PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 2, February 2009, Pages 635–640
S 0002-9939(08)09573-7
Article electronically published on June 20, 2008

FAILURE OF RATIONAL APPROXIMATION ON SOME CANTOR TYPE SETS

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(Communicated by Mario Bonk)

Abstract. Let $A(K)$ be the algebra of continuous functions on a compact set $K \subset \mathbb{C}$ which are analytic on the interior of K, and let $R(K)$ be the closure (with respect to uniform convergence on K) of the functions that are analytic on a neighborhood of K. A counterexample of a question posed by A. O'Farrell about the equality of the algebras $R(K)$ and $A(K)$ when $K = (K_1 \times [0,1]) \cup ([0,1] \times K_2) \subseteq \mathbb{C}$, with K_1 and K_2 compact subsets of $[0,1]$, is given. Also, the equality is proved with the assumption that K_1 has no interior.

1. Introduction

Consider a compact set K of the complex plane. Let $A(K)$ be the algebra of continuous functions on K which are analytic on the interior of K, and let $R(K)$ be the closure (with respect to uniform convergence on K) of the functions that are analytic on a neighborhood of K. Obviously, $R(K) \subseteq A(K)$.

In the 1960s, Vitushkin gave a description in analytic terms of the compact sets K for which $R(K) = A(K)$ (see [Vi]), but there is still no characterization of these compact sets in a geometric way. Nevertheless, there have been important advances in this area recently, as can be seen in the articles of Xavier Tolsa [To1] and [To2] and the one of Guy David [Da]. In this direction, Anthony G. O'Farrell raised the following question (private communication):

Question 1.1. Let K_1 and K_2 be two compact subsets of $[0,1]$ and define $K = (K_1 \times [0,1]) \cup ([0,1] \times K_2) \subseteq \mathbb{C}$. Is it true that $R(K) = A(K)$?

It is known that the identity holds if one of the compact sets K_1 or K_2 has no interior. For completeness, we include a proof of this fact at the end of the paper. However, it was not known whether the identity holds or not in general. In this paper we provide an example of a compact set K which gives a negative answer to the question. The set K is constructed as follows:

Received by the editors February 6, 2008.
2000 *Mathematics Subject Classification.* Primary 30C85; Secondary 31A15.
Key words and phrases. Rational approximation, analytic capacity, Cantor sets.
This work was supported by grant AP2006-02416 (Programa FPU del MEC, España), and also partially supported by grants 2005SGR-007749 (Generalitat de Catalunya) and MTM2007-62817 (MEC, España).

Let $\mathcal{C}(1/3)$ be the ternary Cantor set on the interval $[0,1]$, i.e.,

$$\mathcal{C}(1/3) = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{2^n} I_n^j,$$

where $I_0^1 = [0,1]$ and each I_n^j is an interval of length 3^{-n} obtained by dividing the intervals of length 3^{-n+1} in three equal parts and excluding the central part. Call z_n^j the center of I_n^j. Consider a sequence $\delta_n > 0$ such that $\delta_n < 3^{-n-1}$ and define $J_n^j = (z_n^j - \delta_n/2, z_n^j + \delta_n/2)$, where z_n^j is the center of I_n^j. Let

$$E_m = [0,1] \setminus \bigcup_{n=0}^{m} \bigcup_{j=1}^{2^n} J_n^j.$$

Finally, define $F_m = (E_m \times [0,1]) \cup ([0,1] \times E_m) \subseteq \mathbb{C}$ and put $K = \bigcap_{m=0}^{\infty} F_m$.

With this construction of K we will prove the main result of the paper:

Theorem 1.2. *For a suitable choice of the sequence δ_n, $R(K) \neq A(K)$.*

In the whole paper $\mathcal{M}^1$ stands for the *1-dimensional Hausdorff content* and α denotes the *continuous analytic capacity* (see [Vi]). Remember that, given a compact set $F \subseteq \mathbb{C}$,

$$\alpha(F) = \sup |f'(\infty)|,$$

where the supremum is taken over all continuous functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ which are analytic on $\mathbb{C} \setminus F$ and uniformly bounded by 1 on $\mathbb{C}$. If f satisfies all these properties, we say that f is *admissible* for α and F. By definition, $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$.

2. Proof of the main result

In the following two lemmas, we shall obtain some estimates of the Hausdorff content of $[0,1]^2 \setminus K$ that will be useful in showing that the algebras $R(K)$ and $A(K)$ are not equal for a suitable choice of the sequence δ_n.

Lemma 2.1. *Fix $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\delta < 3^{-n_0+2}$. Define $\tilde{J}_n^j = (z_n^j - \delta/2, z_n^j + \delta/2)$, $R_n^j = \tilde{J}_n^j \times [0,\delta]$ and*

$$R = \bigcup_{n=0}^{n_0} \bigcup_{j=1}^{2^n} R_n^j.$$

Then $\mathcal{M}^1(R) < 8\delta^\eta$, where $\eta = 1 - \frac{1}{\log_2 3} > 0$.

Proof. Since R is the union of the squares R_n^j for $0 \le n \le n_0$ and $1 \le j \le 2^n$ and each square has side length δ, we have

$$\mathcal{M}^1(R) \le \sum_{n=0}^{n_0} \sum_{j=1}^{2^n} \delta = \delta(2^{n_0+1} - 1) \le \delta 2^{n_0+1}.$$

The inequality $\delta < 3^{-n_0+2}$ is equivalent to $n_0 < 2 - \log_3 \delta$. Then, using the fact that $\log_3 \delta = \log_2 \delta / \log_2 3$, we can deduce that

$$\delta 2^{n_0+1} < \delta 2^{3-\log_3 \delta} = \delta 2^{3-\frac{\log_2 \delta}{\log_2 3}} = 8\delta^{1-\frac{1}{\log_2 3}} = 8\delta^\eta.$$

□

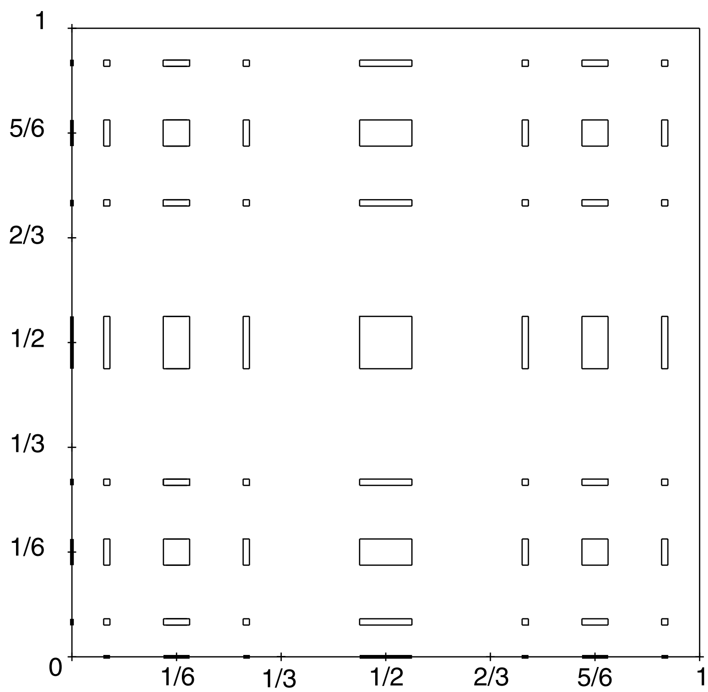

FIGURE 1. This is a picture of the compact set F_2. The rectangles inside $[0,1]^2$ are the holes of F_2, and the bold lines on the sides of $[0,1]^2$ correspond to the subset of the real line $\bigcup_{n=0}^{2}\bigcup_{j=1}^{2^n} J_n^j$.

As we will see in the proof of the following lemma, the important fact of the preceding one is that $\mathcal{M}^1(R)$ is bounded by something that tends to zero as δ decreases rather than the exact value of the bound.

Lemma 2.2. *For every $\varepsilon > 0$ there exists a sequence δ_n such that*

$$\mathcal{M}^1([0,1]^2 \setminus K) < \varepsilon.$$

Proof. Put $G = [0,1]^2 \setminus K$. Consider the crosses P_n^k for $k = 1, \dots, 4^n$ defined in the following way (see Figure 1 to understand the construction):

$P_0^1 = (J_0^1 \times [0,1]) \cup ([0,1] \times J_0^1)$,

$P_1^1 = (J_1^1 \times [0,1/3]) \cup ([0,1/3] \times J_1^1)$, $P_1^2 = (J_1^2 \times [0,1/3]) \cup ([2/3,1] \times J_1^1)$,
$P_1^3 = (J_1^1 \times [2/3,1]) \cup ([0,1/3] \times J_1^2)$, $P_1^4 = (J_1^2 \times [2/3,1]) \cup ([2/3,1] \times J_1^2)$,

$P_2^1 = (J_2^1 \times [0,1/9]) \cup ([0,1/9] \times J_2^1)$, $P_2^2 = (J_2^2 \times [0,1/9]) \cup ([2/9,1/3] \times J_2^1)$,
$P_2^3 = (J_2^3 \times [0,1/9]) \cup ([2/3,7/9] \times J_2^1)$, $P_2^4 = (J_2^4 \times [0,1/9]) \cup ([8/9,1] \times J_2^1)$,
$P_2^5 = (J_2^1 \times [2/9,1/3]) \cup ([0,1/9] \times J_2^2)$, $\dots$.

It is clear that $G \subseteq \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{4^n} P_n^k$. By construction, we also have $\mathcal{M}^1(P_n^1 \cap G) = \mathcal{M}^1(P_n^k \cap G)$ for all $k = 1, \dots, 4^n$. Therefore,

$$\mathcal{M}^1(G) \leq \sum_{n=0}^{\infty} 4^n \mathcal{M}^1(P_n^1 \cap G).$$

Denote by X_n the horizontal strip of the cross P_n^1 and Y_n the vertical one. Because of the symmetry of the compact set K and the subadditivity of $\mathcal{M}^1$,

$$\mathcal{M}^1(P_n^1 \cap G) \leq 2\mathcal{M}^1(X_n \cap G).$$

Observe that G is a countable union of rectangles and on X_n all those rectangles have sides of length less than or equal to δ_n. So, the set $3^n(X_n \cap G) := \{3^n x : \ x \in X_n \cap G\}$ can be included by a translation in a set $R := \bigcup_{n=0}^{n_0} \bigcup_{j=1}^{2^n} R_n^j$ like the one of the preceding lemma if we take $\delta = 3^n \delta_n$ and $n_0 \in \mathbb{N}$ such that $3^{-n_0+1} \leq \delta < 3^{-n_0+2}$. Applying the lemma, we obtain

$$\mathcal{M}^1(X_n \cap G) < 3^{-n} 8 (3^n \delta_n)^{\eta} = 3^{n(\eta-1)} 8 \delta_n^{\eta}$$

with $\eta = 1 - \frac{1}{\log_2 3}$, and then

$$\mathcal{M}^1(G) \leq 8 \sum_{n=0}^{\infty} 4^n \mathcal{M}^1(X_n \cap G) < 8 \sum_{n=0}^{\infty} 4^n 3^{n(\eta-1)} \delta_n^{\eta}.$$

Given $\varepsilon > 0$, it is easy to find a decreasing sequence δ_n that makes the last sum less than ε, because $\eta > 0$. $\square$

Proof of Theorem 1.2. As Vitushkin proved [Vi] (see also [Ga], Theorem VIII.8.2), $R(K) = A(K)$ if and only if $\alpha(D \setminus K) = \alpha(D \setminus \mathrm{int}K)$ for every bounded open set D.

If $\mathcal{C} = \mathcal{C}(1/3) \times \mathcal{C}(1/3)$, we know that $\alpha(\mathcal{C}) > 0$ because $\dim(\mathcal{C}) > 1$, where $\dim(\cdot)$ denotes the Hausdorff dimension. Observe that $\mathcal{C} \subseteq \partial K$ and it does not depend on the chosen sequence δ_n. This implies that $\alpha(\partial K) \geq \alpha(\mathcal{C})$, so it is guaranteed a minimum of continuous analytic capacity on the boundary of K for any sequence δ_n.

Observe also that $\alpha([0,1]^2 \setminus \mathrm{int}K) = \alpha((0,1)^2 \setminus \mathrm{int}K)$ because $\partial([0,1]^2)$ is *negligible* (see [Ga], chapter VIII). Therefore,

$$\alpha(\mathcal{C}) \leq \alpha(\partial K) \leq \alpha([0,1]^2 \setminus \mathrm{int}K) = \alpha((0,1)^2 \setminus \mathrm{int}K).$$

On the other hand, by the preceding lemma we can find a sequence δ_n such that $\mathcal{M}^1([0,1]^2 \setminus K) \leq \alpha(\mathcal{C})/2$. If we take $\alpha \leq \mathcal{M}^1$ into account, we can deduce that

$$\alpha((0,1)^2 \setminus \ K) \leq \mathcal{M}^1([0,1]^2 \setminus K) \leq \alpha(\mathcal{C})/2 < \alpha(\mathcal{C}) \leq \alpha((0,1)^2 \setminus \mathrm{int}K).$$

These inequalities show that the necessary condition for $R(K) = A(K)$ in Vitushkin's theorem does not hold for $D = (0,1)^2$. So, for that sequence δ_n we have $R(K) \neq A(K)$. $\square$

3. $A(K) = R(K)$ when K_1 has no interior

Now, as we said at the beginning of the paper, we proceed to give an affirmative answer to question 1.1 with the assumption that K_1 has no interior. We need an auxiliary lemma that we guess is already known, so we only sketch the proof.

Lemma 3.1. *Fix $\delta > 0$ and $n \in \mathbb{N}$. Let R be a rectangle with sides of length δ and $n\delta$ and put $R = \bigcup_{j=1}^n Q_j$, where Q_j are squares of side length δ with pairwise disjoint interiors. Let $E_j \subseteq Q_j$ and suppose there exists $C_0 > 0$ such that $\alpha(E_j) \geq C_0\delta$ for all j. Then, there exists a constant $C_1 > 0$ depending only on C_0 such that*

$$\sum_{j=1}^n \alpha(E_j) \leq C_1 \alpha(\bigcup_{j=1}^n E_j).$$

Hint of the proof. Given admissible functions f_j for α and E_j, one can find a function f admissible for α and $\bigcup_{j=1}^n E_j$ such that $\sum_j |f_j'(\infty)| = C_1|f'(\infty)|$ by using *Vitushkin's localization scheme* with a modified *triple zero lemma* (see [Ve] or [Vi]), where one uses the fact that the sets E_j are aligned. Then, one can prove the lemma by taking supremums. □

From now on, we shall denote by C an absolute constant that may change its value at different occurrences.

Theorem 3.2. *Let $K_1, K_2 \subseteq [0,1]$ be two compact sets and define $K = (K_1 \times [0,1]) \cup ([0,1] \times K_2)$. Suppose that K_1 has no interior. Then, $R(K) = A(K)$.*

Proof. By Vitushkin's theorem, it is known that $R(K) = A(K)$ if and only if there exists an absolute constant $C > 0$ such that $\alpha(Q \setminus \mathrm{int}K) \leq C\alpha(Q \setminus K)$ for all open squares Q.

Fix a square Q of side length $l > 0$. We can suppose that $Q \setminus K$ is not empty, so there exists a square $F \subseteq Q \setminus K$. Let π_x and π_y be the projections onto the horizontal and vertical coordinate axes respectively. Then, $\pi_y(F) \subseteq \pi_y(Q) \setminus K_2$ and we can find an interval $F_y \subseteq \pi_y(F)$ of length l/n for n big enough.

On the other hand, if we split $\pi_x(Q)$ into intervals I_j, for $j = 1, \ldots, n$, with pairwise disjoint interiors and length l/n, we can also find intervals $F_x^j \subseteq (\pi_x(Q) \setminus K_1) \cap I_j$ for $j = 1, \ldots, n$, because K_1 has no interior. Therefore, $\bigcup_{j=1}^n (F_x^j \times F_y) \subseteq Q \setminus K$ and $\alpha(F_x^j \times F_y) \geq C_0\, l/n$.

Now we are ready to use the preceding lemma with the squares $Q_j = F_y \times I_j$, the subsets $E_j = F_x^j \times F_y$ and $\delta = l/n$, and we obtain

$$l \leq C \sum_{j=1}^n \alpha(F_x^j \times F_y) \leq C\alpha(\bigcup_{j=1}^n (F_x^j \times F_y)) \leq C\alpha(Q \setminus K).$$

We can finally deduce that

$$\alpha(Q \setminus \mathrm{int}K) \leq \alpha(Q) = C\, l \leq C\alpha(Q \setminus K)$$

for every open square Q, so $R(K) = A(K)$. □

We are grateful to Anthony O'Farrell for the communication of another proof of Theorem 3.2 which uses annihilating measures instead of Vitushkin's theorem.

Acknowledgments

The author gratefully acknowledges Mark Melnikov and Xavier Tolsa for the communication of question 1.1 and for useful discussions while preparing this paper.

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