A NOTE ON ZEROES OF REAL POLYNOMIALS IN $C(K)$ SPACES

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Abstract. For real $C(K)$ spaces, we show that being injected in a Hilbert space is a 3-space property. As a consequence, we obtain that, when $K$ does not carry a strictly positive Radon measure, every quadratic continuous homogeneous real-valued polynomial on $C(K)$ admits a linear zero subspace enjoying a property which implies non-separability.

1. Terminology and preliminary results

Throughout what follows $K$ will be a compact Hausdorff topological space and $C(K)$ its associated real-valued function space provided with the supremum norm. By $X$ and $X^*$ we denote a real Banach space and its topological dual, respectively, with the standard duality being represented by $\langle \cdot, \cdot \rangle$. If $A \subset X$ and $B \subset X^*$, we use the notation

$$A^\perp = \{ x^* \in X^*: \langle x^*, x \rangle = 0, x \in A \}, \quad B_\perp = \{ x \in X: \langle x^*, x \rangle = 0, x^* \in B \}.$$ 

Following the terminology introduced in [3], we say that a Banach space $X$ belongs to class $C_H$ whenever $X$ is injected (i.e., there is a one-to-one bounded linear map) into a Hilbert space. Similarly, $W^*$ is defined as the subclass of $C_H$ formed by those spaces which are injected into a separable Hilbert space. The next few auxiliary results are quite straightforward and will be used in the sequel; see [3] for details.

Lemma 1. For a Banach space $X$ the following are equivalent:

(i) $X$ is in $W^*$.
(ii) $X^*$ is weak*-separable.
(iii) $X^*$ contains a countable total subset.
(iv) There is a one-to-one bounded linear map from $X$ into a separable Banach space.

Lemma 2. If $Y$ is a closed linear subspace of $X$ such that $Y \in W^*$ and $X/Y \in C_H$, then $X \in C_H$.

The following result is taken from [1 Corollary 3] and [6 Theorem 4.5].

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Lemma 3. For a compact $K$, the following are equivalent:

(i) $C(K)$ belongs to $C_H$.
(ii) $K$ carries a strictly positive Radon probability.
(iii) $C(K)^*$ contains a weakly compact total subset.

Although the property of being injected into a Hilbert space is not equivalent, in general, to that of being injected into a reflexive space (think of $\ell_p(\Gamma)$, $\Gamma$ uncountable, $2 < p < \infty$), we see next that this equivalence is obtained for $C(K)$ spaces.

Proposition 1. For a compact $K$, $C(K)$ is in $C_H$ if and only if it is injected into a reflexive space.

Proof. Since we only need to prove the sufficiency part, let $T: C(K) \to E$ be a one-to-one bounded linear map, where $E$ is a reflexive Banach space. Then, setting $W := T^*(B_{E^*})$, where $B_{E^*}$ denotes the closed unit ball of the dual space $E^*$ and $T^*$ is the conjugate map of $T$, we have that $W$ is a weakly compact total subset of $C(K)^*$. Lemma 3 now applies. □

Corollary 1. If $\Gamma$ is uncountable, then $c_0(\Gamma)$ is not injected into a reflexive space.

Proof. Since $c_0(\Gamma)$ is isomorphic to $C(K)$, where $K$ is taken to be the one-point compactification of the discrete space $\Gamma$, if $c_0(\Gamma)$ were injected into a reflexive space, we would have after the former proposition that $C(K) \in C_H$. But, this would imply that $c_0(\Gamma) \in C_H$, which is not so; see [3]. □

We say that a closed linear subspace $Y$ of $X$ is splitting whenever there is a bounded linear map $T: X \to Y$ such that $Ker T \cap Y = \{0\}$. Notice that every complemented subspace of $X$ is splitting in $X$.

Proposition 2. Let $Y$ be a closed linear subspace which is splitting in $X$. If $Y$ and $X/Y$ are both injected into a reflexive space, then $X$ is also injected into a reflexive space.

Proof. Let $T: X \to Y$, $S_1: X/Y \to E_1$, $S_2: Y \to E_2$ be bounded linear maps, where $E_1$ and $E_2$ are reflexive spaces, such that $Ker T \cap Y = \{0\}$ and $S_1, S_2$ are one-to-one. Defining $L: X \to E_1 \times E_2$ as

$$Lx := (S_1(x + Y), S_2Tx), \quad x \in X,$$

we obtain a bounded linear map into the reflexive space $E_1 \times E_2$ such that it is also one-to-one: If $Lx = (0, 0)$, then we have that $S_1(x + Y) = 0$ and $S_2Tx = 0$; hence $x \in Y \cap Ker T = \{0\}$. □

2. A three-space property and zeroes of quadratic polynomials on $C(K)$

In [2, Remark 3], the following question is posed: If $Y$ is a closed linear subspace of $X$ such that $Y$ and $X/Y$ are in class $C_H$, does it necessarily follow that $X$ is also in this class? We show that the answer to this question is affirmative for $C(K)$ spaces. Later, we make use of this fact to see that continuous quadratic homogeneous polynomials on $C(K)$, with $K$ not carrying a strictly positive Radon measure,
Proposition 3. Let $K_0$ be a closed subset of the compact space $K$ such that $C_{K_0}(K)$ is injected into a reflexive space. Then $C_{K_0}(K)$ is splitting in $C(K)$.

Proof. Since cozero sets, i.e., complements of the zero-sets of the elements of $C(K)$, form a base for the open sets of $K$, applying Zorn’s Lemma, we can guarantee the existence of a maximal collection of pairwise disjoint cozero sets contained in the open set $K \setminus K_0$. This maximal collection has to be countable; otherwise there would be a copy of $c_0(\Gamma)$, with $\Gamma$ having the cardinality of the collection, contained in $C_{K_0}(K)$, but this would imply that $c_0(\Gamma)$ is injected into a reflexive space, a contradiction after Corollary 1. Thus, let $\{V_j : j \geq 1\}$ be this maximal collection. From the maximality and since $V := \bigcup_{j=1}^{\infty} V_j$ is also a cozero set, there is $z \in C(K)$ such that

$$V \subset K \setminus K_0 \subset \overline{V}, \quad z^{-1}(0) = K \setminus V.$$

Let $T$ be the map from $(C(K))$ into $C_{K_0}(K)$ defined as

$$Tx := x \cdot z, \quad x \in C(K).$$

Then, $T$ is well defined, linear and bounded. Besides, if $x \in Ker T \cap C_{K_0}(K)$, it follows that $x_{|K_0} = 0$ and $xz = 0$; consequently, since $z(t) \neq 0, t \in V$, we have that $x_{|\overline{V}} = 0$ and so $x = 0$. \hfill \square

Proposition 4. Let $Y$ be a closed linear subspace of $C(K)$ such that $(C(K))/Y$ is injected into a reflexive space. Then, there is a Radon probability $\mu$ on $K$ such that $C_{\text{supp } \mu}(K)$ is contained in $Y$.

Proof. Let $T : C(K)/Y \to E$ be a one-to-one bounded linear map, with $E$ reflexive. Let $F$ be the closure of $T^*(E^*)$ in $(C(K)/Y)^* = Y^\perp$. Since $F$ is weakly compactly generated, no copy of $\ell_1(\Gamma)$, $\Gamma$ uncountable, can be contained in $F$; thus, after [6 Lemma 1.3], we have that there is a Radon probability $\mu$ on $K$ such that every element of $F$ is $\mu$-absolutely continuous. Hence, for each $u^* \in E^*$, there is a $\mu$-measurable function $f_{u^*}$ which is the Radon-Nikodym derivative of $T^*u^*$ with respect to $\mu$ and so

$$\langle T^*u^*, x \rangle = \int_{K} x \, d(T^*u^*) = \int_{K} x \, f_{u^*} \, d\mu, \quad x \in C(K).$$

Thus, if $K_0 := \text{supp } \mu$, we have that, for $x \in C_{K_0}(K)$,

$$\langle T^*u^*, x \rangle = \int_{K_0} x \, f_{u^*} \, d\mu = 0,$$

that is

$$C_{K_0}(K) \subset \bigcap_{u^* \in E^*} \text{Ker } T^*u^*.$$

But $T^*$ has weak$^*$-dense range in $Y^\perp$. Therefore

$$Y^\perp = (T^*(E^*))^{\perp^*} = (\bigcap_{u^* \in E^*} \text{Ker } T^*u^*)^{\perp},$$

which leads to $C_{K_0}(K) \subset Y$. \hfill \square
Corollary 2. If $Y$ is a closed linear subspace of $C(K)$ such that $Y$ and $C(K)/Y$ are injected into a reflexive space, then $C(K)$ is in $\mathcal{C}_H$.

Proof. From Proposition 4, we have that there is a closed subset $K_0$ of $K$ supporting a Radon probability such that $C_K_0(K) \subset Y$. Hence, we have that $C_{K_0}(K)$ is injected into a reflexive space and, from Proposition 3, that $C_{K_0}(K)$ is splitting in $C(K)$. After Lemma 3, since $C(K)/C_{K_0}(K)$ is isomorphic to $C(K_0)$, we obtain that $C(K)/C_{K_0}(K)$ is in $\mathcal{C}_H$. The result now follows from Proposition 2 and Proposition 1.

Corollary 3. If $Y$ is a closed linear subspace of $C(K)$ such that $Y$ and $C(K)/Y$ are in $\mathcal{C}_H$, then $C(K)$ is also in $\mathcal{C}_H$; i.e., being injected into a Hilbert space is a 3-space property for $C(K)$ spaces.

Concerning this 3-space property, this author has learned from the editor, Professor Nigel Kalton, that the problem of whether being injected into a Hilbert space is a 3-space property has a negative solution in general. The particular counterexample to this problem (also provided by Professor Kalton) is the following:

Let $\Gamma$ be an uncountable set and consider the Banach space $Z_2(\Gamma)$, defined similarly to $Z_2(\mathbb{N})$ in [5], Section 6. For our purposes, it suffices to recall the following properties of $Z_2(\Gamma)$: It is a reflexive space which can be normed in such a way that it has a closed linear subspace $Y$ isometric to $\ell_2(\Gamma)$ with $Z_2(\Gamma)/Y$ isometric to $\ell_2(\Gamma)$; i.e., $Z_2(\Gamma)$ is a twisted sum of $\ell_2(\Gamma)$ with itself. Furthermore, $Z_2(\Gamma)^*$ is isometric to $Z_2(\Gamma)$. Thus, $Y$ and $Z_2(\Gamma)/Y$ are in class $\mathcal{C}_H$, while we show that $Z_2(\Gamma) \notin \mathcal{C}_H$: Let $T : Z_2(\Gamma) \to H$ be any bounded linear map, where $H$ is a Hilbert space. After [5] Theorem 6.5, Corollary 6.8], it can be deduced that every such map is strictly singular, so the restriction of $T$ to the subspace $\ell_2(\Gamma)$ is therefore compact and is not one-to-one. Consequently, $T$ cannot be one-to-one either.

In [11, Remark 3], the following dichotomy is conjectured: For a real Banach space $X$, either it admits a positive definite continuous quadratic homogeneous polynomial or every continuous quadratic homogeneous polynomial has a zero-set that contains a non-separable linear subspace. In [4], using a weaker form of the 3-space property obtained before, the above conjecture is seen to be correct for $C(K)$ spaces. Our next result is a slight improvement of this.

Let us just recall that if $P$ is a continuous quadratic homogeneous polynomial in $X$, then $P^\prime : X \to X^*$ represents the Fréchet derivative map, i.e., a bounded linear map such that

$$\langle P'(x), y \rangle = 2 \tilde{P}(x, y), \quad x, y \in X,$$

where $\tilde{P}$ stands for the symmetric bilinear functional associated to $P$.

Corollary 4. If $C(K) \notin \mathcal{C}_H$, then for each continuous quadratic homogeneous real-valued polynomial $P$ in $C(K)$, every maximal linear subspace $Z$ contained in $P^{-1}(0)$ satisfies that either $Z \notin \mathcal{W}^*$ or $\overline{P}(\overline{Z})$ is not weakly compactly generated.

Proof. Zorn’s Lemma guarantees the existence of maximal linear subspaces contained in the polynomial’s zero-set $P^{-1}(0)$. Let $Z$ be one of such maximal linear subspaces, which is clearly closed. Seeking a contradiction, let us assume that $Z \in \mathcal{W}^*$ and $\overline{P}(\overline{Z})$ is weakly compactly generated. Again using Rosenthal’s dichotomic result of [6, Lemma 1.3], we know that there is a Radon probability $\mu$ on
$K$ such that every element of $P'(Z)$ is $\mu$-absolutely continuous. Let $Y := P'(Z)_\perp$ and $K_0 := \text{supp } \mu$. Proceeding as in the proof of Proposition 4, it is easy to see that $C_{K_0}(K)$ is contained in $Y$. Besides, $Z \subset Y$ and the maximality of $Z$ yield that $P^{-1}(0) \cap Y = Z$, and so the polynomial $P$ does not change sign in $Y$; thus, defining $P(x + Z) := P(x), x \in Y$, we have a continuous quadratic polynomial $\tilde{P}$ on $Y/Z$ which is positive (or negative) definite. This implies that $Y/Z \in \mathcal{C}_H$; see [H Proposition 2]. Hence, after Lemma 2, it follows that $Y \in \mathcal{C}_H$. Then, we have that $C_{K_0}(K) \in \mathcal{C}_H$ and since $C(K)/C_{K_0}(K)$ is isomorphic to $C(K_0)$,

$$C(K)/C_{K_0}(K) \in \mathcal{C}_H.$$ 

The desired contradiction follows from Corollary 3.

Notice that if $P$ is a continuous quadratic homogeneous real-valued polynomial on the Banach space $X$, then the linear map $P'$ given by the Fréchet derivative satisfies that $\text{Ker } P' \subset P^{-1}(0)$. Thus, our next result gives us a quite large linear subspace contained in the polynomial’s zero-set for a wide class of compacta.

**Corollary 5.** If $C(K) \notin \mathcal{C}_H$ and $C(K)^* \in \mathcal{C}_H$, then for each continuous quadratic homogeneous polynomial $P$, $\text{Ker } P' \notin \mathcal{C}_H$.

**Proof.** Since $C(K)/\text{Ker } P'$ is injected into $C(K)^*$, it follows that $C(K)/\text{Ker } P' \in \mathcal{C}_H$. Hence, from Corollary 3 the result is obtained.

If $K$ is a scattered compact with an uncountable amount of isolated points, then $C(K) \notin \mathcal{C}_H$, while $C(K)^* = \ell_1(K) \in \mathcal{C}_H$. Hence, after the previous corollary, the next result follows.

**Corollary 6.** Let $K$ be a scattered compact space with uncountably many isolated points. Then, for each quadratic continuous homogeneous polynomial $P$, $\text{Ker } P' \notin \mathcal{C}_H$.

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**References**


