PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 2, February 2009, Pages 663–668
S 0002-9939(08)09556-7
Article electronically published on September 5, 2008

EXTRAPOLATION SPACES FOR C-SEMIGROUPS

MIAO LI AND QUAN ZHENG

(Communicated by Joseph A. Ball)

ABSTRACT. Let $\{T(t)\}_{t\geq 0}$ be a C-semigroup on X. We construct an extrapolation space X_s, such that X can be continuously densely imbedded in X_s, and $\{T_s(t)\}_{t\geq 0}$, the extension of $\{T(t)\}_{t\geq 0}$ to X_s, is strongly uniformly continuous and contractive. Using this enlarged space, we give an answer to the question asked in [M. Li, F. L. Huang, Characterizations of contraction C-semigroups, Proc. Amer. Math Soc. 126 (1998), 1063–1069] in the negative.

1. Introduction

Let X be a Banach space, $\mathbf{B}(X)$ the space of all bounded linear operators on X, and C an injective operator in $\mathbf{B}(X)$. A family of linear bounded operators $\{T(t)\}_{t\geq 0} \subset \mathbf{B}(X)$ is called a C-semigroup if $T(\cdot)$ is strongly continuous and $T(0) = C$, $T(t+s)C = T(t)T(s)$ for $t,\ s \geq 0$. Its generator, A, is defined by

$$Ax = C^{-1}\big(\lim_{t\to 0} \frac{T(t)x - Cx}{t}\big)$$

with maximal domain.

A C-semigroup $\{T(t)\}_{t\geq 0}$ is bounded if there is a constant $M > 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$ and is a contraction C-semigroup if $\|T(t)x\| \leq \|Cx\|$ for all $x \in X$ and $t \geq 0$.

It is natural that all bounded C_0-semigroups are strongly uniformly continuous, while for C-semigroups this is far from obvious. However, we show in this paper that for every bounded C-semigroup $\{T(t)\}_{t\geq 0}$ on X, an extrapolation space X_s can be constructed such that the extension of $\{T(t)\}_{t\geq 0}$ to X_s, $\{T_s(t)\}_{t\geq 0}$, is a strongly uniformly continuous contraction C_s-semigroup on X_s, where C_s is the extension of C to X_s. Our extrapolation space is smaller than the one given by deLaubenfels ([1, 2]).

Moreover, we take up the open problem asked in [4]. The question was: Suppose that A is the generator of a contraction C-semigroup on X. Does there exist a restriction of A, A', which is the generator of a contraction C_0-semigroup on $\overline{R(C)}$? If this holds, then $(\lambda - A')^{-1}\overline{R(C)} = \overline{R(C)}$ for all $\lambda > 0$. Hence it is crucial that $\overline{R(C)}$ be an invariant subspace for $(\lambda - A)^{-1}$ since $A' \subseteq A$. So one way to answer the question in the negative is to give a contraction C-semigroup with generator A, in

Received by the editors October 16, 2006, and, in revised form, February 12, 2008.
2000 *Mathematics Subject Classification.* Primary 47D06; Secondary 47D03.
Key words and phrases. C-semigroups, contraction, C_0-semigroups, extrapolation space.
The first author was supported by the NSF of China (Grant No. 10501032), and the second author by TRAPOYT and the NSF of China (Grant No. 10671079).

which $(\lambda - A)^{-1}$ does not leave $\overline{R(C)}$ invariant. It is easier to construct a bounded C-semigroup than a contraction one. Now the extrapolation space is helpful. By making use of it, we can obtain contraction C-semigroups from bounded ones.

Throughout this paper, for an operator A on X, we write $D(A)$ for its domain, $R(A)$ for its range, and the closure of $R(A)$ is denoted by $\overline{R(A)}$. The C-resolvent set of A, $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } R(C) \subset R(\lambda - A)\}$, and the C-resolvent of A is $R_C(\lambda, A) := (\lambda - A)^{-1}C$ for $\lambda \in \rho_C(A)$. For Y a subspace of X and A a linear operator on X, we denote by $A|_Y$ the part of A in Y, i.e., $A|_Y \subset A$ with maximal domain. For the properties of C-semigroups and of contractions, we refer to [2, 4].

2. Main results

First we give a positive answer to the question mentioned above under some additional assumptions. The following result also improves Theorem 3.4 in [4].

Theorem 2.1. *Let $A = C^{-1}AC$, $\overline{CD(A)} = \overline{R(C)}$ and $D(A) \subseteq R(r - A)$ for some $r > 0$. Then the following are equivalent:*

(a) *A generates a contraction C-semigroup on X.*

(b) *$(0, \infty) \subseteq \rho_C(A)$ and $\lambda \|R_C(\lambda, A)x\| \le \|Cx\|$ for $\lambda > 0$ and $x \in X$.*

(c) *$A|_{\overline{R(C)}}$ generates a contraction C_0-semigroup on $\overline{R(C)}$.*

Proof. (a) $\Rightarrow$ (b) follows from Theorem 3.3 in [4].

(b) $\Rightarrow$ (c). Define $B \subseteq A$ with $D(B) = CD(A)$. Then B is a densely defined closable operator on $\overline{R(C)}$. By (b), $\|(\lambda - A)x\| \ge \lambda \|x\|$ for $\lambda > 0$ and $x \in D(B)$; i.e., B is dissipative. This implies that $\overline{B}$ is also dissipative and $R(\lambda - \overline{B})$ is a closed subspace of $\overline{R(C)}$. To show that $R(\lambda - \overline{B}) = \overline{R(C)}$, let $x \in D(A)$. Since $D(A) \subseteq R(r - A)$, $x = (r - A)y$ for some $y \in D(A)$ and $ACy = CAy$ due to the assumption $A = C^{-1}AC$,

$$Cx = (r - A)Cy = (r - B)Cy \in R(r - \overline{B}).$$

This implies that $\overline{R(C)} = \overline{CD(A)} \subseteq R(r - \overline{B})$, as desired. It now follows from the Lumer-Phillips theorem that $\overline{B}$ generates a contraction C_0-semigroup on $\overline{R(C)}$. It remains to show that $\overline{B} = A|_{\overline{R(C)}}$. It is clear that $\overline{B} \subseteq A|_{\overline{R(C)}}$, and so $\overline{R(C)} \subseteq R(r - A|_{\overline{R(C)}})$. Also, the injectivity of $r - A$ implies that of $r - A|_{\overline{R(C)}}$. Thus, $\overline{B} = A|_{\overline{R(C)}}$ follows from the identity that $(r - \overline{B})^{-1} = (r - A|_{\overline{R(C)}})^{-1}$.

(c) $\Rightarrow$ (a). Let $T(t) = S(t)C$, where $S(t)$ is the contraction C_0-semigroup generated by $A|_{\overline{R(C)}}$ on $\overline{R(C)}$. It is easy to show that $T(t)$ is a contraction C-semigroup; we only need to show that A is the generator. If $x \in D(A)$, then since $ACx = CAx \in R(C)$ by the assumption that $A = C^{-1}AC$, we know that $Cx \in D(A|_{\overline{R(C)}})$ and

$$\frac{T(t)x - Cx}{t} = \frac{S(t)Cx - Cx}{t} \to A|_{\overline{R(C)}}Cx = CA|_{\overline{R(C)}}x = CAx$$

as $t \to 0$, so an extension of A is the generator. Suppose that $\lambda > 0$; if $(\lambda - A)x = 0$, then, since $Cx \in D(A|_{\overline{R(C)}})$,

$$(\lambda - A|_{\overline{R(C)}})Cx = (\lambda - A)Cx = C(\lambda - A)x = 0.$$

Thus $x = 0$; i.e., $\lambda - A$ is injective. Also, for $x \in X$, let $y = R(\lambda, A|_{\overline{R(C)}})Cx$. Then $Cx = (\lambda - A|_{\overline{R(C)}})y = (\lambda - A)y$. This implies that $R(C) \subseteq R(\lambda - A)$ and so $\lambda \in \rho_C(A)$. Then it follows from Corollary 3.12 in [2] that $C^{-1}AC = A$ is the generator. $\square$

Now we turn to the construction of the extrapolation space. For simplicity, we only consider bounded C-semigroups.

Let $\{T(t)\}_{t\geq 0}$ be a bounded C-semigroup on X with generator A, so there exists some constant $M > 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. For each $x \in X$, define $\|x\|_s = \sup_{t\geq 0} \|T(t)x\|$. Then

$$\|Cx\| \leq \|x\|_s \leq M\|x\|. \tag{2.1}$$

Since C is injective, $\|\cdot\|_s$ is a norm on X. Denote by X_s the completion of X with respect to the norm $\|\cdot\|_s$. Extend $T(t)$ to X_s by defining $T_s(t)y = \lim_{n\to\infty} T(t)x_n$ for all $t \geq 0$, with the limit taken in X, whenever $\{x_n\}$ is a sequence in X converging to y, in X_s. We also denote by C_s the extension of C to X_s. It is not hard to see that $T_s(t)$ is bounded on X_s for each $t \geq 0$, and C_s is injective.

Theorem 2.2. *Let X_s, $T_s(t)$, C_s be as above. Then*

(a) *For all $t \geq 0$, $R(T_s(t))$ is contained in $\overline{R(T(t))}$, the closure of $R(T(t))$ in X. In particular, $R(T_s(t)) \subseteq X$ and $R(C_s) \subseteq \overline{R(C)}$, the closure of $R(C)$ in X.*

(b) *$\{T_s(t)\}_{t\geq 0}$ is a strongly uniformly continuous contraction C_s-semigroup.*

(c) *Suppose that A_s is the generator of $\{T_s(t)\}_{t\geq 0}$. Then*

(c_1) *$A \subseteq A_s$;*

(c_2) *$A_s = C_s^{-1}AC_s$;*

(c_3) *$A = A_s|_X$.*

Proof. (a) follows immediately from the definition of $T_s(t)$.

(b). First, we show that $T_s(t_1 + t_2)C_s = T_s(t_1)T_s(t_2)$ for all $t_1, t_2 \geq 0$. Let $y \in X_s$. Then there exists $\{x_n\} \subset X$ such that x_n converges to y in X_s, which means that $T(t)x_n$ converges in X for all $t \geq 0$. Also, by the definition of $T_s(t)$ and (a), we have

$$\begin{aligned} C_sT_s(t_1+t_2)y &= C\lim_{n\to\infty} T(t_1+t_2)x_n = \lim_{n\to\infty} CT(t_1+t_2)x_n \\ &= \lim_{n\to\infty} T(t_1)T(t_2)x_n = T(t_1)\lim_{n\to\infty} T(t_2)x_n \\ &= T(t_1)T_s(t_2)y = T_s(t_1)T_s(t_2)y \end{aligned}$$

with the four limits taken in X.

Next, for every $x \in X$,

$$\|T_s(t)x\|_s = \|T(t)x\|_s = \sup_{r\geq 0}\|T(r)T(t)x\| = \sup_{r\geq 0}\|T(r+t)Cx\| \leq \|Cx\|_s = \|C_sx\|_s;$$

therefore, $\{T_s(t)\}_{t\geq 0}$ is a family of contractions since X is dense in X_s.

Finally, we show that $\{T_s(t)\}_{t\geq 0}$ is strongly uniformly continuous. Now let $y \in X_s$. Then there exists a sequence $\{x_n\} \subset X$ satisfying $\|x_n - y\|_s \to 0$ as

$n \to \infty$. Thus

$$\begin{aligned}
&\|T_s(t+h)y - T_s(t)y\|_s \\
\le\ & \|T_s(t+h)y - T_s(t+h)x_n\|_s + \|T_s(t+h)x_n - T_s(t)x_n\|_s \\
&\quad + \|T_s(t)x_n - T_s(t)y\|_s \\
\le\ & 2\|C_s(x_n - y)\|_s + \sup_{r\ge 0}\|T(t+r+h)Cx_n - T(t+r)Cx_n\| \\
\le\ & 2\|C_s(x_n - y)\|_s + M\|T(h)x_n - Cx_n\|.
\end{aligned}$$

We already use the contractivity of $T_s(t)$ in the above. Note that the right side is independent of t, so $\{T_s(t)\}_{t\ge 0}$ is strongly uniformly continuous.

(c_1). Suppose that $x \in D(A)$. Then by (2.1), we know

$$\begin{aligned}
\Big\|\frac{T_s(t)x - C_s x}{t} - C_s Ax\Big\|_s &= \Big\|\frac{T(t)x - Cx}{t} - CAx\Big\|_s \\
&\le M\Big\|\frac{T(t)x - Cx}{t} - CAx\Big\| \to 0 \text{ as } t \to 0;
\end{aligned}$$

it follows that $x \in D(A_s)$ with $A_s x = Ax$.

(c_2). If $y \in D(A_s)$, then $\|\frac{T_s(t)y - C_s y}{t} - C_s A_s y\|_s \to 0$ as $t \to 0$. Since $R(T_s(t)) \subseteq X$, by the definition of $\|\cdot\|_s$, we have

$$\Big\|\frac{T(h)(T_s(t)y - C_s y)}{t} - T(h)C_s A_s y\Big\| \to 0 \text{ as } t \to 0$$

uniformly in h. Set $h = 0$. Noting that C_s commutes with A_s and $T_s(t)$, we have

$$\Big\|\frac{T_s(t)C_s y - C_s C_s y}{t} - C_s A_s C_s y\Big\| \to 0 \text{ as } t \to 0.$$

Since $C_s y \in D(A_s) \cap X$ and $A_s C_s y = C_s A_s y \in X$, this means

$$\Big\|\frac{T(t)C_s y - CC_s y}{t} - CA_s C_s y\Big\| \to 0 \text{ as } t \to 0,$$

which implies that $C_s y \in D(A)$ and $AC_s y = A_s C_s y = C_s A_s y$, i.e., $A_s y = C_s^{-1}AC_s y$. So we get $A_s \subseteq C_s^{-1}AC_s$.

On the other hand, $C_s^{-1}AC_s \subseteq C_s^{-1}A_s C_s = A_s$ since A_s is the generator.

(c_3). If $x \in D(A_s) \cap X$ and $A_s x \in X$, then $Cx = C_s x \in D(A)$ by (b) and $ACx = A_s Cx = C_s A_s x = CA_s x$, which implies that $A_s x = C^{-1}ACx$. So the claim follows from the fact that $A = C^{-1}AC$. $\square$

Remark 2.3. (a) It should be mentioned that the extrapolation space, W, of [1, 2], is defined only when $R(C)$ is dense; in [3] it is defined when $R(C)$ is dense or $\rho(A)$ contains a half-line. When $R(C)$ is dense, generating a contraction C-semigroup is equivalent to generating a strongly continuous semigroup of contractions by Theorem 4.6 in [4]; thus A_s of Theorem 2.2 is such a generator when $R(C)$ is dense.

(b) Recall the definition of W in [1] or [2]: for $x \in X$, $\|x\|_W = \sup_{t\ge 0}\|T(t)x\| = \|x\|_s$. Since W is a Banach space containing X, and X_s is the completion of X under the norm $\|\cdot\|_s$, it is clear that X_s is contained in W when $R(C)$ is dense and W is defined.

(c) $T_s(t)$ from Theorem 2.2 is a nonincreasing C-semigroup:

$$\|T_s(r)x\|_s = \sup_{t\ge 0}\|T(t)T(r)x\| = \sup_{t\ge r}\|T(t)Cx\|,$$

which is nonincreasing as a function of r. This implies that e^{rA}, at least formally a strongly continuous semigroup generated by A, is a contraction on $\bigcup_{t\geq 0} R(T(t))$, defined by $e^{rA}T(t)x \equiv T(t+r)x$.

(d) As a consequence of (c), when $\bigcup_{t\geq 0} R(T(t))$ is dense, A_s of Theorem 2.2 generates a strongly continuous semigroup of contractions. This is a weaker hypothesis than $R(C)$ being dense.

Now we use the extrapolation space to give a negative answer to the question mentioned in the Introduction.

Example 2.4. Let $X = c_0(\mathbb{N})$ and C_0 be the right shift on X, that is,

$$C_0 : (x_1, x_2, x_3 \cdots) \to (0, x_1, x_2, x_3, \cdots).$$

Next let

$$A = \begin{pmatrix} i & C_0^{-1} \\ 0 & -i \end{pmatrix} \text{ with } D(A) = X \times R(C_0)$$

and

$$C = \begin{pmatrix} C_0 & 0 \\ 0 & C_0 \end{pmatrix}.$$

It is not hard to show that A generates a bounded C-semigroup on $X \times X$ given by

$$T(t) = \begin{pmatrix} e^{it}C_0 & \frac{1}{2i}(e^{it} - e^{-it}) \\ 0 & e^{-it}C_0 \end{pmatrix},$$

but $\{T(t)\}_{t\geq 0}$ is not contractive. For every $\lambda \neq 0$, if $x_2 \notin \overline{R(C_0)}$, then

$$(\lambda - A)^{-1}C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\lambda - i)^{-1}C_0x_1 + (\lambda + i)^{-1}(\lambda - i)^{-1}x_2 \\ (\lambda + i)^{-1}C_0x_2 \end{pmatrix} \notin \overline{R(C)}.$$

So $(\lambda - A)^{-1}$ does not leave $\overline{R(C)}$ invariant. Since $\|C_0x\| = \|x\|$, for all $x \in X$, so $X_s = X$, and $\|\cdot\|_s$ is a topologically equivalent renorming of X. Thus $T_s(t) = T(t)$, $(\lambda - A)^{-1} = (\lambda - A_s)^{-1}$. Therefore $(\overline{R(C_s)})_s$ is not an invariant space of $(\lambda - A_s)^{-1}$. Thus no restriction of A_s generates a contraction C_0-semigroup on $(\overline{R(C_s)})_s$.

Remark 2.5. (a) The result is true for any injective $C_0 \in \mathbf{B}(X)$, X an arbitrary complex Banach space, satisfying $\overline{R(C_0)} \neq X$ and $0 \notin \sigma_a(C_0)$; i.e., $C_0x_n \to 0$ implies $x_n \to 0$.

(b) Although A_s of Example 2.4 does not generate a strongly continuous semigroup on $\overline{R(C)}$, there does exist a subspace, Y, between $\overline{R(C)}$ and X_s, on which A_s generates a strongly continuous semigroup, namely, $Y = X \times \overline{R(C_0)}$.

We end this paper with some open questions:

1. Is every contraction C-semigroup a nonincreasing C-semigroup (meaning $t \mapsto \|T(t)x\|$ is nonincreasing, for all $x \in X$)? This is true for C being isometric; that is, $\|Cx\| = \|x\|$ for all $x \in X$, since in this case,

$$\|T(t+s)x\| = \|CT(t+s)x\| = \|T(t+s)Cx\| = \|T(t)T(s)x\| \leq \|CT(s)x\| = \|T(s)x\|.$$

We conjecture that it is not true in general cases.

2. If A generates a contraction C-semigroup on X, does there exist a closed subspace Y such that $\overline{R(C)} \subseteq Y \subseteq X$ and $A|_Y$ generates a strongly continuous semigroup of contractions? Example 2.4 of this paper shows that the answer is no if Y is replaced by $R(C)$, but as remarked above in the section on Example 2.4, the answer is yes (in Example 2.4) with a different choice of Y.

3. Does every generator of a bounded C-semigroup have an extension, possibly on a larger space, that generates a strongly continuous semigroup of contractions? If 2 is true, then it and Theorem 2.2 would imply the answer is yes; when $R(C)$ is dense or $\rho(A)$ contains a half-line, it is known ([1, 2, 3]) that the answer is yes.

4. Is there a minimal Banach space in which X is embedded on which an extension of A generates a bounded, strongly continuous semigroup? Even when $R(C)$ is dense, so that an extension as in 3 exists, it is not known if a minimal one exists. In contrast, the interpolation space is maximal (see Chapter V in [2]).

5. Does $X_s = W$ always (when $R(C)$ is dense, so that both are defined)? Or is there an example where W is strictly larger than X_s?

Acknowledgement

We are very grateful to the referees for some improvement of the results, a simplification of Example 2.4, and for addressing the last five open questions.

References

1. R. deLaubenfels, *C-semigroups and strongly continuous semigroups*, Israel J. Math. **81** (1993), 227-255. MR1231189 (95d:47047)
2. R. deLaubenfels, Existence Families, Functional Calculi and Evolution Equations, Lecture Notes in Math. **1570**, Springer-Verlag, Berlin, 1994. MR1290783 (96b:47047)
3. V. Keyantuo, *Interpolation and extrapolation of C-semigroups*, Publ. Math. Fac. Sci. Besançon, Anal. Nonlinéaire **12** (1990), 103-121.
4. M. Li and F.L. Huang, *Characterizations of contraction C-semigroups*, Proc. Amer. Math. Soc. **126** (1998), 1063-1069. MR1443839 (98f:47048)

Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, People's Republic of China

E-mail address: limiao1973@hotmail.com

Department of Mathematics, Huazhnog University of Science and Technology, Wuhan 430074, People's Republic of China

E-mail address: qzheng@hust.edu.cn