following exact sequences:

$$0 \to \mathrm{Ker} f \to M \to \mathrm{Im} f \to 0,$$
$$0 \to \mathrm{Im} f \to N \to \mathrm{Coker} f \to 0.$$ □

Proposition 3.5. *Let $\mathfrak{a}$ be an ideal of R. Let M be an R-module such that* $\mathrm{Supp}\, M \subseteq V(\mathfrak{a})$ *and $0 :_M \mathfrak{a}$ has finite Goldie dimension. Then M has finite Goldie dimension.*

Proof. Since $0 :_M \mathfrak{a}$ has finite Goldie dimension and $\mathrm{Supp}\, M \subseteq V(\mathfrak{a})$, it follows from Bourbaki's Theorem (see [4, Exercise 1.2.27]) that $\mathrm{Ass}_R M$ is finite. On the other hand, for any $\mathfrak{p} \in \mathrm{Ass}_R M$ we have

$$\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \cong \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), 0 :_{M_\mathfrak{p}} \mathfrak{a}R_\mathfrak{p}),$$

as $k(\mathfrak{p})$-vector spaces, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Therefore $\mu^0(\mathfrak{p}, M)$ is finite, and so $G \dim M < \infty$. □

Corollary 3.6. *Let $\mathfrak{a}$ be an ideal of R, and let M be an $\mathfrak{a}$-cominimax R-module. Then M has finite Goldie dimension. In particular the set of associated primes of M is finite.*

Proof. This is immediate from Proposition 3.5. □

Proposition 3.7. *Let $\mathfrak{a}$ be an ideal of R. Let M be an R-module such that* $\mathrm{H}^i_\mathfrak{a}(M)$ *is $\mathfrak{a}$-cominimax for all i. Then* $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is $\mathfrak{a}$-minimax for all i.*

Proof. The case $i = 0$ is clear, so let $i > 0$ and do induction on i. We first reduce to the case $\Gamma_\mathfrak{a}(M) = 0$. To do this, let $\bar{M} = M/\Gamma_\mathfrak{a}(M)$. Then we have the long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \Gamma_\mathfrak{a}(M)) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \bar{M}) \to \cdots$$

and the isomorphism $\mathrm{H}^i_\mathfrak{a}(M) \cong \mathrm{H}^i_\mathfrak{a}(\bar{M})$ for $i > 0$. So in view of Proposition 2.3, we may assume that M is $\mathfrak{a}$-torsion free. Let E be the injective envelope of M and put $L = E/M$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, E) = 0$, and we therefore get the isomorphisms $\mathrm{H}^i_\mathfrak{a}(L) \cong \mathrm{H}^{i+1}_\mathfrak{a}(M)$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, L) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ for all $i \geq 0$. Now the assertion follows by induction. □

Proposition 3.8. *Let $\mathfrak{a}$ be an ideal of R. Let M be an R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax for all i. If t is a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i \neq t$, then $\mathrm{H}^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax.*

Proof. We use induction on t. Let $\bar{M} = M/\Gamma_{\mathfrak{a}}(M)$. Then $\mathrm{H}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{a}}(\bar{M})$ for all $i > 0$. If $t = 0$, then $\mathrm{H}^i_{\mathfrak{a}}(\bar{M})$ is $\mathfrak{a}$-cominimax for all i. Hence by Proposition 3.7, $\operatorname{Ext}^i_R(R/\mathfrak{a}, \bar{M})$ is $\mathfrak{a}$-minimax for all i. It follows that $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax. So let $t > 0$ and suppose that the result has been proved for $t-1$. Since $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax, the exact sequence

$$\cdots \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \bar{M}) \to \cdots$$

allows us to assume that M is $\mathfrak{a}$-torsion free. Let E be the injective envelope of M and put $L = E/M$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, E) = 0$ and $\Gamma_{\mathfrak{a}}(E) = 0$, and we therefore get the isomorphisms $\mathrm{H}^i_{\mathfrak{a}}(L) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M)$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, L) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ for all $i \geq 0$. Now the assertion follows by induction. □

Corollary 3.9. *Let $\mathfrak{a}$ be an ideal of R and M an $\mathfrak{a}$-minimax R-module. If t is a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i \neq t$, then $\mathrm{H}^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax.*

Proof. This follows from Corollary 2.5 and Proposition 3.8. □

Corollary 3.10. *Let $\mathfrak{a}$ be a principal ideal of R and M an $\mathfrak{a}$-minimax R-module. Then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i \geq 0$.*

Proof. Since $\mathrm{H}^0_{\mathfrak{a}}(M)$ is a submodule of M, it turns out that $\mathrm{H}^0_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax by Proposition 2.3 and Example 3.2(i). Also $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i > 1$. Therefore, the result follows from Corollary 3.9. □

4. Finiteness of associated primes

It will be shown in this section that the subjects of the previous sections can be used to prove a finiteness result about local cohomology modules. In fact, we will generalize the main result of Brodmann and Lashgari to $\mathfrak{a}$-minimax modules. The main result is Theorem 4.2. The following theorem will serve to shorten the proof of the main theorem.

Theorem 4.1. *Let $\mathfrak{a}$ be an ideal of R and let M be an R-module. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i < t$, and $\operatorname{Ext}^t_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax. Then for any $\mathfrak{a}$-minimax submodule N of $\mathrm{H}^t_{\mathfrak{a}}(M)$ and for any finitely generated R-module L with $\operatorname{Supp} L \subseteq V(\mathfrak{a})$, the R-module $\operatorname{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax.*

Proof. The exact sequence

$$0 \to N \to \mathrm{H}^t_{\mathfrak{a}}(M) \to \mathrm{H}^t_{\mathfrak{a}}(M)/N \to 0$$

provides the following exact sequence:

$$\operatorname{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)) \to \operatorname{Hom}_R(L, H^t_{\mathfrak{a}}(M)/N) \to \operatorname{Ext}^1_R(L, N) \to \cdots .$$

Since by Corollary 2.5, $\operatorname{Ext}^1_R(L, N)$ is $\mathfrak{a}$-minimax, so in view of Proposition 2.3 it is thus sufficient for us to show that the R-module $\operatorname{Hom}_R(L, H^t_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. To this end, in view of Corollary 2.8, it is enough for us to show that the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax.

We use induction on t. When $t = 0$, the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax, by assumption. Since

$$\operatorname{Hom}_R(R/\mathfrak{a}, H^0_{\mathfrak{a}}(M)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, M),$$

it follows that $\operatorname{Hom}_R(R/\mathfrak{a}, H^0_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax.

Now suppose, inductively, that $t > 0$ and that the result has been proved for $t-1$. Since $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax, it follows that $\operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax for all $i \geq 0$. On the other hand, the exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0$$

induces the exact sequence

$$\operatorname{Ext}^t_R(R/\mathfrak{a}, M) \to \operatorname{Ext}^t_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \to \operatorname{Ext}^{t+1}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)).$$

Hence, by Proposition 2.3 and the assumption, the R-module $\operatorname{Ext}^t_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. Also since $H^0_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$ and $H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \cong H^i_{\mathfrak{a}}(M)$ for all $i > 0$, it follows that $H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-cominimax for all $i < t$. Therefore we may assume that M is $\mathfrak{a}$-torsion free. Let E be an injective envelope of M and put $M_1 = E/M$. Then also $\Gamma_{\mathfrak{a}}(E) = 0$ and $\operatorname{Hom}_R(R/\mathfrak{a}, E) = 0$. Consequently, $\operatorname{Ext}^i_R(R/\mathfrak{a}, M_1) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ and $H^i_{\mathfrak{a}}(M_1) \cong H^{i+1}_{\mathfrak{a}}(M)$ for all $i \geq 0$ (including the case $i = 0$). The induction hypothesis applied to M_1 yields that $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M_1))$ is $\mathfrak{a}$-minimax. Hence $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. □

Now we are prepared to prove the main theorem of this section, which is a generalization of the main result of Brodmann and Lashgari.

Theorem 4.2. *Let $\mathfrak{a}$ be an ideal of R and let M be an $\mathfrak{a}$-minimax R-module. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-minimax for all $i < t$. Then for any $\mathfrak{a}$-minimax submodule N of $\mathrm{H}^t_{\mathfrak{a}}(M)$, the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax. In particular, the Goldie dimension of $\mathrm{H}^t_{\mathfrak{a}}(M)/N$ is finite, and so the set $\operatorname{Ass}_R \mathrm{H}^t_{\mathfrak{a}}(M)/N$ is finite.*

Proof. Apply Theorem 4.1 and Corollary 2.5. □

Nhan, in [19, Proposition 5.5], established the following corollary in the case R is local. The following result provides a slight generalization of [19, Proposition 5.5] and [1, Theorem 2.2].

Corollary 4.3. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module. Let $\operatorname{Obj}(\mathcal{N})$ (resp. $\operatorname{Obj}(\mathcal{A})$) denote the category of all Noetherian (resp. Artinian) R-modules and R-homomorphisms. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M) \in \operatorname{Obj}(\mathcal{N}) \cup \operatorname{Obj}(\mathcal{A})$ for all $i < t$. Then the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax, and so the set $\operatorname{Ass}_R \mathrm{H}^t_{\mathfrak{a}}(M)$ is finite.*

Proof. Apply Theorem 4.1 and the fact that the class of $\mathfrak{a}$-minimax modules contains all Noetherian and Artinian modules. □

Corollary 4.4. *Let $(R, \mathfrak{m})$ be a local (Noetherian) ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module. Assume that $\mathfrak{a}$ contains an M-filter regular sequence of length t. Then $\mathrm{H}^t_{\mathfrak{a}}(M)$ has finite Goldie dimension.*

Proof. According to Melkersson [17, Theorem 3.1], $\mathrm{H}^i_{\mathfrak{a}}(M)$ is Artinian for all $i < t$. Hence, it follows from Corollary 4.3 that $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax, and so $G \dim H^t_{\mathfrak{a}}(M)$ is finite. □

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