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BOUNDS FOR HILBERT COEFFICIENTS

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ABSTRACT. We compute the Hilbert coefficients of a graded module with pure resolution and prove lower and upper bounds for these coefficients for arbitrary graded modules.

INTRODUCTION

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables, and let N be any graded S-module of dimension d. Then for $i \gg 0$, the numerical function $H(N, i) = \sum_{j \le i} \dim_K N_j$ is a polynomial function of degree d; see [1, 4.1.6]. In other words, there exists a polynomial $P_N(x) \in \mathbb{Z}[x]$ such that

$$H(N, i) = P_N(i) \quad \text{for all} \quad i \gg 0.$$

The polynomial $P_N(x)$ is called the *Hilbert polynomial* of N. It can be written in the form

$$P_N(x) = \sum_{i=0}^{d} (-1)^i e_i(N) \binom{x+d-i}{d-i}$$

with integer coefficients $e_i(N)$, called the *Hilbert coefficients* of N.

In the first section we will give explicit formulas for the $e_i(N)$ in the case where N has a pure resolution. In the second section we use these formulas and a recent result of Eisenbud and Schreyer, who succeeded in proving a conjecture by Boij and Söderberg [2, Conjecture 2.4] asserting that the Betti diagram of a Cohen-Macaulay module over a polynomial ring is a positive linear combination of Betti diagrams of modules with pure resolutions. As an application of the Eisenbud-Schreyer theorem one obtains (as already noted by Boij and Söderberg) a proof of the multiplicity conjecture of Huneke and Srinivasan; see [6]. The result is now the following: let M be a graded Cohen-Macaulay S-module of codimension s generated in degree 0. Let β_{ij} be the graded Betti-numbers of M and set $m_i = \min\{j \colon \beta_{ij} \neq 0\}$ and $M_i = \max\{j \colon \beta_{ij} \neq 0\}$ for $i = 1, \dots s$. Then

$$\beta_0 \frac{m_1 m_2 \cdots m_s}{s!} \le e_0(N) \le \beta_0 \frac{M_1 M_2 \cdots M_s}{s!}.$$

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As a main result of this paper we present in Theorem 2.1 similar inequalities for all the Hilbert coefficients of M, where the upper and lower bounds are again expressed only as functions of the lowest and highest shifts m_i and M_i of the minimal graded free resolution of M.

The Cohen-Macaulay condition is crucial for the multiplicity bounds. Already in [6] it is noted that the lower bound does not hold without this hypothesis. On the other hand, the upper bound may hold for arbitrary graded S-modules generated in degree 0. Indeed, there are many special cases in which the upper bound is proved in general. A rather complete survey of the multiplicity conjecture can be found in [4].

The Cohen-Macaulay condition is also crucial for the bounds for the $e_i(N)$, since in the proof we use the explicit formulas for the Hilbert coefficients of modules with pure resolution obtained in Theorem 1.1. If one drops the Cohen–Macaulay condition, then the Hilbert coefficients are no longer determined by the shifts in the resolution.

1. The Hilbert coefficients of a module with pure resolution

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables, and let N be a finitely generated graded S-module. We say N has a pure resolution of type $(d_0, d_1, \ldots, d_s)$ if its minimal graded free S-resolution is of the form

$$0 \longrightarrow S^{\beta_s}(-d_s) \longrightarrow \cdots \longrightarrow S^{\beta_1}(-d_1) \longrightarrow S^{\beta_0}(-d_0) \longrightarrow 0.$$

The main result of this section is

Theorem 1.1. *Let N be a finitely generated graded Cohen-Macaulay S-module of codimension s with pure resolution of type $(d_0, d_1, \ldots, d_s)$ with $d_0 = 0$. Then the Hilbert coefficients of N are*

$$e_i(N) = \beta_0 \frac{\prod_{j=1}^{s} d_j}{(s+i)!} \sum_{1 \le j_1 \le j_2 \cdots \le j_i \le s} \prod_{k=1}^{i} (d_{j_k} - (j_k + k - 1)), \quad i = 0, \ldots, n-s.$$

Proof. We first recall a few facts about Hilbert series and multiplicities as described in [1]. The Hilbert series $H_N(t) = \sum_i H(N,i)t^i$ is a rational function of the form

$$H_N(t) = \frac{Q_N(t)}{(1-t)^{d+1}},$$

where $d = n - s$ is the dimension of N. The Hilbert coefficients $e_i = e_i(N)$ of N can be computed according to the formula

$$e_i = \frac{Q_N^{(i)}(1)}{i!}, \quad i = 0, \ldots, d.$$

On the other hand, by using the additivity of Hilbert functions, the free resolution of N yields the presentation

$$H_N(t) = \frac{P_N(t)}{(1-t)^{n+1}} \quad \text{with} \quad P_N(t) = \sum_{j=0}^{s} (-1)^j \beta_j t^{d_j}.$$

Thus we see that $P_N(t) = Q_N(t)(1-t)^s$. This yields

$$e_i = (-1)^s \frac{P_N^{(s+i)}(1)}{(s+i)!}, \quad i = 0, \ldots, d. \tag{1}$$

For any two integers $0 \leq a \leq b$ we set

$$g_a(b) = \sum_{1 \leq i_1 < i_2 < \cdots < i_a \leq b} i_1 i_2 \cdots i_a.$$

Then we have

$$\begin{aligned} P_N^{(s+i)}(1) &= \sum_{j=0}^{s} (-1)^j \beta_j \prod_{k=0}^{s+i-1} (d_j - k) \\ &= \sum_{j=0}^{s} (-1)^j \beta_j \sum_{k=1}^{s+i} (-1)^{s+i-k} g_{s+i-k}(s+i-1) d_j^k \\ &= \sum_{k=1}^{s+i} (-1)^{s+i-k} g_{s+i-k}(s+i-1) \sum_{j=0}^{s} (-1)^j \beta_j d_j^k. \end{aligned}$$

Hence if we set $a_k = \sum_{j=0}^{s} (-1)^j \beta_j d_j^{k+s}$ for all $k \geq 0$ and observe that for all $k < s$, $\sum_{j=0}^{s} (-1)^j \beta_j d_j^k = 0$ (see [6], where the proof of this fact is given in the cyclic case), we obtain together with (1) the following identities:

$$(2) \qquad (-1)^s (s+i)! e_i = \sum_{k=0}^{i} (-1)^{i-k} g_{i-k}(s+i-1) a_k, \quad i = 0, \ldots, d.$$

In order to compute the a_i we consider for each i the following matrix:

$$B_i = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_s d_s \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_s d_s^2 \\ \vdots & \vdots & & \vdots \\ \beta_1 d_1^{s-1} & \beta_2 d_2^{s-1} & \cdots & \beta_s d_s^{s-1} \\ \beta_1 d_1^{s+i} & \beta_2 d_2^{s+i} & \cdots & \beta_s d_s^{s+i} \end{pmatrix}.$$

Replacing the last column of B_i by the alternating sum of its columns we obtain the matrix B_i' for which $\det B_i' = (-1)^s \det B_i$ and whose last column is the transpose of $(0, 0, \ldots, a_i)$. It follows that

$$(3) \qquad a_i = (-1)^s \det B_i / \det C,$$

where

$$C = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_{s-1} d_{s-1} \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_{s-1} d_{s-1}^2 \\ \vdots & \vdots & & \vdots \\ \beta_1 d_1^{s-1} & \beta_2 d_2^{s-1} & \cdots & \beta_{s-1} d_{s-1}^{s-1} \end{pmatrix}.$$

Note that $\det C = \beta_1 \cdots \beta_{s-1} d_1 \cdots d_{s-1} \det V(d_1, \cdots, d_{s-1})$, where $V(d_1, \cdots, d_{s-1})$ is the Vandermonde matrix for the sequence $d_1, d_2, \ldots, d_{s-1}$. Hence we obtain

$$\det C = \beta_1 \cdots \beta_{s-1} d_1 \cdots d_{s-1} \prod_{1 \leq i < j \leq s-1} (d_j - d_i).$$

On the other hand we have

$$\det B_i = \beta_1 \cdots \beta_s d_1 \cdots d_s \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_s \\ \vdots & \vdots & & \vdots \\ d_1^{s-2} & d_2^{s-2} & \cdots & d_s^{s-2} \\ d_1^{s+i-1} & d_2^{s+i-1} & \cdots & d_s^{s+i-1} \end{pmatrix}.$$

According to the subsequent Lemma 1.2 we have

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_s \\ \vdots & \vdots & & \vdots \\ d_1^{s-2} & d_2^{s-2} & \cdots & d_s^{s-2} \\ d_1^{s+i-1} & d_2^{s+i-1} & \cdots & d_s^{s+i-1} \end{pmatrix} = f_i(d_1, \ldots, d_s) \cdot \prod_{1 \le j < k \le s} (d_k - d_j),$$

where for each integer $k \ge 0$ we set

$$f_k(g_1, \ldots, g_s) = \sum g_1^{c_1} \cdots g_s^{c_s}.$$

Here the sum is taken over all integer vectors $c = (c_1, \ldots, c_s)$ with $c_i \ge 0$ for all i and $|c| = \sum_{i=1}^s c_i = k$.

Thus by (3) we have

$$a_i = (-1)^s \beta_s d_s f_i(d_1, \ldots, d_s) \prod_{j=1}^{s-1} (d_s - d_j).$$

Now we use the fact that $\beta_s = \beta_0 \prod_{j=1}^{s-1} d_j / \prod_{j=1}^{s-1}(d_s - d_j)$ (see [5] or [2]) and obtain

$$a_i = (-1)^s \beta_0 d_1 \cdots d_s f_i(d_1, \ldots, d_s).$$

This result together with (2) yields the formulas

$$e_i = \beta_0 \frac{d_1 \cdots d_s}{(s+i)!} \sum_{j=0}^{i} (-1)^{i-j} g_{i-j}(s+i-1) f_j(d_1, \ldots, d_s). \tag{4}$$

Expanding the products in the sum

$$\sum_{1 \le j_1 \le j_2 \cdots \le j_i \le s} \prod_{k=1}^{i} (d_{j_k} - (j_k + k - 1))$$

yields

$$\sum_{1 \le j_1 \le j_2 \cdots \le j_i \le s} \prod_{k=1}^{i} (d_{j_k} - (j_k + k - 1)) = \sum_{j=0}^{i} (-1)^{i-j} g_{i-j}(s+i-1) f_j(d_1, \ldots, d_s).$$

Hence the desired formulas for the e_i follow from (4). $\square$

It remains to prove

Lemma 1.2. *For all $k \geq s-1 \geq 0$ one has*

$$\det \begin{pmatrix} 1 & \dots & 1 \\ d_1 & \dots & d_s \\ \vdots & & \vdots \\ d_1^{s-2} & \dots & d_s^{s-2} \\ d_1^k & \dots & d_s^k \end{pmatrix} = f_{k-s+1}(d_1, \dots, d_s) \cdot \prod_{1 \leq i < j \leq s} (d_j - d_i).$$

Proof. Given integers $1 \leq r \leq s$ and $k \geq s$ we define the matrix

$$A_r^{(k)} = (a_{ij}^{(k)})_{\substack{i=1,\dots,s-r+1 \\ j=1,\dots,s-r+1}}$$

with

$$a_{ij}^{(k)} = \begin{cases} f_{i-1}(d_1, \dots, d_{r-1}, d_{r-1+j}), & \text{for } i \leq s-r,\ j = 1, \dots, s-r+1 \\ f_{k-r+1}(d_1, \dots, d_{r-1}, d_{r-1+j}), & \text{for } i = s-r+1,\ j = 1, \dots, s-r+1. \end{cases}$$

Notice that $A_1^{(k)}$ is the matrix whose determinant we want to compute, while $A_s^{(k)}$ is the 1×1 matrix with entry $f_{k-s+1}(d_1, \dots, d_{s-1}, d_s)$.

Next observe that for each integer $\ell > 0$ and all $j > 1$ one has

$$\begin{aligned} &f_\ell(d_1, \dots, d_{r-1}, d_{r-1+j}) - f_\ell(d_1, \cdots, d_{r-1}, d_r) \\ &\qquad = (d_{r-1+j} - d_r) \cdot f_{\ell-1}(d_1, \dots, d_r, d_{r-1+j}). \end{aligned}$$

Hence if we subtract the first column from the other columns of $A_r^{(k)}$ and then expand this new matrix with respect to the first row (which is $(1, 0, \cdots, 0)$), we see that

$$\det A_r^{(k)} = (d_{r+1} - d_r)(d_{r+2} - d_r) \cdots (d_s - d_r) \det A_{r+1}^{(k)}.$$

From this we obtain that

$$\det A_1^{(k)} = \det A_s^{(k)} \cdot \prod_{1 \leq i < j \leq s} (d_j - d_i) = f_{k-s+1}(d_1, \dots, d_{s-1}, d_s) \cdot \prod_{1 \leq i < j \leq s} (d_j - d_i),$$

as desired. □

For $i = 0, 1, 2$ the formulas for the Hilbert coefficients read as follows:

$e_0(N) = \beta_0 \frac{\prod_{i=1}^s d_i}{s!}$;

$e_1(N) = \beta_0 \frac{\prod_{i=1}^s d_i}{(s+1)!} \sum_{i=1}^s (d_i - i)$;

$e_2(N) = \beta_0 \frac{\prod_{i=1}^s d_i}{(s+2)!} \sum_{1 \leq i \leq j \leq s} (d_i - i)(d_j - j - 1)$.

In the special case that N has a d-linear resolution, our formulas yield

$$e_i(N) = \beta_0 \binom{d+s-1}{s+i} \binom{s+i-1}{i}.$$

Remark 1.3. The assumption made in Theorem 1.1 that d_0 should be zero is not essential. It is only made to simplify the formulas for the Hilbert coefficients. While for the multiplicity we have $e_0(N) = e_0(N(a))$ for any shift a, the other Hilbert coefficients transform as follows: if N has a pure resolution of type $(d_0, d_1, \dots, d_s)$, then $N(d_0)$ has a pure resolution of type $(0, d_1 - d_0, \dots, d_s - d_0)$ whose Hilbert coefficients we know by Theorem 1.1.

On the other hand we have $P_N(x) = P_{N(d_0)}(x - d_0)$, from which one deduces that

$$(5)\qquad \sum_{i=0}^{d}(-1)^i e_i(N)\binom{x+d-i}{d-i} = \sum_{i=0}^{d}(-1)^i e_i(N(d_0))\binom{x-d_0+d-i}{d-i}.$$

Hence if we want to express the $e_i(N)$ by the $e_i(N(d_0))$, we have to express the right-hand side polynomial as a linear combination of the binomials $\binom{x+d-i}{d-i}$. To do this, first notice that

$$\binom{x-d_0+k}{k} = \begin{cases} \sum_{j=0}^{k}(-1)^{k-j}\binom{d_0}{k-j}\binom{x+j}{j}, & \text{if } d_0 > 0,\\ \sum_{j=0}^{k}\binom{-d_0+k-j-1}{k-j}\binom{x+j}{j}, & \text{if } d_0 < 0.\end{cases}$$

Substituting these expressions for $\binom{x-d_0+k}{k}$ in the right-hand side of (5) and comparing coefficients, we obtain

$$e_{d-j}(N) = \begin{cases} \sum_{i=j}^{d}\binom{d_0}{i-j}e_{d-i}(N(d_0)), & \text{if } d_0 > 0,\\ \sum_{i=j}^{d}(-1)^{i-j}\binom{-d_0-i-j-1}{i-j}e_{d-i}(N(d_0)), & \text{if } d_0 < 0.\end{cases}$$

2. Upper and lower bounds

Given a sequence $d_1, d_2, \ldots, d_s$ of integers. We set

$$h_i(d_1,\ldots,d_s) = \sum_{1\le j_1\le j_2\cdots\le j_i\le s}\ \prod_{k=1}^{i}(d_{j_k} - (j_k + k - 1))$$

for $i = 0,\ldots,n-s$, where $h_0(d_1,\ldots,d_s) = 1$. This definition will simplify notation in the following discussions.

Let N be any finitely generated graded Cohen-Macaulay S-module of projective dimension s and graded Betti numbers β_{ij}. For each $i = 1,\ldots s$, the minimal and maximal shifts of N in homological degree i are defined by $m_i = \min\{j\colon \beta_{ij} \neq 0\}$ and $M_i = \max\{j\colon \beta_{ij} \neq 0\}$.

When N is generated in degree 0 and has a pure resolution of type $(d_1,\ldots,d_s)$, we have $m_i = M_i = d_i$ for all i, and Theorem 1.1 tells us that

$$e_i(N) = \beta_0\frac{d_1d_2\cdots d_s}{(s+i)!}h_i(d_1,\ldots,d_s) \quad \text{for } i = 0,1,\ldots,n-s.$$

In analogy to the multiplicity bounds proved by Eisenbud and Schreyer [3], we now state

Theorem 2.1. *Let N be a finitely generated graded Cohen-Macaulay S-module of codimension s generated in degree* 0. *Then*

$$\beta_0\frac{m_1m_2\cdots m_s}{(s+i)!}h_i(m_1,\ldots,m_s) \le e_i(N) \le \beta_0\frac{M_1M_2\cdots M_s}{(s+i)!}h_i(M_1,\ldots,M_s)$$

for $i = 0,1,\ldots,n-s$.

Proof. For the proof of the theorem we make essential use of a theorem of Eisenbud and Schreyer [3, Theorem 0.2], whose statement was conjectured by Boij and Söderberg in [2, Conjecture 2.4]. The theorem says that each normalized Betti diagram of a graded module is a rational convex linear combination of pure diagrams.

For any strictly increasing sequence of integers $d = (d_0, d_1, \ldots, d_s)$, the matrix $\pi(d)$ defined by

$$\pi(d)_{i,j} = \begin{cases} (-1)^{i+1} \prod_{\substack{k\neq i \\ k\neq 0}} \frac{d_k - d_0}{d_k - d_i}, & \text{if } j = d_i, \\ 0, & \text{if } j \neq d_i, \end{cases}$$

is called a pure diagram.

Let $D = (\beta_{ij}/\beta_0)$ be the normalized Betti diagram of N, and let $m = (m_1, \ldots, m_s)$ and $M = (M_1, \ldots, M_s)$ be the sequences of minimal and maximal shifts of N. We denote by $\Pi_{m,M}$ the set of all pure diagrams $\pi(d)$ with $m_i \leq d_i \leq M_i$. Then

$$D = \sum_{\pi(d)\in\Pi_{m,M}} c_{\pi(d)}\pi(d) \quad \text{with} \quad c_{\pi(d)} \in \mathbb{Q} \quad \text{and} \sum_{\pi(d)\in\Pi_{m,M}} c_{\pi(d)} = 1.$$

It follows that

$$e_i(N) = \beta_0 \cdot \sum_{\pi(d)\in\Pi_{m,M}} c_{\pi(d)} e_i(\pi(d)). \tag{6}$$

Let $\prod_{k=1}^{i}(d_{j_k} - (j_k + k - 1))$ be one of the summands in $h_i(d)$. We claim that either $\prod_{k=1}^{i}(d_{j_k} - (j_k + k - 1)) = 0$, or else $d_{j_k} - (j_k + k - 1) > 0$ for $k = 1, \ldots, i$. The claim will then imply that

$$e_i(\pi(d)) \leq e_i(\pi(d')) \tag{7}$$

whenever we have $d_i \leq d'_i$ for $i = 1, \ldots, s$.

In order to prove the claim, suppose that $\prod_{k=1}^{i}(d_{j_k} - (j_k + k - 1)) \neq 0$. Since $d_i \geq i$ for all i, we must then have that $d_{j_1} - j_1 > 0$. Assume that not all factors $d_{j_k} - (j_k + k - 1)$ are positive and let ℓ be the smallest integer with $d_{j_\ell} - (j_\ell + \ell - 1) < 0$. Then $\ell > 1$ and $d_{j_{\ell-1}} - (j_{\ell-1} + \ell - 2) > 0$. It follows that

$$d_{j_{\ell-1}} - (j_{\ell-1} + \ell - 2) - (d_{j_\ell} - (j_\ell + \ell - 1)) \geq 2$$

or equivalently

$$j_\ell - j_{\ell-1} \geq d_{j_\ell} - d_{j_{\ell-1}} + 1.$$

This is a contradiction, since $d_1 < d_2 < \cdots < d_s$.

Now (6) and (7) imply that

$$\begin{aligned} e_i(\pi(m)) &= \min\{e_i(\pi(d)) \ \pi(d) \in \Pi_{m,M}\} \\ &\leq \frac{e_i(N)}{\beta_0} \leq \max\{e_i(\pi(d)) \ \pi(d) \in \Pi_{m,M}\} = e_i(\pi(M)), \end{aligned}$$

as desired. □

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