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A NOTE ON ZEROES OF REAL POLYNOMIALS IN $C(K)$ SPACES

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ABSTRACT. For real $C(K)$ spaces, we show that being injected in a Hilbert space is a 3-space property. As a consequence, we obtain that, when K does not carry a strictly positive Radon measure, every quadratic continuous homogeneous real-valued polynomial on $C(K)$ admits a linear zero subspace enjoying a property which implies non-separability.

1. TERMINOLOGY AND PRELIMINARY RESULTS

Throughout what follows K will be a compact Hausdorff topological space and $C(K)$ its associated real-valued function space provided with the supremum norm. By X and X^* we denote a real Banach space and its topological dual, respectively, with the standard duality being represented by $\langle \cdot, \cdot \rangle$. If $A \subset X$ and $B \subset X^*$, we use the notation

$$A^{\perp} = \{x^* \in X^* : \langle x^*, x \rangle = 0, x \in A\}, \quad B_{\perp} = \{x \in X : \langle x^*, x \rangle = 0, x^* \in B\}.$$

Following the terminology introduced in [3], we say that a Banach space X belongs to class $\mathcal{C}_H$ whenever X is injected (i.e., there is a one-to-one bounded linear map) into a Hilbert space. Similarly, $\mathcal{W}^*$ is defined as the subclass of $\mathcal{C}_H$ formed by those spaces which are injected into a separable Hilbert space. The next few auxiliary results are quite straightforward and will be used in the sequel; see [3] for details.

Lemma 1. *For a Banach space X the following are equivalent:*

(i) *X is in $\mathcal{W}^*$.*
(ii) *X^* is weak*-separable.*
(iii) *X^* contains a countable total subset.*
(iv) *There is a one-to-one bounded linear map from X into a separable Banach space.*

Lemma 2. *If Y is a closed linear subspace of X such that $Y \in \mathcal{W}^*$ and $X/Y \in \mathcal{C}_H$, then $X \in \mathcal{C}_H$.*

The following result is taken from [1, Corollary 3] and [6, Theorem 4.5].

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Lemma 3. *For a compact K, the following are equivalent:*

(i) *$C(K)$ belongs to $\mathcal{C}_H$.*
(ii) *K carries a strictly positive Radon probability.*
(iii) *$C(K)^*$ contains a weakly compact total subset.*

Although the property of being injected into a Hilbert space is not equivalent, in general, to that of being injected into a reflexive space (think of $\ell_p(\Gamma)$, Γ uncountable, $2 < p < \infty$), we see next that this equivalence is obtained for $C(K)$ spaces.

Proposition 1. *For a compact K, $C(K)$ is in $\mathcal{C}_H$ if and only if it is injected into a reflexive space.*

Proof. Since we only need to prove the sufficiency part, let $T : C(K) \to E$ be a one-to-one bounded linear map, where E is a reflexive Banach space. Then, setting

$$W \;:=\; T^*(B_{E^*}),$$

where B_{E^*} denotes the closed unit ball of the dual space E^* and T^* is the conjugate map of T, we have that W is a weakly compact total subset of $C(K)^*$. Lemma 3 now applies. □

Corollary 1. *If Γ is uncountable, then $c_0(\Gamma)$ is not injected into a reflexive space.*

Proof. Since $c_0(\Gamma)$ is isomorphic to $C(K)$, where K is taken to be the one-point compactification of the discrete space Γ, if $c_0(\Gamma)$ were injected into a reflexive space, we would have after the former proposition that $C(K) \in \mathcal{C}_H$. But, this would imply that $c_0(\Gamma) \in \mathcal{C}_H$, which is not so; see [3]. □

We say that a closed linear subspace Y of X is *splitting* whenever there is a bounded linear map $T : X \to Y$ such that $Ker\ T\ \cap Y = \{0\}$. Notice that every complemented subspace of X is splitting in X.

Proposition 2. *Let Y be a closed linear subspace which is splitting in X. If Y and X/Y are both injected into a reflexive space, then X is also injected into a reflexive space.*

Proof. Let $T : X \to Y$, $S_1 : X/Y \to E_1$, $S_2 : Y \to E_2$ be bounded linear maps, where E_1 and E_2 are reflexive spaces, such that $Ker\ T\ \cap Y = \{0\}$ and S_1, S_2 are one-to-one. Defining $L : X \to E_1 \times E_2$ as

$$Lx := (S_1(x + Y), S_2Tx), \quad x \in X,$$

we obtain a bounded linear map into the reflexive space $E_1 \times E_2$ such that it is also one-to-one: If $Lx = (0, 0)$, then we have that $S_1(x + Y) = 0$ and $S_2Tx = 0$; hence $x \in Y \cap Ker\ T\ = \{0\}$. □

2. A three-space property and zeroes of quadratic polynomials on $C(K)$

In [2, Remark 3], the following question is posed: *If Y is a closed linear subspace of X such that Y and X/Y are in class $\mathcal{C}_H$, does it necessarily follow that X is also in this class?* We show that the answer to this question is affirmative for $C(K)$ spaces. Later, we make use of this fact to see that continuous quadratic homogeneous polynomials on $C(K)$, with K not carrying a strictly positive Radon measure, have a zero-set that contains linear subspaces with a property which implies non-separability. This slightly improves Corollary 8 of [4], where an affirmative answer to the conjecture stated in [1, Remark 3] is given for $C(K)$ spaces. If A is a subset of the compact K, by $C_A(K)$ we mean the closed linear subspace of $C(K)$ formed by those elements which vanish in A.

Proposition 3. *Let K_0 be a closed subset of the compact space K such that $C_{K_0}(K)$ is injected into a reflexive space. Then $C_{K_0}(K)$ is splitting in $C(K)$.*

Proof. Since cozero sets, i.e., complements of the zero-sets of the elements of $C(K)$, form a base for the open sets of K, applying Zorn's Lemma, we can guarantee the existence of a maximal collection of pairwise disjoint cozero sets contained in the open set $K \setminus K_0$. This maximal collection has to be countable; otherwise there would be a copy of $c_0(\Gamma)$, with Γ having the cardinality of the collection, contained in $C_{K_0}(K)$, but this would imply that $c_0(\Gamma)$ is injected into a reflexive space, a contradiction after Corollary 1. Thus, let $\{V_j : \ j \geq 1\}$ be this maximal collection. From the maximality and since $V := \bigcup_{j=1}^{\infty} V_j$ is also a cozero set, there is $z \in C(K)$ such that

$$V \ \subset \ K \setminus K_0 \ \subset \overline{V}, \quad z^{-1}(0) \ = \ K \setminus V.$$

Let T be the map from $C(K)$ into $C_{K_0}(K)$ defined as

$$Tx := x \cdot z, \quad x \in C(K).$$

Then, T is well defined, linear and bounded. Besides, if $x \in Ker \ T \ \cap C_{K_0}(K)$, it follows that $x_{|K_0} = 0$ and $xz = 0$; consequently, since $z(t) \neq 0$, $t \in V$, we have that $x_{|\overline{V}} = 0$ and so $x = 0$. □

Proposition 4. *Let Y be a closed linear subspace of $C(K)$ such that $C(K)/Y$ is injected into a reflexive space. Then, there is a Radon probability μ on K such that $C_{supp\ \mu}(K)$ is contained in Y.*

Proof. Let $T : C(K)/Y \to E$ be a one-to-one bounded linear map, with E reflexive. Let F be the closure of $T^*(E^*)$ in $(C(K)/Y)^* = Y^{\perp}$. Since F is weakly compactly generated, no copy of $\ell_1(\Gamma)$, Γ uncountable, can be contained in F; thus, after [6, Lemma 1.3], we have that there is a Radon probability μ on K such that every element of F is μ-absolutely continuous. Hence, for each $u^* \in E^*$, there is a μ-measurable function f_{u^*} which is the Radon-Nikodym derivative of T^*u^* with respect to μ and so

$$\langle T^*u^*, x\rangle \ = \ \int_K x \ d(T^*u^*) \ = \ \int_K x \ f_{u^*} \ d\mu, \quad x \in C(K).$$

Thus, if $K_0 := supp\ \mu$, we have that, for $x \in C_{K_0}(K)$,

$$\langle T^*u^*, x\rangle \ = \ \int_{K_0} x \ f_{u^*} \ d\mu \ = \ 0,$$

that is

$$C_{K_0}(K) \ \subset \ \bigcap_{u^* \in E^*} Ker \ T^*u^*.$$

But T^* has weak*-dense range in $Y^{\perp}$. Therefore

$$Y^{\perp} \ = \ \overline{T^*(E^*)}^{w^*} \ = \ (\bigcap_{u^* \in E^*} Ker \ T^*u^*)^{\perp},$$

which leads to $C_{K_0}(K) \subset Y$. □

Corollary 2. *If Y is a closed linear subspace of $C(K)$ such that Y and $C(K)/Y$ are injected into a reflexive space, then $C(K)$ is in $\mathcal{C}_H$.*

Proof. From Proposition 4, we have that there is a closed subset K_0 of K supporting a Radon probability such that $C_{K_0}(K) \subset Y$. Hence, we have that $C_{K_0}(K)$ is injected into a reflexive space and, from Proposition 3, that $C_{K_0}(K)$ is splitting in $C(K)$. After Lemma 3, since $C(K)/C_{K_0}(K)$ is isomorphic to $C(K_0)$, we obtain that $C(K)/C_{K_0}(K)$ is in $\mathcal{C}_H$. The result now follows from Proposition 2 and Proposition 1. □

Corollary 3. *If Y is a closed linear subspace of $C(K)$ such that Y and $C(K)/Y$ are in $\mathcal{C}_H$, then $C(K)$ is also in $\mathcal{C}_H$; i.e., being injected into a Hilbert space is a 3-space property for $C(K)$ spaces.*

Concerning this 3-space property, this author has learned from the editor, Professor Nigel Kalton, that the problem of whether being injected into a Hilbert space is a 3-space property has a negative solution in general. The particular counterexample to this problem (also provided by Professor Kalton) is the following:

Let Γ be an uncountable set and consider the Banach space $Z_2(\Gamma)$, defined similarly to $Z_2(\mathbb{N})$ in [5, Section 6]. For our purposes, it suffices to recall the following properties of $Z_2(\Gamma)$: It is a reflexive space which can be normed in such a way that it has a closed linear subspace Y isometric to $\ell_2(\Gamma)$ with $Z_2(\Gamma)/Y$ isometric to $\ell_2(\Gamma)$; i.e., $Z_2(\Gamma)$ is a twisted sum of $\ell_2(\Gamma)$ with itself. Furthermore, $Z_2(\Gamma)^*$ is isometric to $Z_2(\Gamma)$. Thus, Y and $Z_2(\Gamma)/Y$ are in class $\mathcal{C}_H$, while we show that $Z_2(\Gamma) \notin \mathcal{C}_H$: Let $T : Z_2(\Gamma) \to H$ be any bounded linear map, where H is a Hilbert space. After [5, Theorem 6.5, Corollary 6.8], it can be deduced that every such map is strictly singular, so the restriction of T to the subspace $\ell_2(\Gamma)$ is therefore compact and is not one-to-one. Consequently, T cannot be one-to-one either.

In [1, Remark 3], the following dichotomy is conjectured: *For a real Banach space X, either it admits a positive definite continuous quadratic homogeneous polynomial or every continuous quadratic homogeneous polynomial has a zero-set that contains a non-separable linear subspace*. In [4], using a weaker form of the 3-space property obtained before, the above conjecture is seen to be correct for $C(K)$ spaces. Our next result is a slight improvement of this.

Let us just recall that if P is a continuous quadratic homogeneous polynomial in X, then $P' : X \to X^*$ represents the Fréchet derivative map, i.e., a bounded linear map such that

$$\langle P'(x), y\rangle = 2 \overset{\vee}{P} (x, y), \quad x, y \in X,$$

where $\overset{\vee}{P}$ stands for the symmetric bilinear functional associated to P.

Corollary 4. *If $C(K) \notin \mathcal{C}_H$, then for each continuous quadratic homogeneous real-valued polynomial P in $C(K)$, every maximal linear subspace Z contained in $P^{-1}(0)$ satisfies that either $Z \notin \mathcal{W}^*$ or $\overline{P'(Z)}$ is not weakly compactly generated.*

Proof. Zorn's Lemma guarantees the existence of maximal linear subspaces contained in the polynomial's zero-set $P^{-1}(0)$. Let Z be one of such maximal linear subspaces, which is clearly closed. Seeking a contradiction, let us assume that $Z \in \mathcal{W}^*$ and $\overline{P'(Z)}$ is weakly compactly generated. Again using Rosenthal's dichotomic result of [6, Lemma 1.3], we know that there is a Radon probability μ on

K such that every element of $\overline{P'(Z)}$ is μ-absolutely continuous. Let $Y := P'(Z)_\perp$ and $K_0 := supp\ \mu$. Proceeding as in the proof of Proposition 4, it is easy to see that $C_{K_0}(K)$ is contained in Y. Besides, $Z \subset Y$ and the maximality of Z yield that $P^{-1}(0) \cap Y = Z$, and so the polynomial P does not change sign in Y; thus, defining $\widetilde{P}(x+Z) := P(x)$, $x \in Y$, we have a continuous quadratic polynomial $\widetilde{P}$ on Y/Z which is positive (or negative) definite. This implies that $Y/Z \in \mathcal{C}_H$; see [1, Proposition 2]. Hence, after Lemma 2, it follows that $Y \in \mathcal{C}_H$. Then, we have that $C_{K_0}(K) \in \mathcal{C}_H$ and since $C(K)/C_{K_0}(K)$ is isomorphic to $C(K_0)$,

$$C(K)/C_{K_0}(K) \ \in \ \ \mathcal{C}_H.$$

The desired contradiction follows from Corollary 3. □

Notice that if P is a continuous quadratic homogeneous real-valued polynomial on the Banach space X, then the linear map P' given by the Fréchet derivative satisfies that $Ker\ P' \subset P^{-1}(0)$. Thus, our next result gives us a quite large linear subspace contained in the polynomial's zero-set for a wide class of compacta.

Corollary 5. *If $C(K) \notin \mathcal{C}_H$ and $C(K)^* \in \mathcal{C}_H$, then for each continuous quadratic homogeneous polynomial P, $Ker\ P' \notin \mathcal{C}_H$.*

Proof. Since $C(K)/Ker\ P'$ is injected into $C(K)^*$, it follows that $C(K)/Ker\ P' \in \mathcal{C}_H$. Hence, from Corollary 3 the result is obtained. □

If K is a scattered compact with an uncountable amount of isolated points, then $C(K) \notin \mathcal{C}_H$, while $C(K)^* = \ell_1(K) \in \mathcal{C}_H$. Hence, after the previous corollary, the next result follows.

Corollary 6. *Let K be a scattered compact space with uncountably many isolated points. Then, for each quadratic continuous homogeneous polynomial P, $Ker\ P' \notin \mathcal{C}_H$.*

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